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A Duchon framework for the sphere

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Abstract

In his fundamental paper [2] Jean Duchon presented a strategy for analysing the accuracy of surface spline interpolants to sufficiently smooth target functions. In the mid-nineties Duchon's strategy was revisited by Light and Wayne [4], who successfully used it to provide useful error estimates for radial basis function interpolation in Euclidean space. A relatively new and closely related area of interest is to investigate how well radial basis functions interpolate data which are restricted to the surface of a unit sphere. In this paper we present a modified version Duchon's strategy for the sphere; this is used in our follow up paper [3] to provide new L_p error estimates ($p \in [1,\infty]$) for radial basis function interpolation on the sphere.

1 Introduction

In the Euclidean space setting the so-called Duchon framework is established as a useful strategy for proving error bounds for radial basis function interpolation, see [4]. In this paper we specialise the framework to the sphere. Our purpose is two fold. First, we will show that it is possible to cover S^{d-1} with a finite collection of geodesic balls G_i such that, for any f belonging to the Sobolev space $W_2^\beta(S^{d-1})$, we have

$$\sum_{G_i} \|f|_{G_i}\|^2_{W_2^\beta(G_i)} \leq C\|f\|^2_{W_2^\beta(S^{d-1})}, \tag{1.1}$$

where the constant C is independent of f. Second, we construct a continuous linear operator E which extends any function f, belonging to the local Sobolev space $W_2^\beta(G(z,\theta))$ on a geodesic ball, to a function Ef, belonging to $W_2^\beta(S^{d-1})$. The norm of this operator can be bounded above by a constant $\mathcal{K}$ which necessarily depends upon θ. We will show that if E is restricted to a certain Sobolev subspace then its norm can be bounded by a constant $\widetilde{\mathcal{K}}$ independent of θ.

1.1 Interpolation theory of Banach spaces: basic results

Let A_0 and A_1 be Banach spaces such that there is a continuous inclusion $A_1 \subset A_0$. For $f \in A_0$ and $t > 0$, define

$$K(t,f) = \inf_{g \in A_1} (\|f-g\|_{A_0} + t\|g\|_{A_1}). \tag{1.2}$$

For $\tau \in (0,1)$, $A_\tau = (A_0, A_1)_\tau$ is defined to be the Banach space with norm

$$\|f\|_\tau = \|K(t,f)t^{-\tau}\|_{L_2((0,\infty),\frac{dt}{t})} = \left(\int_0^\infty \left(\frac{K(t,f)}{t^\tau}\right)^2 \frac{dt}{t}\right)^{1/2}. \tag{1.3}$$

⋄ **Operator interpolation property:** Suppose that we have two interpolation pairs (A_0, A_1) and (B_0, B_1) as above, and a linear operator that maps A_i to B_i, such that

$$\|Tf\|_{B_i} \le C_i \cdot \|f\|_{A_i}, \quad \text{for all } f \in A_i,\ i \in \{0,1\}.$$

Then the *operator interpolation property* says that T may be viewed as a bounded linear map of A_τ to B_τ and

$$\|Tf\|_{B_\tau} \le C_0^{1-\tau} C_1^\tau \cdot \|f\|_{A_\tau}, \quad \text{for all } f \in A_\tau. \tag{1.4}$$

The above results may be found, [8]. Our main interest lies with the Sobolev spaces $W_2^k(\Omega)$, defined for a bounded open set $\Omega \subset \mathbb{R}^d$ and, initially, for a non-negative integer k, as the Hilbert space of functions $f \in L_2(\Omega)$ with norm

$$\|f\|_{W_2^k(\Omega)} = (f,f)^{1/2}_{W_2^k(\Omega)} = \left(\sum_{0 \le |\alpha| \le k} \|D^\alpha f\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}, \tag{1.5}$$

(for details see [7]). The importance of interpolation spaces comes from the fact (cf. [5]) that $(W_2^k(\Omega), W_2^m(\Omega))_\tau$ is norm equivalent to $W_2^{(1-\tau)k+\tau m}(\Omega)$, provided that $(1-\tau)k + \tau m$ is an integer. For this reason, the fractional Sobolev space $W_2^{k+\tau}(\Omega)$, $0 < \tau < 1$, is defined as follows:

$$W_2^{k+\tau}(\Omega) = \left(W_2^k(\Omega), W_2^{k+1}(\Omega)\right)_\tau = \{f \in W_2^k(\Omega) : \|f\|_{W_2^{k+\tau}(\Omega)} < \infty\}, \tag{1.6}$$

where, by (1.3), we have

$$\|f\|_{W_2^{k+\tau}(\Omega)} = \left(\int_0^\infty \left(\frac{K(t,f)}{t^\tau}\right)^2 \frac{dt}{t}\right)^{1/2}, \tag{1.7}$$

and

$$K(t,f) = \inf_{g \in W_2^{k+1}(\Omega)} \left(\|f-g\|_{W_2^k(\Omega)} + t\|g\|_{W_2^{k+1}(\Omega)} \right). \tag{1.8}$$

The development of Sobolev space theory begins with a study of the global spaces, where $\Omega = \mathbb{R}^d$. In order to generalise the various results established for $\mathbb{R}^d$ to the case of a bounded domain Ω, it is important to know whether there exists a continuous linear extension operator

$$E : W_2^k(\Omega) \to W_2^k(\mathbb{R}^d), \text{ satisfying } (Ef)|_\Omega = f, \text{ for all } f \in W_2^k(\Omega). \tag{1.9}$$

In [7], Stein proved the following remarkable theorem, see [7].

Theorem 1.1. *Let Ω be a bounded open connected set with sufficiently smooth boundary. There exists an extension operator (1.9) defined for all nonnegative integers k, such that*

$$\|Ef\|_{W_2^k(\mathbb{R}^n)} \le \mathcal{C}_{ext} \cdot \|f\|_{W_2^k(\Omega)} \text{ where } \mathcal{C}_{ext} \text{ is independent of } f.$$

Furthermore, if $\Omega = B(x,r)$ is an open ball, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ E can be chosen so that the support of Ef is contained in $B(x,(1+\epsilon)r)$.

We conclude with the following useful results.

(i) The space of compactly supported, infinitely differentiable test functions $C_0^\infty(\Omega)$ is contained in each one of the Sobolev spaces $W_2^k(\Omega)$.

(ii) For any $\gamma \in C_0^\infty(\mathbb{R}^d)$ and $f \in W_2^k(\mathbb{R}^d)$, we have that $f \mapsto \gamma f$ is a continuous linear mapping of $W_2^k(\mathbb{R}^d)$ to itself.

2 Sobolev spaces on the sphere

In order to construct a Sobolev extension operator for the sphere we must, first of all, define the relevant local and global spherical Sobolev spaces. There are several (equivalent) ways of defining these spaces, however the definition that we shall use relies on the fact that the sphere is a $(d-1)$−dimensional differentiable manifold. The notion of defining a Sobolev space on a differentiable manifold was considered in [5], we shall give an account of this theory before specialising it to the sphere.

2.1 Differentiable manifolds

Let $\mathbb{M}$ denote a $(d-1)-$dimensional compact differentiable manifold, and suppose that $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^n$ is an atlas for $\mathbb{M}$, i.e., a finite collection of **charts** (U_i, ϕ_i), where U_i are open subsets of $\mathbb{M}$, covering $\mathbb{M}$, and where ϕ_i are infinitely differentiable mappings $\phi_i : U_i \to B(0,1) \subset \mathbb{R}^{d-1}$, whose inverses ϕ_i^{-1} are also infinitley differentiable. Also, let $\{\chi_i : \mathbb{M} \to \mathbb{R}\}_{i=1}^n$ be a **partition of unity** subordinated to the atlas, i.e., a set of infinitely differentiable functions χ_i on $\mathbb{M}$ vanishing outside of compact subsets of the U_i, such that $\sum_i \chi_i = 1$.

For any function $f : \mathbb{M} \to \mathbb{R}$, we can use a partition of unity to write

$$f = \sum_{i=1}^{n} (\chi_i f), \text{ where } (\chi_i f)(m) = \chi_i(m) f(m), \; m \in \mathbb{M}. \tag{2.1}$$

This gives us a decomposition of f in terms of local functions $\chi_i f$, which are compactly supported in U_i. For any function $f : \mathbb{M} \to \mathbb{R}$ with compact support in U_i, we can define its projection $\pi_i(f) : \mathbb{R}^{d-1} \to \mathbb{R}$ onto $\mathbb{R}^{d-1}$ by

$$\pi_i(f)(x) = \begin{cases} f \circ \phi_i^{-1}(x), & \text{if } x \in B(0,1); \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

With this in place, we define the Sobolev space $W_2^\beta(\mathbb{M})$ $(\beta > 0)$ to be the set

$$W_2^\beta(\mathbb{M}) := \left\{ f \in L_2(\mathbb{M}) : \pi_i(\chi_i f) \in W_2^\beta(\mathbb{R}^{d-1}) \text{ for } i = 1, \ldots, n \right\}, \tag{2.3}$$

which is equipped with the norm

$$\|f\|_{W_2^\beta(\mathbb{M})} = \left(\sum_{i=1}^{n} \|\pi_i(\chi_i f)\|^2_{W_2^\beta(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}}. \tag{2.4}$$

2.2 Application to the sphere

Let $\hat{n} = (0, \ldots, 1)$ and $\hat{s} = (0, \ldots, -1)$ denote the north and south poles of the S^{d-1} respectively. Then a simple open cover for the sphere is provided by

$$U_1 = G(\hat{n}, \theta_0) \quad \text{and} \quad U_2 = G(\hat{s}, \theta_0), \text{ where } \theta_0 \in \left(\frac{\pi}{2}, \frac{2\pi}{3} \right). \tag{2.5}$$

Definition 2.1. *The stereographic projection $\sigma_{\hat{n}}$ of the punctured sphere $S^{d-1} \setminus \{\hat{n}\}$ onto $\mathbb{R}^{d-1}$ is defined as the mapping that takes $\xi \in S^{d-1} \setminus \{\hat{n}\}$ to the intersection of the equatorial hyperplane $\{\xi_d = 0\}$, and the extended line that passes through ξ and $\hat{n}$.*

We remark that the stereographic projection $\sigma_{\hat{s}}$ based on $\hat{s}$ can be defined analogously. Using elementary trigonometry we can set

$$\phi_1 = \frac{1}{\tan(\theta_0/2)} \cdot \sigma_{\hat{s}} \quad \text{and} \quad \phi_2 = \frac{1}{\tan(\theta_0/2)} \cdot \sigma_{\hat{n}}, \tag{2.6}$$

and conclude that $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ is a C^∞ atlas of covering coordinate charts for the sphere. Hence, S^{d-1} is a $(d-1)-$dimensional differentiable manifold and so we define the Sobolev space $W_2^\beta(S^{d-1})$ to be the set

$$\Big\{ f \in L_2(S^{d-1}) : \pi_i(\chi_i f) \in W_2^k(\mathbb{R}^{d-1}) \ \text{ for } \ i = 1, 2 \Big\}, \tag{2.7}$$

which is equipped with the norm

$$\|f\|_{W_2^\beta(S^{d-1})} = \left(\sum_{i=1}^{2} \|\pi_i(\chi_i f)\|^2_{W_2^\beta(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}}. \tag{2.8}$$

This definition seems to depend on the choice of atlas used to define S^{d-1}. However, it can be shown that any two spaces defined using two different atlases coincide as sets, and the norms (2.8) are equivalent, see [5] for details.

In order to define the spaces on some geodesic ball, $G(z, \theta)$, we use the coordinate charts to specify open sets in $\mathbb{R}^{d-1}$ by

$$\Omega_i = \phi_i(G(z,\theta) \cap U_i) \quad \text{for} \quad i \in \{1, 2\}. \tag{2.9}$$

The local Sobolev space $W_2^\beta(G(z,\theta))$ of order β is defined to be the set

$$\Big\{ f \in L_2(G(z,\theta)) : \pi_i(\chi_i f)|_{\Omega_i} \in W_2^\beta(\Omega_i) \ \text{ for } \ i \in \{1, 2\}, \ \Omega_i \neq \emptyset \Big\}, \tag{2.10}$$

which is equipped with the norm

$$\|f\|_{W_2^\beta(G(z,\theta))} = \left(\sum_{i=1}^{2} \|\pi_i(\chi_i f)|_{\Omega_i}\|^2_{W_2^\beta(\Omega_i)} \right)^{\frac{1}{2}} \tag{2.11}$$

where, if $\Omega_i = \emptyset$, then we adopt the convention that $\|\cdot\|_{W_2^\beta(\Omega_i)} = 0$.

2.3 On specific local Sobolev spaces

Let $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ denote a fixed atlas for S^{d-1} and let $\{\chi_i\}_{i=1}^2$ denote a corresponding partition of unity. Our aim here is to present a closer analysis of the local spaces $W_2^k(G(z,\theta))$, for k a non-negative integer.

For any function $f \in W_2^k(G(z,\theta))$, we can use the partition of unity and the atlas to write

$$f = \sum_{i=1}^{2} (\chi_i f)|_{G(z,\theta)\cap U_i} = \sum_{i=1}^{2} (\chi_i f) \circ \phi_i^{-1}|_{\Omega_i=\phi_i(G(z,\theta)\cap U_i)}. \tag{2.12}$$

Further, we have the following useful observation.

Observation 2.2. *Each χ_i has compact support,* $\operatorname{supp}\{\chi_i\}$, *in U_i. Thus, there exists a positive constant $C_{\mathcal{A}}$, depending only on $\mathcal{A}$ and $\{\chi_i\}_{i=1}^2$, such that the geodesic distance of* $\operatorname{supp}\{\chi_i\}$ *from the boundary of U_i is strictly greater than $C_{\mathcal{A}}$, for $i \in \{1,2\}$.*

Let $\alpha \in (0,1)$ and let $V_i(\alpha)$ denote the $\alpha C_{\mathcal{A}}-$geodesic neighbourhood of $\operatorname{supp}\{\chi_i\}$, for $i \in \{1,2\}$. Furthermore, if $\theta < C_{\mathcal{A}}/3$ and $z \in S^{d-1}$, then we have the following cases (Figure 1)

1. $z \notin \overline{V_2(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_1(2/3) \subset U_1$, and $\operatorname{supp}\{\chi_2\} \cap G(z,\theta) = \emptyset$,

2. $z \notin \overline{V_1(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_2(2/3) \subset U_2$, and $\operatorname{supp}\{\chi_1\} \cap G(z,\theta) = \emptyset$,

3. $z \in \overline{V_1(1/3)} \cap \overline{V_2(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_i(2/3) \subset U_i$, for $i \in \{1,2\}$, and

- **a)** either $\operatorname{supp}\{\chi_1\} \cap G(z,\theta)$ or $\operatorname{supp}\{\chi_2\} \cap G(z,\theta)$ is nonempty,
- **b)** both $\operatorname{supp}\{\chi_1\} \cap G(z,\theta)$ and $\operatorname{supp}\{\chi_2\} \cap G(z,\theta)$ are nonempty.

We note that the condition $\theta < C_{\mathcal{A}}/3$ is sufficient to guarantee that closure of $G(z,\theta)$ is a subset of at least one of the U_i, $i \in \{1,2\}$. In this case we can use the fact that the coordinate charts $\{\phi_i\}_{i=1}^2$ (2.6) map geodesic balls to Euclidean balls, to deduce that

$$\Omega_i = \phi_i(G(z,\theta)) = B(x_i, r_i). \tag{2.13}$$

For further illustration, suppose that $z \in S^{d-1}$ is positioned as in case 3 above. In this case we have the following strict inclusions

$$\overline{G(z,\theta)} \subset \overline{G(z,C_{\mathcal{A}}/3)} \subset V_i(2/3) \subset U_i, \quad i \in \{1,2\},$$

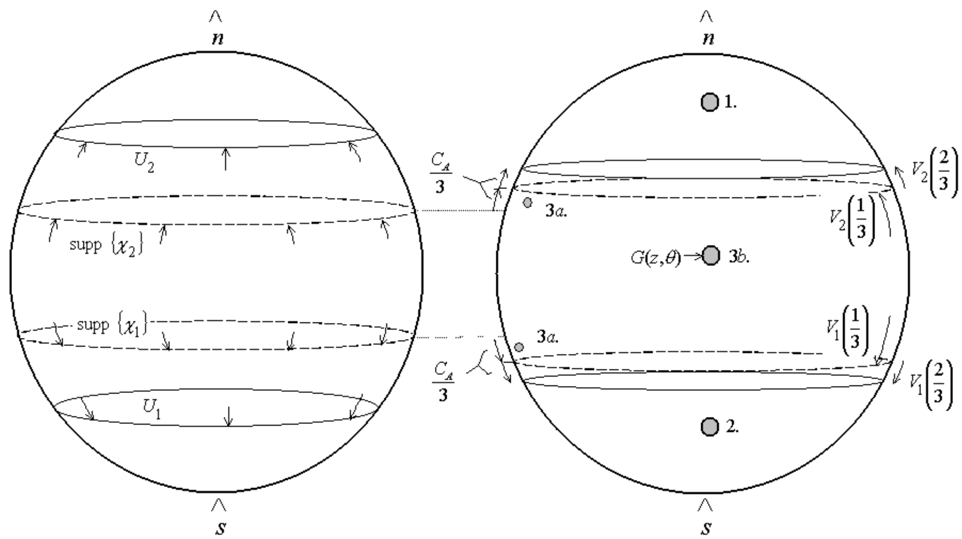

Figure 1: To illustrate the positioning of a geodesic ball of radius $\theta < C_{\mathcal{A}}/3$.

and thus both Ω_1 and Ω_2 are Euclidean balls. Indeed, taking the ϕ_i images gives:

$$\Omega_i \subset \underbrace{\overline{\phi_i(G(z,\theta))}}_{=\overline{B(x_i,r_i)}} \subset \underbrace{\overline{\phi_i(G(z,C_{\mathcal{A}}/3))}}_{=\overline{B(x_i^{\mathcal{A}},r_i^{\mathcal{A}})}} \subset \phi_i(V_i(2/3)) \subset B(0,1), \quad i \in \{1,2\}.$$

We observe that concentric geodesic balls are not, in general, mapped to concentric Euclidean balls. However, since these inclusions are strict there exists a positive constant $e_{\mathcal{A}}$ such that

$$\overline{B(x_i^{\mathcal{A}},r_i^{\mathcal{A}})} \subset \phi_i(V_i(2/3)) \subset B(0,1-e_{\mathcal{A}}) \subset B(0,1), \quad i \in \{1,2\}.$$

Thus, for any positive $\epsilon < e_{\mathcal{A}}$, we have

$$\begin{aligned} \Omega_i = B(x_i,r_i) &\subset \overline{\phi_i(G(z,\theta))} \subset B(x_i,r_i+\epsilon) \\ &\subset B(x_i^{\mathcal{A}},r_i^{\mathcal{A}}+\epsilon) \subset B(0,1), \quad i \in \{1,2\}. \end{aligned} \tag{2.14}$$

Also, when $\overline{G(z,\theta)}$ is **not** completely contained in one of the U_i (a possibility covered by cases 1 and 2 above) then we note that $\mathrm{supp}\{\chi_i\} \cap G(z,\theta) = \emptyset$. In

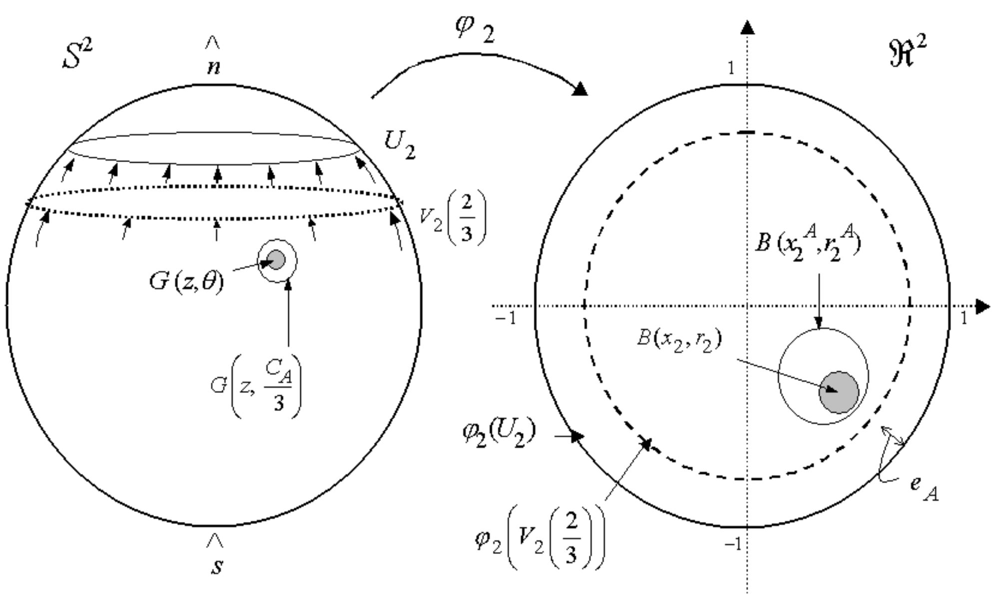

Figure 2: Illustration of Lemma 2.3

this case, for any given $f \in W_2^k(G(z,\theta))$, we can deduce that

$$f_i = (\chi_i f) \circ \phi_i^{-1}|_{\Omega_i} = (\chi_i f)|_{G(z,\theta)\cap U_i} = 0. \tag{2.15}$$

In summary we have:

Lemma 2.3. *Let $C_{\mathcal{A}}$ be as in Observation 2.2. Then for any $z \in S^{d-1}$ and $\theta < C_{\mathcal{A}}/3$ we have*

(i) at least one of Ω_i (2.9) is an open Euclidean ball, $B(x_i, r_i)$;

(ii) there exists a positive constant $e_{\mathcal{A}}$, depending only on the atlas $\mathcal{A}$, such that, if $\Omega_i = B(x_i, r_i)$, then $B(x_i, r_i + \epsilon) \subset B(0,1)$ for all $0 < \epsilon < e_{\mathcal{A}}$.

(iii) if Ω_i is not an open Euclidean ball then $(\chi_i f) \circ \phi_i^{-1}|_{\Omega_i} = 0$, $f \in W_2^k(G(z,\theta))$.

Lemma 2.3 allows us to take the view that a Sobolev space on a suitable geodesic ball in S^{d-1} essentially "behaves" like a Sobolev space on a Euclidean ball in $\mathbb{R}^{d-1}$. The following result shows that the radii of the geodesic and Euclidean balls are comparable.

Lemma 2.4. *Assume that* $\overline{G(z,\theta)} \subset U_i$, $i \in \{1,2\}$, *and let* $B(x_i, r_i)$ *be as in* (2.13)*. Then there exist positive constants* c_ϕ *and* C_ϕ *such that*

$$c_\phi \cdot \theta \le r_i \le C_\phi \cdot \theta, \quad i \in \{1,2\}. \tag{2.16}$$

Proof. The Euclidean and geodesic distances between any two points $\xi, \eta \in S^{d-1}$ are related by the formula

$$d(\xi,\eta) = \|\xi - \eta\| = 2\sin\left(\frac{g(\xi,\eta)}{2}\right).$$

Furthermore, if $\xi, \eta \in S^{d-1} \setminus \{\hat{n}\}$, we have the following relationship from [6]

$$\|\xi - \eta\| = \frac{2\|\sigma_{\hat{s}}(\xi) - \sigma_{\hat{s}}(\eta)\|}{(1 + \|\sigma_{\hat{s}}(\xi)\|^2)^{1/2}(1 + \|\sigma_{\hat{s}}(\eta)\|^2)^{1/2}}.$$

We remark that the analogous relation holds for $\xi, \eta \in S^{d-1} \setminus \{\hat{s}\}$. Let $\xi, \eta \in G(z,\theta)$ then, without loss, we shall establish (2.16) for $i = 1$. The above relations and (2.6) yield

$$\sin\left(\frac{g(\xi,\eta)}{2}\right) = \frac{\tan(\theta_0/2)\|\phi_1(\xi) - \phi_1(\eta)\|}{(1 + \tan^2(\theta_0/2)\|\phi_1(\xi)\|^2)^{1/2}(1 + \tan^2(\theta_0/2)\|\phi_1(\eta)\|^2)^{1/2}}.$$

Since $\phi_1(U_1) = B(0,1)$, we can maximise and minimise this expression by assuming that $\|\phi_1(\xi)\| = \|\phi_1(\eta)\|$ equals 0 and 1 respectively. This gives

$$\sin\theta_0\|\phi_1(\xi) - \phi_1(\eta)\| \le 2\sin\left(\frac{g(\xi,\eta)}{2}\right) \le 2\tan(\theta_0/2)\|\phi_1(\xi) - \phi_1(\eta)\|.$$

For any $\alpha \in (0, \pi/3)$, the small angle result, $\alpha/2 \le \sin\alpha \le \alpha$, implies that

$$\sin\theta_0\|\phi_1(\xi) - \phi_1(\eta)\| \le g(\xi,\eta) \le 2\theta$$

and

$$g(\xi,\eta) \le 4\tan(\theta_0/2)\|\phi_1(\xi) - \phi_1(\eta)\| \le 8\tan(\theta_0/2) r_1.$$

Since we can write

$$2r_1 = \sup_{\xi,\eta \in G(z,\theta)} \|\phi_1(\xi) - \phi_1(\eta)\| \quad \text{and} \quad 2\theta = \sup_{\xi,\eta \in G(z,\theta)} g(\xi,\eta),$$

the proof is completed by taking the supremum on the LHS of the two inequalities above. We find that $c_\phi = (4\tan(\theta_0/2))^{-1}$ and $C_\phi = (\sin\theta_0)^{-1}$. ☐

3 Duchon's inequality for the sphere

The original Duchon framework makes use of a scaled integer lattice in $\mathbb{R}^d$ to provide a regular mesh with a specified spacing. While we do not have quite such a reular mesh on the sphere, we can find a quasi uniform mesh that will satisfy our requirement. An example can be obtained by inscribing a $d-$dimensional cube inside S^{d-1} with a scaled integer lattice embedded on each side, then radially projecting the lattice points to S^{d-1}.

Lemma 3.1. *Let $d \geq 2$, be an integer and set $M = 2\sqrt{d-1}$. Let $\theta \in (0, \pi - \delta_d)$, where $\delta_d = \frac{1}{4d^{3/2}}$ and let M_1 be an arbitrary positive number. Let*

$$h_0 := \min\left\{\frac{\theta}{M + M_1}, 1\right\}. \tag{3.1}$$

Then, for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset S^{d-1}$ such that

$$S^{d-1} = \cup_{z \in Z_h} G(z, Mh).$$

Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. There exists a positive integer Q independent of h such that

$$\sum_{z \in Z_h} F_{G(z, \overline{M}h)} \leq Q, \quad \text{where} \quad \overline{M} = M + M_1. \tag{3.2}$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-(d-1)}$, where C_Q is independent of h.

Proof. Let M_1 be a given positive constant and let $\theta \in (0, \pi - \delta_d)$. Let h_0 be as in (3.1) and choose $h \in (0, h_0)$. To specify a mesh Z_h for the sphere we inscribe a $d-$dimensional cube inside S^{d-1}. This cube will have side $2/\sqrt{d}$. Let $n_h \geq 2$ denote the integer such that

$$\frac{1}{n_h} \leq h < \frac{1}{n_h - 1}. \tag{3.3}$$

On each face of the inscribed cube we place a regular mesh of dimension $d-1$ such that each subcube has side $2/(n_h\sqrt{d})$. That is, we place a lattice of points isomorphic to

$$\frac{2}{n_h\sqrt{d}} \cdot \mathbf{Z}^{d-1} \cap \left[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right]^{d-1}$$

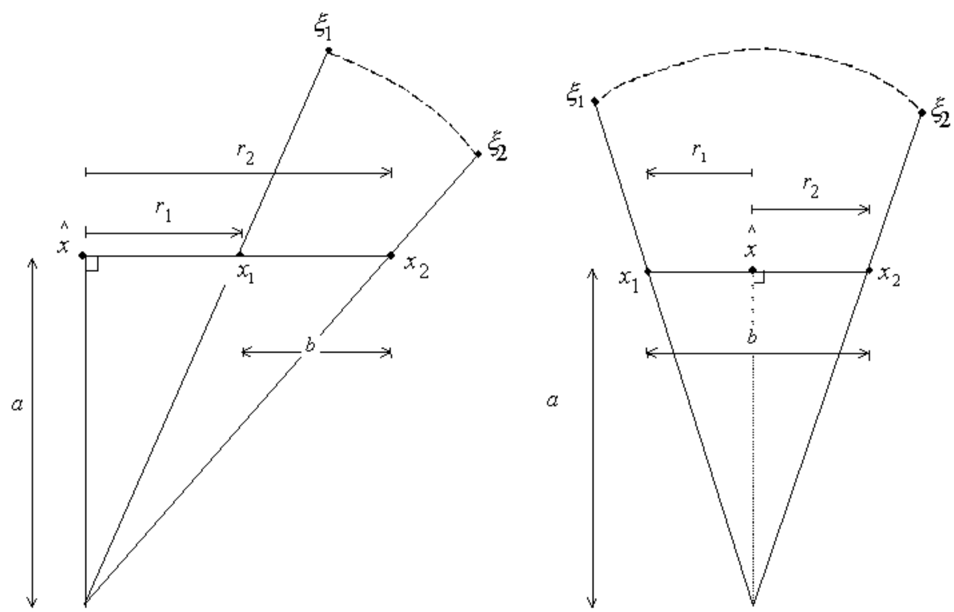

Figure 3: Illustration of quasi-uniform mesh proof

on each side of the cube. We now define Z_h to be the radial projection of these points on the cube onto S^{d-1}.

Suppose that two points x_1 and x_2 lie on one side of the inscribed cube and are a distance b apart. Let ξ_1 and ξ_2 be the radial projections onto S^{d-1} of x_1 and x_2 respectively. Consider Figure 3. The point $\hat{x}$ is the closest point to the origin from the (extended) line connecting x_1 and x_2. Further, r_1 and r_2 represent the respective distances of x_1 and x_2 relative to $\hat{x}$ (with the convention that $r_2 \geq 0$). We observe that the distance a of $\hat{x}$ to the origin satisfies

$$\frac{1}{\sqrt{d}} \leq a < 1. \tag{3.4}$$

The geodesic distance $g(\xi_1, \xi_2)$ is given by

$$g(\xi_1, \xi_2) = \left| \tan^{-1} \frac{r_1}{a} - \tan^{-1} \frac{r_2}{a} \right|.$$

Employing the mean value theorem we can deduce that

$$g(\xi_1, \xi_2) = \frac{b}{a} \cdot \frac{1}{1+\xi^2}, \quad \text{for some } \xi \in \left(-\frac{1}{a}\sqrt{\frac{d-1}{d}}, \frac{1}{a}\sqrt{\frac{d-1}{d}}\right) \tag{3.5}$$

Maximising the RHS of (3.5) and using (3.4) gives $g(\xi_1, \xi_2) \leq \frac{b}{a} < b\sqrt{d}$. Similarly, minimising the RHS gives

$$g(\xi_1, \xi_2) \geq \frac{b}{a} \cdot \frac{ad^2}{d(a^2+1)-1} > \frac{b}{a} \cdot \frac{1}{2d-1} > \frac{b}{2d}.$$

Thus, we have shown

$$\frac{b}{2d} \leq g(\xi_1, \xi_2) \leq b\sqrt{d}. \tag{3.6}$$

For any $\xi \in S^{d-1}$, let z_ξ denote its closest point from Z_h. Let $\mathcal{C}^d$ denote the d-dimensional cube inscribed into S^{d-1}, then we can view the radial projection as a mapping $Proj : \mathcal{C}^d \to S^{d-1}$. In particular, let $x \in \mathcal{C}^d$ be such that

$$Proj\,(x) = \xi.$$

The point x necessarily lies within a $(d-1)$-dimensional subcube, which contains the lattice point $\hat{x}$ such that

$$Proj\,(\hat{x}) = \xi_z.$$

The furthest that x can be away from $\hat{x}$ is bounded above by $\frac{2}{n_h}\sqrt{\frac{d-1}{d}}$, the diameter of the subcube. Hence, by (3.6) and (3.3), any $\xi \in S^{d-1}$ is such that

$$\min_{z \in Z_h} g(z, \xi) = g(z_\xi, \xi) < \frac{2}{n_h}\sqrt{d-1} \leq 2h\sqrt{d-1}.$$

This proves the first part of the theorem for $M = 2\sqrt{d-1}$.

The minimum separation distance of the lattice points on the surface of the inscribed cube is $2/(n_h\sqrt{d})$. Therefore by (3.6) and (3.3)

$$\min_{z,z' \in Z_h} g(z, z') > \frac{1}{2}\left(\frac{2}{n_h\sqrt{d}}\right)\frac{1}{d} > \frac{1}{2}\frac{h}{d^{3/2}} = 2\delta_d h. \tag{3.7}$$

We now show that the second part of the theorem holds by an elementary surface area argument. Let $\xi \in S^{d-1}$ and suppose that

$$\sum_{z \in Z_h} F_{G(z,(M+M_1)h)}(\xi) = N,$$

that is,

$$g(\xi, z_i) < (M + M_1)h = \overline{M}h, \quad \text{for } z_i \in Z_h, \ i = 1, \dots, N.$$

This implies that

$$G(z_i, \delta_d h), \subset G(\xi, \overline{M}h + \delta_d h), \quad \text{for } i = 1, \dots, N. \tag{3.8}$$

For any geodesic ball $G(\xi, \theta) \subset S^{d-1}$, there exists positive constants $\mathcal{C}_1^a$ and $\mathcal{C}_2^a$, depending only on d, such that its surface area is bounded by

$$\mathcal{C}_1^a \cdot \theta^{d-1} \leq \int_{G(\xi,\theta)} d\omega_{d-1} \leq \mathcal{C}_2^a \cdot \theta^{d-1}, \quad \xi \in S^{d-1}.$$

We note that, as a consequence of (3.7), the balls $G(z_i, \delta_d h)$ $(i = 1, \dots, N)$ are disjoint. Thus, by (3.8), we can conclude that the area of $G(\xi, \overline{M}h + \delta_d h)$, which is bounded above by $\mathcal{C}_2^a \cdot (\overline{M}h + \delta_d h)^{d-1}$, must be at least $N\mathcal{C}_1^a \cdot (\delta_d h)^{d-1}$. This implies that

$$N\mathcal{C}_1^a \cdot \delta_d^{d-1} \leq \mathcal{C}_2^a \cdot (\overline{M} + \delta_d)^{d-1}.$$

Therefore, there exists an integer Q that is independent of ξ and h such that

$$N \leq \frac{\mathcal{C}_2^a}{\mathcal{C}_1^a} \left(\frac{\overline{M}}{\delta_d} + 1 \right)^{d-1} \leq Q.$$

To finish the proof we let $|Z_h|$ denote the cardinality of the mesh. Then we have

$$\begin{aligned} |Z_h| \cdot \mathcal{C}_1^a (\overline{M}h)^{d-1} &= \sum_{z \in Z_h} \mathcal{C}_1^a (\overline{M}h)^{d-1} \\ &\leq \sum_{z \in Z_h} \int_{G(z,\overline{M}h)} d\omega_{d-1}(\xi) = \sum_{z \in Z_h} \int_{S^{d-1}} F_{G(z,\overline{M}h)}(\xi) d\omega_{d-1}(\xi) \\ &= \int_{S^{d-1}} \underbrace{\sum_{z \in Z_h} F_{G(z,\overline{M}h)}(\xi)}_{\leq Q} \, d\omega_{d-1}(\xi) \leq \ Q\omega_{d-1}. \end{aligned}$$

That is,

$$|Z_h| \leq \left(\frac{Q\omega_{d-1}}{\mathcal{C}_1^a \overline{M}^{d-1}} \right) h^{-(d-1)} \ = \ C_Q h^{-(d-1)},$$

where C_Q is independent of h. □

We are now in position to prove the main result of this section.

Theorem 3.2. *Let $\beta > 0$ and let M_1 be any positive number. Let h_0 be as in* (3.1) *with $\theta = C_{\mathcal{A}}/3$, that is*

$$h_0 = C_{\mathcal{A}}/(3\overline{M}) \quad \text{where } \overline{M} = 2\sqrt{d-1} + M_1.$$

Let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for S^{d-1} from Lemma 3.1. *Then, for any $f \in W_2^{\beta}(S^{d-1})$, we have*

$$\sum_{z \in Z_h} \|f\|^2_{W_2^{\beta}(G(z,\overline{M}h))} \le Q\|f\|^2_{W_2^{\beta}(S^{d-1})} \tag{3.9}$$

where Q is the constant (independent of h) from Lemma 3.1.

Proof. For $z \in Z_h$ and $i \in \{1, 2\}$ we shall set

$$\Omega_i(z) = \phi_i(G(z, \overline{M}h) \cap U_i) \subset B(0, 1). \tag{3.10}$$

We begin by proving the result in the integer case. Thus, for any $f \in W_2^k(S^{d-1})$, k a non-negative integer, we use (2.11) and consider

$$\sum_{z \in Z_h} \|f\|^2_{W_2^k(G(z,\overline{M}h))} = \sum_{i=1}^{2} \sum_{z \in Z_h} \|\pi_i(\chi_i f)|_{\Omega_i(z)}\|^2_{W_2^k(\Omega_i(z))}. \tag{3.11}$$

Let F_A denote the characteristic function of a set $A \subset \mathbb{R}^d$. For $i \in \{1, 2\}$, consider any function $g \in W_2^k(\mathbb{R}^{d-1})$. Using (3.10) and Lemma 3.1 we can write

$$\begin{aligned}
\sum_{z \in Z_h} \|g|_{\Omega_i(z)}\|^2_{W_2^k(\Omega_i(z))} &= \sum_{z \in Z_h} \sum_{|\alpha| \le k} \|D^{\alpha} g|_{\Omega_i(z)}\|^2_{L_2(\Omega_i(z))} \\
&= \sum_{|\alpha| \le k} \sum_{z \in Z_h} \int_{\Omega_i(z)} \left((D^{\alpha} g|_{\Omega_i(z)})(x)\right)^2 dx \\
&= \sum_{|\alpha| \le k} \int_{\mathbb{R}^{d-1}} \sum_{z \in Z_h} F_{\Omega_i(z)}(x)(D^{\alpha} g(x))^2 dx \\
&\le Q \sum_{|\alpha| \le k} \|D^{\alpha} g\|^2_{L_2(\mathbb{R}^{d-1})} = Q\|g\|^2_{W_2^k(\mathbb{R}^{d-1})}.
\end{aligned} \tag{3.12}$$

Applying these arguments to $g = \pi_i(\chi_i f) \in W_2^k(\mathbb{R}^{d-1})$, and substituting into (3.11) provides the result for the integer order spaces.

Since $\overline{M}h < \overline{M}h_0 < C_{\mathcal{A}}/3$, we can use Lemma 2.3 (iii) to conclude that if $\Omega_i(z)$ is not an open Euclidean ball then $\pi_i(\chi_i f)|_{\Omega_i(z)}$ is the zero function. In particular, for any $f \in W_2^{k+\tau}(S^{d-1})$, where $\tau \in (0,1)$, we can write

$$\sum_{z\in Z_h} \|f\|^2_{W_2^{k+\tau}(G(z,\overline{M}h))} == \sum_{i=1}^{2} \sum_{z\in\mathcal{E}_h(i)} \|\pi_i(\chi_i f)|_{\Omega_i(z)}\|^2_{W_2^{k+\tau}(\Omega_i(z))}, \tag{3.13}$$

where

$$\mathcal{E}_h(i) = \Big\{ z \in Z_h : \Omega_i(z) \text{ is an open Euclidean ball} \Big\}. \tag{3.14}$$

Fix $i \in \{1,2\}$ and let $f_i = \pi_i(\chi_i f)$. Then using (1.7) we have

$$\begin{aligned}\sum_{z\in\mathcal{E}_h(i)} \|f_i|_{\Omega_i(z)}\|^2_{W_2^{k+\tau}(\Omega_i(z))} &= \sum_{z\in\mathcal{E}_h(i)} \int_0^\infty \big(K(t, f_i|_{\Omega_i(z)})\big)^2 \frac{dt}{t^{2\tau+1}} \\ &= \int_0^\infty \sum_{z\in\mathcal{E}_h(i)} \big(K(t, f_i|_{\Omega_i(z)})\big)^2 \frac{dt}{t^{2\tau+1}},\end{aligned} \tag{3.15}$$

Since $\Omega_i(z)$ is an open Euclidean ball for $z \in \mathcal{E}_h(i)$ we have that

$$W_2^{k+1}(\Omega_i(z)) = W_2^{k+1}(\mathbb{R}^{d-1})|_{\Omega_i(z)}, \quad (\text{ see [1]}).$$

Furthermore, the "restriction of functions from $\mathbb{R}^{d-1}$ to $\Omega_i(z)$" can be viewed as a continuous linear operator from $W_2^{k+1}(\mathbb{R}^{d-1})$ to $W_2^{k+1}(\Omega_i(z))$. This implies that we can rewrite the K-functional as

$$K(t, f_i|_{\Omega_i(z)}) = \inf_{g\in W_2^{k+1}(\mathbb{R}^{d-1})} (\|(f_i - g)|_{\Omega_i(z)}\|_{W_2^k(\Omega_i(z))} + t\|g|_{\Omega_i(z)}\|_{W_2^{k+1}(\Omega_i(z))}).$$

Thus, by choosing any $\tilde{g} \in W_2^{k+1}(\mathbb{R}^{d-1})$ we have the following bound

$$\begin{aligned}&\sum_{z\in\mathcal{E}_h(i)} \big(K(t, f_i|_{\Omega_i(z)})\big)^2 \\ &\le \sum_{z\in\mathcal{E}_h(i)} (\|(f_i - \tilde{g})|_{\Omega_i(z)}\|_{W_2^k(\Omega_i(z))} + t\|\tilde{g}|_{\Omega_i(z)}\|_{W_2^{k+1}(\Omega_i(z))})^2.\end{aligned}$$

Expanding the square in the above inequality gives rise to two square terms and a cross term. We investigate these individually.

◇ **Square terms:**

$$\sum_{z\in\mathcal{E}_h(i)} \|(f_i-\tilde{g})|_{\Omega_i(z)}\|^2_{W_2^k(\Omega_i(z))} \le Q\cdot\|f_i-\tilde{g}\|^2_{W_2^k(\mathrm{IR}^{d-1})}.$$

This follows by the integer order argument (3.12), and similarly we have

$$t^2\sum_{z\in\mathcal{E}_h(i)} \|\tilde{g}|_{\Omega_i(z)}\|^2_{W_2^{k+1}(\Omega_i(z))} \le Q\cdot t^2\|\tilde{g}\|^2_{W_2^{k+1}(\mathrm{IR}^{d-1})}.$$

◇ **Cross term:**

$$\begin{aligned}
&2t\sum_{z\in\mathcal{E}_h(i)} \|(f_i-\tilde{g})|_{\Omega_i(z)}\|_{W_2^k(\Omega_i(z))}\|\tilde{g}|_{\Omega_i(z)}\|_{W_2^{k+1}(\Omega_i(z))}\\
&\le 2t\sqrt{\sum_{z\in\mathcal{E}_h(i)} \|(f_i-\tilde{g})|_{\Omega_i(z)}\|^2_{W_2^k(\Omega_i(z))}\sum_{z\in\mathcal{E}_h(i)} \|\tilde{g}|_{\Omega_i(z)}\|^2_{W_2^{k+1}(\Omega_i(z))}}\\
&\le Q\cdot 2t\|f_i-\tilde{g}\|_{W_2^k(\mathrm{IR}^{d-1})}\|\tilde{g}\|_{W_2^{k+1}(\mathrm{IR}^{d-1})}.
\end{aligned}$$

This follows by applying the Cauchy-Schwarz inequality and then employing integer order argument (3.12). Piecing these individual bounds together allows us to conclude that

$$\left(\sum_{z\in\mathcal{E}_h(i)}\left(K(t,f_i|_{\Omega_i(z)})\right)^2\right)^{1/2} \le Q^{1/2}(\|f_i-\tilde{g}\|_{W_2^k(\mathrm{IR}^{d-1})}+t\|\tilde{g}\|_{W_2^{k+1}(\mathrm{IR}^{d-1})}).$$

Taking the infimum over $\tilde{g}\in W_2^{k+1}(\mathrm{IR}^{d-1})$ allows us to deduce that

$$\sum_{z\in\mathcal{E}_h(i)}\left(K(t,f_i|_{\Omega_i(z)})\right)^2 \le Q\cdot K(t,f_i)^2. \tag{3.16}$$

Substituting (3.16) into (3.15) allows us to conclude

$$\sum_{z\in Z_h}\|f\|^2_{W_2^{k+\tau}(G(z,\overline{M}h))} \le Q\sum_{i=1}^{2}\int_0^\infty\left(\frac{K(t,f_i)}{t^\tau}\right)^2\frac{dt}{t} = Q\|f\|^2_{W_2^{k+\tau}(S^{d-1})}.$$

□

4 A Sobolev extension theorem for the sphere

The aim of this section is to construct a continuous extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$, with the property that

$$(E_{G(z,\theta)}f)|_{G(z,\theta)} = f \quad \text{for all } f \in W_2^k(G(z,\theta)).$$

In view of (2.12) we start by extending the local functions $(\chi_i f) \circ \phi_i^{-1}|_{\Omega_i} \in W_2^k(\Omega_i)$ to $W_2^k(\mathbb{R}^{d-1})$ for $i \in \{1,2\}$.

Remark 4.1. *If $\theta < C_{\mathcal{A}}/3$ then, by Lemma 2.3, we can restrict attention to the case where Ω_i is an open Euclidean ball, since otherwise $(\chi_i f) \circ \phi_i^{-1}|_{\Omega_i}$ is the zero function and thus has a trivial extension.*

For the unit ball $B(0,1)$ and for $\epsilon > 0$ sufficiently small, we can appeal to Theorem 1.1 for a continuous extension operator $E_{B(0,1)} : W_2^k(B(0,1)) \to W_2^k(\mathbb{R}^{d-1})$, where

$$\text{supp}(E_{B(0,1)}f) \subset B(0, 1+\epsilon) \text{ for all } f \in W_2^k(B(0,1)).$$

To define an extension operator on $B(x,r)$ we use the coordinate transform

$$\sigma(y) = ry + x, \quad \text{for } r > 0 \text{ and } y \in \mathbb{R}^{d-1}, \tag{4.1}$$

and set

$$E_{B(x,r)}f(y) = \Big(E_{B(0,1)}(f \circ \sigma)\Big) \circ \sigma^{-1}(y) \quad y \in \mathbb{R}^{d-1}. \tag{4.2}$$

In addition, we have that,

$$\text{supp}(E_{B(x,r)}f) \subset B(x, r(1+\epsilon)) \text{ for all } f \in W_2^k(B(x,r)). \tag{4.3}$$

Remark 4.2. *Let $z \in S^{d-1}$, $\theta < C_{\mathcal{A}}/3$, and assume that $\Omega_i = B(x_i, r_i) \subset B(0,1)$. We can use Lemma 2.3 (ii) to choose any $\epsilon < e_{\mathcal{A}}$ such that*

$$\text{supp}(E_{B(x_i,r_i)}f) \subset B(x_i, r_i(1+\epsilon)) \subset B(x_i, r_i+\epsilon) \subset B(0,1).$$

Thus, by choosing a fixed positive $\epsilon < e_{\mathcal{A}}$, which is independent of the centre z of the geodesic ball, we can ensure that $E_{B(x_i,r_i)}f$ is compactly supported in $B(0,1)$, for all $f \in W_2^k(B(x_i,r_i))$.

We are now in a position to prove the first extension theorem.

Theorem 4.3. *Let $z \in S^{d-1}$ and $\theta < C_{\mathcal{A}}/3$, then there exists an extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$ satisfying:*

1. $(E_{G(z,\theta)}f)|_{G(z,\theta)} = f$*, for every* $f \in W_2^k(G(z,\theta))$*,*
2. $\|E_{G(z,\theta)}f\|_{W_2^k(S^{d-1})} \le \mathcal{K}\|f\|_{W_2^k(G(z,\theta))}$*, where* $\mathcal{K}$ *is independent of* f *and* z*.*

Proof. Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. Let $f \in W_2^k(G(z,\theta))$ and $p \in S^{d-1}$, then we define a candidate extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$ by

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} E_{\Omega_i}\Big((\chi_i f) \circ \phi_i^{-1}|_{\Omega_i}\Big)(\phi_i(p)) \cdot F_{U_i}(p). \tag{4.4}$$

Using Remark 4.1, we need only focus on the case where Ω_i is a Euclidean ball. Thus, we shall assume that z is located as in case $(3b)$ see Figure 1. That is, $z \in U_1 \cap U_2$ and $\Omega_i = B(x_i, r_i)$, for $i \in \{1,2\}$. In this case we have

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} E_{B(x_i,r_i)}\Big((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)(\phi_i(p)), \tag{4.5}$$

where $E_{B(x_i,r_i)} : W_2^k(B(x_i,r_i)) \to W_2^k(\mathbb{R}^{d-1})$ is given by (4.2). In addition, we also choose $\epsilon > 0$ as in Remark 4.1 to ensure that

$$\operatorname{supp}\Big\{E_{B(x_i,r_i)}\Big((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)\Big\} \subset B(0,1), \ \text{ for } i \in \{1,2\}. \tag{4.6}$$

To prove part 1 we assume that $p \in G(z,\theta)$, that is, $\phi_i(p) \in B(x_i,r_i)$, for $i \in \{1,2\}$. Then, by Theorem 1.1, we have

$$E_{B(x_i,r_i)}\Big((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)(\phi_i(p)) = (\chi_i f) \circ \phi_i^{-1}(\phi_i(p)) = (\chi_i f)(p),$$

hence

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} (\chi_i f)(p) = f(p), \qquad \text{as required.}$$

To prove part 2 we use (2.8) and consider

$$\begin{aligned} \|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} &= \sum_{j=1}^{2} \|\pi_j(\chi_j E_{G(z,\theta)} f)\|^2_{W_2^k(\mathbb{R}^{d-1})} \\ &= \sum_{j=1}^{2} \|\pi_j(\chi_j)\pi_j(E_{G(z,\theta)} f)\|^2_{W_2^k(\mathbb{R}^{d-1})}. \end{aligned}$$

We note that $\pi_j(\chi_j) \in C_0^\infty(\mathbb{R}^{d-1})$ and so there exists a constant $\mathcal{K}_\chi$ depending only on $\mathcal{A}$ and the partition of unity $\{\chi_j\}_{j=1}^2$ such that

$$\|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} \le \mathcal{K}_\chi \sum_{j=1}^{2} \|\pi_j(E_{G(z,\theta)} f)\|^2_{W_2^k(\mathbb{R}^{d-1})}.$$

$$= \mathcal{K}_\chi \sum_{j=1}^{2} \Big\| \sum_{i=1}^{2} E_{B(x_i,r_i)}\Big((\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)(\phi_i \circ \phi_j^{-1})\Big\|^2_{W_2^k(\mathbb{R}^{d-1})} \qquad \text{by (4.5)}$$

$$= \mathcal{K}_\chi \sum_{j=1}^{2} \Big\| \sum_{i=1}^{2} E_{B(x_i,r_i)}\Big((\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)(\phi_i \circ \phi_j^{-1})\Big\|^2_{W_2^k(B(0,1))} \qquad \text{by (4.6)}$$

$$\le 2\mathcal{K}_\chi \sum_{j=1}^{2}\sum_{i=1}^{2} \Big\| E_{B(x_i,r_i)}\Big((\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)(\phi_i \circ \phi_j^{-1})\Big\|^2_{W_2^k(B(0,1))}.$$

Since $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ is an atlas for S^{d-1}, the coordinate changes, $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$, for $i \neq j \in \{1,2\}$, are infinitely differentiable. Therefore there exists a constant $\mathcal{K}_\mathcal{A}$, depending only on $\mathcal{A}$, such that

$$\|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} \le 2\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \sum_{j=1}^{2}\sum_{i=1}^{2} \Big\| E_{B(x_i,r_i)}\Big((\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)\Big\|^2_{W_2^k(B(0,1))}$$

$$\le 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \sum_{i=1}^{2} \Big\| E_{B(x_i,r_i)}\Big((\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big)\Big\|^2_{W_2^k(\mathbb{R}^{d-1})}. \qquad (4.7)$$

The function $(\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}$ belongs to $W_2^k(B(x_i,r_i))$, for $i \in \{1,2\}$. Thus, we can appeal to Theorem 1.1 to deduce the existence of a constant $\mathcal{C}_{ext}$, independent of $(\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}$ (and therefore of f), such that

$$\|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} \le 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \mathcal{C}_{ext}^2 \sum_{i=1}^{2} \Big\| (\chi_i f)\circ \phi_i^{-1}|_{B(x_i,r_i)}\Big\|^2_{W_2^k(B(x_i,r_i))}.$$

Taking square roots gives

$$\|E_{G(z,\theta)} f\|_{W_2^k(S^{d-1})} \le \mathcal{K}\|f\|_{W_2^k(G(z,\theta))}, \quad \text{where } \mathcal{K} = \mathcal{C}_{ext}\sqrt{4\mathcal{K}_\chi \mathcal{K}_\mathcal{A}}, \qquad (4.8)$$

and this proves the theorem for z as in case $(3b)$; the proof for the other cases follows in a similar, but simpler, fashion. □

We now turn to the extension constant $\mathcal{K}$ (4.8) of the operator $E_{G(z,\theta)}$. In particular, we shall investigate its dependence upon the geodesic radius θ. By inspection, it is clear that the factors $\mathcal{K}_\chi$ and $\mathcal{K}_\mathcal{A}$ are both independent of θ. However, the factor $\mathcal{C}_{ext}$ may depend upon the radii r_1 or r_2 and these are both related to θ by (2.16). Indeed, this dependence is established as follows.

Lemma 4.4. *Let $k > \frac{d-1}{2}$ be a non-negative integer and let $E_{B(x,r)}$: $W_2^k(B(x,r)) \to W_2^k(\mathbb{R}^{d-1})$ be a linear extension operator such that, for all $f \in W_2^k(B(x,r))$,*

$$\|E_{B(x,r)}f\|_{W_2^k(\mathbb{R}^{d-1})} \leq \mathcal{C}_{ext}\|f\|_{W_2^k(B(x,r))}. \tag{4.9}$$

Then the extension constant $\mathcal{C}_{ext} > 0$ is necessarily dependent upon the radius r of the ball.

Proof. Since $k > \frac{d-1}{2}$ the Sobolev embeding theorem [1] tells us that $W_2^k(\mathbb{R}^{d-1})$ is a space of continuous functions. Hence, there exists a constant $c > 0$ such that

$$c \cdot |f(y)| \leq \|f\|_{W_2^k(\mathbb{R}^{d-1})}, \text{ for all } f \in W_2^k(\mathbb{R}^{d-1}) \text{ and } y \in \mathbb{R}^{d-1}. \tag{4.10}$$

Let E denote a continuous linear extension operator and choose $f \in W_2^k(B(x,r))$ to be $f = 1$. A combination of (4.9) and (4.10) yields

$$c \leq \|E_{B(x,r)}f\|_{W_2^k(\mathbb{R}^{d-1})} \leq \mathcal{C}_{ext}\|f\|_{W_2^k(B(x,r))}. \tag{4.11}$$

Now since $\|f\|_{W_2^k(B(x,r))} \to 0$ as $r \to 0$ we can deduce, from (4.11), that the constant $\mathcal{C}_{ext}$ must grow to ∞ as $r \to 0$. Hence $\mathcal{C}_{ext}$ is necessarily dependent on the radius of the ball. □

5 A restricted Sobolev extension theorem

We will now show that if $E_{G(z,\theta)}$ is restricted to a certain Sobolev subspace, then it has an extension constant $\widetilde{\mathcal{K}}$ which is independent of both z and θ. We begin, again, by investigating the analogous problem in the Euclidean setting. Specifically, let $k > \frac{d-1}{2}$, and consider the extension of continuous functions from $W_2^k(B(x^*,r))$ to $W_2^k(\mathbb{R}^{d-1})$. To simplify matters, we shall assume that $B(x^*,r) \subset B(0,1)$ and focus only on the operator

$$E_{B(x^*,r)}f(y) = \Big(E_{B(0,1)}(f \circ \sigma)\Big) \circ \sigma^{-1}(y) \quad y \in \mathbb{R}^{d-1},$$

where $E_{B(0,1)}$ is supplied by Theorem 1.1, and where σ is given by (4.1). One advantage of this construction is that, for any $f \in W_2^k(B(x^*,r))$, we can consider the translated and scaled function $f \circ \sigma \in W_2^k(B(0,1))$. In particular, we have the following useful change of variables result.

Lemma 5.1. *Let $\alpha \in \mathbb{N}_0^d$ be a multi-index with $|\alpha| \le k$. Then, for any $f \in W_2^k(B(x^*,r))$, we have*

$$\|D^\alpha f\|^2_{L_2(B(x^*,r))} = r^{(d-1)-2|\alpha|}\|D^\alpha(f\circ\sigma)\|^2_{L_2(B(0,1))}, \tag{5.1}$$

and similarly, for any $f \in W_2^k(\mathbb{R}^{d-1})$, we have

$$\|D^\alpha f\|^2_{L_2(\mathbb{R}^{d-1})} = r^{(d-1)-2|\alpha|}\|D^\alpha(f\circ\sigma)\|^2_{L_2(\mathbb{R}^{d-1})}. \tag{5.2}$$

Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*,r)$. We measure the density of X in $B(x^*,r)$ by assigning the local Euclidean mesh norm

$$\rho := \rho(X, B(x^*,r)) = \sup_{y\in B(x^*,r)} \min\{\|x-y\| : x \in X\}. \tag{5.3}$$

Remark 5.2. *Assuming that the set $X = \{x_i\}_{i=1}^N$ has mesh-norm ρ in $B(x^*,r)$, then its inverse image $\sigma^{-1}(X)$ has mesh norm ρ/r in $B(0,1)$.*

Consider the following Sobolev subspace

$$\widetilde{W}_2^k(B(x^*,r)) = \left\{ f \in W_2^k(B(x^*,r)) : f(x) = 0, \ \ x \in X \right\} \tag{5.4}$$

where X is a set of distinct points in $B(x^*,r)$. Our first aim is show that if the mesh norm ρ of X is small enough, then the restriction of $E_{B(x^*,r)}$ to $\widetilde{W}_2^k(B(x^*,r))$ has an extension constant $\widetilde{\mathcal{C}}_{ext}$ independent of both x and r.

We begin by providing some background material. Given a non-negative integer k we let $\Pi_k(\mathbb{R}^d)$ denote the space of $d-$variate polynomials of degree at most k. We say that a set of distinct points $\{x_i\}_{i=1}^{M_{k-1}} \subset \mathbb{R}^{d-1}$, where

$$M_{k-1} = \dim \Pi_{k-1}(\mathbb{R}^{d-1}), \tag{5.5}$$

is $\Pi_{k-1}-$unisolvent if the only element of $\Pi_{k-1}(\mathbb{R}^{d-1})$ to vanish at every x_i is the zero polynomial. In connection with this, we have the following definition.

Definition 5.3. *Let $k \in \mathbb{N}$ and M_{k-1} be as in (5.5). We say that an $M_{k-1}-$tuple $(x_1,\ldots,x_{M_{k-1}}) \in B(x^*,r)^{M_{k-1}}$ is "$\Pi_{k-1}-$unisolvent in $B(x^*,r)$" whenever $\{x_1,\ldots,x_{M_{k-1}}\}$ is a $\Pi_{k-1}-$unisolvent set of distinct points.*

We observe that the set

$$\mathcal{U} = \left\{ (x_1,\ldots,x_{M_{k-1}}) \in (\mathbb{R}^{d-1})^{M_{k-1}} : \{x_1,\ldots,x_{M_{k-1}}\} \text{ is } \Pi_{k-1} - \text{unisolvent} \right\}$$

is open (its complement is the set of solutions of algebraic equations). Thus, for a given $(x_1, \dots, x_{M_{k-1}}) \in \mathcal{U}$, there exists $\delta > 0$, such that

$$\overline{B(x_1, \delta)} \times \cdots \times \overline{B(x_{M_{k-1}}, \delta)} \subset \mathcal{U}. \tag{5.6}$$

With this observation, we can formulate the following result.

Lemma 5.4. *Let $(\hat{x}_1, \dots, \hat{x}_{M_{k-1}}) \in B(0,1)^{M_{k-1}}$ be $\Pi_{k-1}-$unisolvent in $B(0,1)$. Then there exists $\delta_{k-1} > 0$ such that $\overline{B(\hat{x}_i, \delta_{k-1})} \subset B(0,1)$, for $1 \le i \le M_{k-1}$, and so that each element of*

$$\mathcal{W} = \overline{B(\hat{x}_1, \delta_{k-1})} \times \cdots \times \overline{B(\hat{x}_{M_{k-1}}, \delta_{k-1})},$$

is $\Pi_{k-1}-$unisolvent in $B(0,1)$.

The next result shows how a sufficiently dense set of points in $B(x^*, r)$ can be mapped, under σ^{-1}, to a set in $B(0,1)$ containing a $\Pi_{k-1}-$unisolvent subset.

Proposition 5.5. *Let $\mathcal{W}$, M_{k-1} and δ_{k-1} be as in Lemma 5.4. Let $X = \{x_i\}_{i=1}^N$ denote a set of $N \ge M_{k-1}$ distinct points in $B(x^*, r)$, with mesh norm ρ (5.3). If $\rho/r < \delta_{k-1}$, then there exists M_{k-1} distinct points,*

$$\{w_1, \dots, w_{M_{k-1}}\} \subset \sigma^{-1}(X) = \{\sigma^{-1}(x_1), \dots, \sigma^{-1}(x_N)\},$$

such that $(w_1, \dots, w_{M_{k-1}}) \in \mathcal{W}$.

Proof. By Remark 5.2, the mesh norm of $\sigma^{-1}(X)$ in $B(0,1)$ is ρ/r. Thus, for each of the points $\hat{x}_i \in B(0,1)$ from Lemma 5.4, we have that

$$\min_{1 \le j \le N} \|\sigma^{-1}(x_j) - \hat{x}_i\| < \rho/r < \delta_{k-1}, \quad \text{for } 1 \le i \le M_{k-1}.$$

In other words, each $\overline{B(\hat{x}_i, \delta_{k-1})}$ contains at least one element, w_i say, from $\sigma^{-1}(X)$, and hence the result follows from Lemma 5.4. □

The following lemma is a specialisation of a key result due to Duchon [2].

Lemma 5.6. (**Duchon**) *Let $k > \frac{d-1}{2}$ be a positive integer, let $B(0,1) \subset \mathbb{R}^{d-1}$ and let $\mathcal{W}$ be as in Lemma 5.4. Then, for each $i \le k$, there exists a constant $C_{\mathcal{W}}(i)$ depending on $B(0,1)$, $\mathcal{W}$, k and i such that*

$$\sum_{|\alpha|=i} \|D^\alpha f\|^2_{L_2(B(0,1))} \le C_{\mathcal{W}}(i) \cdot \sum_{|\alpha|=k} \|D^\alpha f\|^2_{L_2(B(0,1))} \tag{5.7}$$

for all $f \in W_2^k(B(0,1))$ *such that* $f(w_i) = 0$, $i = 1, \dots, M_{k-1}$, *for some* $(w_1, \dots, w_{M_{k-1}}) \in \mathcal{W}$

We are now able to prove the following restricted extension theorem.

Theorem 5.7. *Let* $k > \frac{d-1}{2}$ *be a positive integer, and let* $\mathcal{W}$, M_{k-1} *and* δ_{k-1} *be as in Lemma* 5.4. *Let* X *denote a set of distinct points in* $B(x^*, r) \subset \mathbb{R}^{d-1}$ *whose mesh norm* ρ (5.3) *satisfies*

$$\rho/r < \delta_{k-1}.$$

Let $E_{B(x^*,r)} : W_2^k(B(x^*, r)) \to W_2^k(\mathbb{R}^{d-1})$ *be the extension operator given by* (4.2). *Then there exists a constant* $\widetilde{\mathcal{C}}_{ext}$ *independent of* x^* *and* r *such that*

$$\|E_{B(x^*,r)} f\|_{W_2^k(\mathbb{R}^{d-1})} \le \widetilde{\mathcal{C}}_{ext} \|f\|_{W_2^k(B(x^*,r))} \quad \text{for all } f \in \widetilde{W}_2^k(B(x^*, r)). \quad (5.8)$$

Proof. Let $f \in \widetilde{W}_2^k(B(x^*, r))$, then, applying (1.5), (5.2), and (4.2) respectively, we have

$$\|E_{B(x^*,r)} f\|^2_{W_2^k(\mathbb{R}^{d-1})} = \sum_{0 \le |\alpha| \le k} r^{(d-1)-2|\alpha|} \left\| D^\alpha \Big(E_{B(0,1)}(f \circ \sigma) \Big) \right\|^2_{L_2(\mathbb{R}^{d-1})}.$$

Furthermore, since $r \in (0,1)$ we can deduce that

$$\|E_{B(x^*,r)} f\|^2_{W_2^k(\mathbb{R}^{d-1})} \le r^{(d-1)-2k} \cdot \|E_{B(0,1)}(f \circ \sigma)\|^2_{W_2^k(\mathbb{R}^{d-1})}.$$

By Theorem 1.1 there exists a constant $C_{B(0,1)}$, independent of x and r, such that

$$\|E_{B(x^*,r)} f\|^2_{W_2^k(\mathbb{R}^{d-1})} \le C_{B(0,1)} \cdot r^{(d-1)-2k} \cdot \|f \circ \sigma\|^2_{W_2^k(B(0,1))}. \quad (5.9)$$

We observe that:

(*i*) The assumption $\rho/r < \delta_{k-1}$ implies, by Lemma 5.5, that there exists a set of distinct points $\{w_1, \dots, w_{M_{k-1}}\} \subset \sigma^{-1}(X)$, such that $(w_1, \dots, w_{M_{k-1}}) \in \mathcal{W} \subset B(0,1)^{M_{k-1}}$.

(*ii*) The function $f \circ \sigma \in W_2^k(B(0,1))$ vanishes at each point in $\sigma^{-1}(X)$, and hence at each w_i for $i = 1, \dots, M_{k-1}$.

Together, (i) and (ii) allow us to apply Lemma 5.6. Thus, setting $A^{\mathcal{W}} = \max\{C_{\mathcal{W}}(i) : i = 0, \dots, k\}$, we can employ (1.5), (5.7) and (5.1) to deduce that

$$\begin{aligned}\|f \circ \sigma\|^2_{W_2^k(B(0,1))} &\leq A^{\mathcal{W}} r^{-(d-1)+2k} \sum_{|\alpha|=k} \|D^\alpha f\|^2_{L_2(B(x^*,r))} \\ &\leq A^{\mathcal{W}} r^{-(d-1)+2k} \sum_{0\leq|\alpha|\leq k} \|D^\alpha f\|^2_{L_2(B(x^*,r))}.\end{aligned}$$

That is, we have

$$\|f \circ \sigma\|^2_{W_2^k(B(0,1))} \leq A^{\mathcal{W}} \cdot r^{-(d-1)+2k} \|f\|^2_{W_2^k(B(x^*,r))}. \tag{5.10}$$

Substituting (5.10) into (5.9) and taking square roots yields,

$$\|E_{B(x^*,r)} f\|_{W_2^k(\mathbb{R}^{d-1})} \leq \sqrt{C_{B(0,1)} \cdot A^{\mathcal{W}}} \cdot \|f\|_{W_2^k(B(x^*,r))}.$$

We note that the constant $A^{\mathcal{W}}$ is independent of x^* and r, and hence, setting $\widetilde{\mathcal{C}}_{ext} = \sqrt{C_{B(0,1)} \cdot A^{\mathcal{W}}}$ completes the proof. □

5.1 The extension theorem revisited

In Theorem 4.3, we constructed an extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$. Following the proof of this theorem to equation (4.7), we have

$$\|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} \leq 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \sum_{i=1}^{2} \|E_{B(x_i,r_i)} f_i\|^2_{W_2^k(\mathbb{R}^{d-1})} \tag{5.11}$$

where

$$f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \quad \text{for} \;\; i \in \{1,2\},$$

and where the constants $\mathcal{K}_\chi$ and $\mathcal{K}_\mathcal{A}$ are independent of z and θ.

Observation 5.8. *If, for a given $f \in W_2^k(G(z,\theta))$, the projected functions $f_i \in W_2^k(B(x_i,r_i))$, $i \in \{1,2\}$, vanish on a sufficiently dense set of points in $B(x_i,r_i)$, then we could use a combination of Theorem* 5.7 *and Lemma* 2.4 *to conclude that $E_{G(z,\theta)}$ extends f independently of z and θ.*

The above observation provides us with the strategy to prove the first restricted extension theorem for the sphere. To begin with we let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set distinct points on S^{d-1} whose density is measured using the mesh norm

$$h := h(\Xi, S^{d-1}) := \sup_{\eta\in S^{d-1}} \min\{g(\eta,\xi_i) = \cos^{-1}(\eta^T \xi_i) : \xi_i \in \Xi\}. \tag{5.12}$$

For a positive integer $k > \frac{d-1}{2}$, we consider the following subspace

$$\widetilde{W}_2^k(G(z,\theta)) = \Big\{ f \in W_2^k(G(z,\theta)) : f(\xi_i) = 0, \;\; \xi_i \in \Xi \Big\}. \tag{5.13}$$

Suppose that $f \in \widetilde{W}_2^k(G(z,\theta))$. Then, for $i \in \{1,2\}$, the projected functions $f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \in W_2^k(B(x_i,r_i))$ vanish on the transformed set of points given by

$$X_\theta^{(i)} = \Big\{ \phi_i(\xi) : \xi \in \Xi \cap G(z,\theta) \Big\} \subset B(x_i,r_i). \tag{5.14}$$

In summary, we conclude that

$$f \in \widetilde{W}_2^k(G(z,\theta)) \Rightarrow f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \in \widetilde{W}_2^k(B(x_i,r_i)), \quad \text{for } \; i \in \{1,2\},$$

where

$$\widetilde{W}_2^k(B(x_i,r_i)) = \Big\{ f \in W_2^k(B(x_i,r_i)) : f(x) = 0, \; \text{for } x \in X_\theta^{(i)} \Big\}. \tag{5.15}$$

For $i \in \{1,2\}$, we measure the density of $X_\theta^{(i)}$, which we assume to be non-empty, by assigning the local Euclidean mesh norm

$$\rho_i = \sup_{x \in B(x_i,r_i)} \min \Big\{ \|x - \phi_i(\xi)\| : \xi \in \Xi \cap G(z,\theta) \Big\}. \tag{5.16}$$

Remark 5.9. *Let $k > \frac{d-1}{2}$ be a positive integer and let δ_{k-1} be as in Lemma* 5.4*. Let $f \in \widetilde{W}_2^k(G(z,\theta))$ and assume that the local Euclidean mesh norms* (5.16) *satisfy*

$$\frac{\rho_i}{r_i} < \delta_{k-1}, \quad \text{for } \; i \in \{1,2\}. \tag{5.17}$$

Then, using Theorem 5.7*, there exist a constant $\widetilde{\mathcal{C}}_{ext}$, independent of x_i and r_i, $i \in \{1,2\}$, such that*

$$\|E_{G(z,\theta)} f\|^2_{W_2^k(S^{d-1})} \le 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \widetilde{\mathcal{C}}^2_{ext} \sum_{i=1}^{2} \|f_i\|^2_{W_2^k(B(x_i,r_i))} = \widetilde{\mathcal{K}}^2 \|f\|^2_{W_2^k(G(z,\theta))}.$$

where $\widetilde{\mathcal{K}} = \sqrt{4\mathcal{K}_\chi \mathcal{K}_\mathcal{A}} \widetilde{\mathcal{C}}_{ext}$ is independent of z and θ.

Remark 5.9 provides the route to the initial restriced extension theorem for the sphere. In view of this, we present the geometrical arguments which relate the density of Ξ in S^{d-1} to the densities of the $X_\theta^{(i)}$ in $B(x_i,r_i)$, for $i \in \{1,2\}$.

5.2 Geometrical arguments

Let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points on S^{d-1} whose density is measured by the mesh norm h, given by (5.12). To measure the density of Ξ locally, on some $G(z,\theta)$ say, we assign a local mesh norm by

$$h_L = \sup_{\eta \in G(z,\theta)} \min\Big\{ g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta) \Big\}. \tag{5.18}$$

The following result provides a relationship between h and h_L.

Lemma 5.10. *Let Ξ be a set of points on S^{d-1} with mesh norm $h \in (0, \pi/6)$. Let $z \in S^{d-1}$, $\theta \geq 3h$ and let h_L denote the local mesh norm of $\Xi \cap G(z,\theta)$. Then*

$$h_L \leq 4h. \tag{5.19}$$

Proof. Let $\eta \in G(z,\theta)$ and let ξ be a closest point to η from Ξ. Then, by (5.12), we have $g(\eta,\xi) \leq h$. We prove the lemma by splitting into two cases based on the position of ξ.

(a) If $\xi \in G(z,\theta)$, then $\min\{g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta)\} \leq h < 4h$.

(b) If $\xi \notin G(z,\theta)$, then $g(\xi,z) \geq \theta \geq 3h$. Thus, there exists a point $\eta' \in G(z,\theta)$, lying on the intersection between the boundary of $G(\xi, 2h)$ and the geodesic arc connecting z and ξ, (see Figure 4). That is, η' satisfies

$$g(z,\xi) = g(z,\eta') + g(\eta',\xi) = g(z,\eta') + 2h.$$

Furthermore, there must exist a $\xi' \in \Xi$, such that $g(\eta',\xi') \leq h$. The triangle inequality allows us to deduce

$$\begin{aligned} g(z,\xi') &\leq g(z,\eta') + g(\eta',\xi') = g(z,\xi) - 2h + g(\eta',\xi') \\ &\leq g(z,\eta) + g(\eta,\xi) - h < \theta + h - h = \theta. \end{aligned}$$

Thus, $\xi' \in G(z,\theta)$, and this implies

$$g(\eta,\xi') \leq g(\eta,\xi) + g(\xi,\eta') + g(\eta',\xi') \leq h + 2h + h = 4h.$$

Hence $\min\{g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta)\} \leq 4h$. These arguments hold for any $\eta \in G(z,\theta)$ and so, by (5.18), the proof is complete. ∎

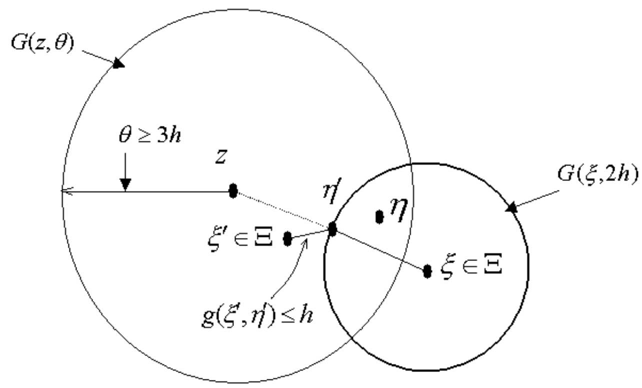

Figure 4: Illustration of Lemma 5.10

The next result shows how the geodesic mesh norm of Ξ relates to the local Euclidean mesh norms of the $X_\theta^{(i)}$, for $i \in \{1,2\}$.

Lemma 5.11. *Let Ξ be a set of points on S^{d-1} with mesh norm $h \in (0, \pi/6)$. Let $z \in S^{d-1}$, $\theta \geq 3h$ and assume that $\overline{G(z,\theta)} \subset U_i$, $i \in \{1,2\}$. Let ρ_i* (5.16) *denote the Euclidean mesh norm of $X_\theta^{(i)} = \phi_i(\Xi \cap G(z,\theta))$, then*

$$\rho_i \leq 4C_\phi h, \quad \text{for } i \in \{1,2\}, \tag{5.20}$$

where C_ϕ is as in Lemma 2.4.

Proof. Let $x \in B(x_i, r_i)$, then $\eta = \phi_i^{-1}(x) \in G(z,\theta)$. By Lemma 5.10, there exists a point $\xi \in \Xi$ such that $g(\eta,\xi) \leq 4h$. We note that $4h \leq 2\pi/3$ and so we can use Lemma 2.4 to deduce that

$$\min_{\xi \in \Xi \cap G(z,\theta)} \|\phi_i(\eta) - \phi_i(\xi)\| = \min_{\xi \in \Xi \cap G(z,\theta)} \|x - \phi_i(\xi)\| \leq 4C_\phi h.$$

This result holds for all $x \in B(x_i, r_i)$ and so proves the lemma. $\square$

5.3 The final theorem for integer order spaces

Let Ξ denote the usual set of distinct points in S^{d-1} with mesh norm $h \in (0, \pi/6)$, given by (5.12). Assume that $\theta < C_{\mathcal{A}}/3$, and let $R > 0$, be such that $\theta = Rh$. Using Lemma 5.11, we can deduce that if $R \geq 3$, then $\rho_i \leq 4C_\phi \cdot h$, for $i \in \{1, 2\}$. Furthermore, using Lemma 2.4, there exists a constant $c_\phi > 0$, such that

$$\frac{\rho_i}{r_i} \leq \frac{4C_\phi \cdot h}{r_i} \leq \frac{4C_\phi \cdot h}{c_\phi Rh} = \frac{4C_\phi}{c_\phi R}, \quad \text{for } i \in \{1, 2\}. \tag{5.21}$$

Thus, if $R \geq 3$ and $R > 4C_\phi / c_\phi \delta_{k-1}$ then condition (5.17) holds. In view of this, by setting

$$\mathcal{R}_0 = \max\left\{3, \frac{4C_\phi}{c_\phi \delta_{k-1}}\right\}, \tag{5.22}$$

we are able to formulate the following extension theorem.

Theorem 5.12. *Let $k > \frac{d-1}{2}$ be a positive integer and let Ξ denote a set of distinct points on S^{d-1} whose mesh norm h satisfies*

$$(i)\ \ h \in (0, \pi/6), \quad (ii)\ \ \mathcal{R}_0 h < C_{\mathcal{A}}/3,$$

where $\mathcal{R}_0$ is given by (5.22). Let $z \in S^{d-1}$, $\theta \in (\mathcal{R}_0 h, C_{\mathcal{A}}/3)$, and let $E_{G(z,\theta)}$ denote the continuous extension operator given by (4.5). Then there exists a constant $\widetilde{\mathcal{K}} > 0$, independent of z and θ, such that

$$\|E_{G(z,\theta)} f\|_{W_2^k(S^{d-1})} \leq \widetilde{\mathcal{K}} \|f\|_{W_2^k(G(z,\theta))} \quad f \in \widetilde{W}_2^k(G(z,\theta)). \tag{5.23}$$

Proof. The conditions of the theorem guarantee that the local mesh norms $\{\rho_i\}_{i=1}^2$ of the transformed point sets $\{X_\theta^{(i)}\}_{i=1}^2$ satisfy (5.17). The theorem then follows from the arguments set out in Remark 5.9. □

5.4 The final theorem for fractional order spaces

We motivate this section by recalling some standard Banach space theory.

Definition 5.13. *A closed linear subspace $\widetilde{A}$ of a Banach space A is said to be a* **complemented subspace** *of A if and only if there exists a continuous projection $\mathcal{P}$ on A with $\mathcal{P}(A) = \widetilde{A}$.*

Let (A_0, A_1) be an interpolation pair and let $\mathcal{P}$ denote a projection operator acting upon both A_0 and A_1. Since a complemented subspace of a Banach space is closed, we can consider the spaces $\widetilde{A}_0 = \mathcal{P}(A_0)$ and $\widetilde{A}_1 = \mathcal{P}(A_1)$ as

Banach space in their own right, with the inherited norms $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ respectively. Furthermore, $(\widetilde{A}_0, \widetilde{A}_1)$ is itself a valid interpolation pair. The following result is due to Triebel, see [8]; Section 1.17.

Theorem 5.14. *Let $\{A_0, A_1\}$ and $\{\widetilde{A}_0, \widetilde{A}_1\}$ denote two interpolation couples, where $\widetilde{A}_0$ and $\widetilde{A}_1$ are the complemented subspaces of A_0 and A_1 respectively, with common projection operator $\mathcal{P}$. Then, for $\tau \in (0,1)$, we have*

$$\widetilde{A}_\tau = (\widetilde{A}_0, \widetilde{A}_1)_\tau = \mathcal{P}(A_0, A_1)_\tau = \mathcal{P}(A_\tau). \tag{5.24}$$

That is, $\widetilde{A}_\tau$ is the complemented subspace of A_τ with the same projection $\mathcal{P}$.

We illustrate this material with the following Sobolev space result.

Proposition 5.15. *Let $k > \frac{d-1}{2}$, be a positive integer. Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*, r) \subset \mathbb{R}^{d-1}$. The familiar Sobolev subspace*

$$\widetilde{W}_2^k(B(x^*, r)) = \Big\{ f \in W_2^k(B(x^*, r)) : f(x) = 0, \;\; x \in X \Big\},$$

is a complemented subspace of $W_2^k(B(x^*, r))$.

Proof. We can choose a set of N linearly independent cardinal functions $\hat{\gamma}_i \in C_0^\infty(B(x^*, r)) \subset W_2^k(B(x^*, r))$ with the following properties

(i) $\hat{\gamma}_i(x_i) = 1$, for $i = 1, \ldots, N$,

(ii) $\hat{\gamma}_i$ has compact support $K_i \subset B(x^*, r)$ and $K_i \cap K_j = \emptyset$ whenever $i \neq j$.

Then, since $k > \frac{d-1}{2}$, we can define a projection operator on $W_2^k(B(x^*, r))$ by

$$\mathcal{Q}_X : f \mapsto \sum_{i=1}^N f(x_i)\hat{\gamma}_i. \tag{5.25}$$

We note that the null space of $\mathcal{Q}_X$ is precisely $\widetilde{W}_2^k(B(x^*, r))$. Thus, setting $\mathcal{P}_X := \mathcal{I} - \mathcal{Q}_X$, where $\mathcal{I}$ denotes the identity, completes the proof. □

For $k > \frac{d-1}{2}$, we can deduce that $\Big(\widetilde{W}_2^k(B(x^*, r)), \widetilde{W}_2^{k+1}(B(x^*, r))\Big)$ is a valid interpolation pair. Thus, for $\tau \in (0,1)$, we can define its interpolation space

$$\widetilde{W}_2^{k+\tau}(B(x^*, r)) = \Big(\widetilde{W}_2^k(B(x^*, r)), \widetilde{W}_2^{k+1}(B(x^*, r))\Big)_\tau. \tag{5.26}$$

In particular, as a corollary of Theorem 5.14, we have the following result.

Theorem 5.16. *For $k > \frac{d-1}{2}$, and $\tau \in (0,1)$, we have that*

$$\widetilde{W}_2^{k+\tau}(B(x^*,r)) = \Big\{ f \in W_2^{k+\tau}(B(x^*,r)) : f(x) = 0, \text{ for } x \in X \Big\}.$$

The final aim of this paper is to prove a fractional version of Theorem 5.12. We begin by considering the Euclidean setting where we have the following intermediate results.

Proposition 5.17. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $E_{B(x^*,r)}$ be the extension operator given by (4.2), which maps $W_2^{k+i}(B(x^*,r))$ to $W_2^{k+i}(\mathbb{R}^{d-1})$, for $i = 0,1$. Then*

(i) $E_{B(x^*,r)} : W_2^{k+\tau}(B(x^*,r)) \to W_2^{k+\tau}(\mathbb{R}^{d-1})$,

(ii) $(E_{B(x^*,r)}f)|_{B(x,r)} = f$, *for all* $f \in W_2^{k+\tau}(B(x^*,r))$,

(iii) $\|Ef\|_{W_2^{k+\tau}(\mathbb{R}^{d-1})} \le \mathcal{C}_{ext}^{(\tau)} \|f\|_{W_2^{k+\tau}(B(x^*,r))}$, *where $\mathcal{C}_{ext}^{(\tau)}$ is independent of f.*

Proof. Parts (i) and (iii) are true by the operator interpolation property. Also, property (ii) holds for all of the integer order spaces (cf. Theorem 1.1), it also holds for the fractional spaces since $W_2^{k+\tau}(B(x^*,r)) \subset W_2^k(B(x^*,r))$. □

Proposition 5.18. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*,r) \subset B(0,1)$ whose mesh norm ρ satisfies*

$$\frac{\rho}{r} < \delta_k, \tag{5.27}$$

where δ_k is as in Lemma 5.4. Then there exists a constant $\widetilde{\mathcal{C}}_{ext}^{(\tau)}$ independent of x^* and r such that

$$\|E_{B(x^*,r)}f\|_{W_2^{k+\tau}(\mathbb{R}^{d-1})} \le \widetilde{\mathcal{C}}_{ext}^{(\tau)} \|f\|_{W_2^{k+\tau}(B(x^*,r))}, \quad f \in \widetilde{W}_2^{k+\tau}(B(x^*,r)). \tag{5.28}$$

Proof. We recall, from Theorem 5.16, that

$$\widetilde{W}_2^{k+\tau}(B(x^*,r)) := \Big\{ f \in W_2^{k+\tau}(B(x^*,r)) : f(x) = 0 \quad x \in X \Big\}.$$

The assumption $\rho/r < \delta_k$, allows us to deduce, from Theorem 5.7, that

$$\|E_{B(x^*,r)}f\|_{W_2^{k+i}(\mathbb{R}^{d-1})} \le C_i \|f\|_{W_2^{k+i}(B(x^*,r))}, \text{ for all } f \in \widetilde{W}_2^{k+i}(B(x^*,r)),$$

for $i \in \{0,1\}$, where the constants C_0 and C_1 are independent of x^* and r. Let $A_i = \widetilde{W}_2^{k+i}(B(x,r))$ and $B_i = W_2^{k+i}(\mathbb{R}^{d-1})$, $i \in \{0,1\}$, then the result follows from the operator interpolation property, which shows that $\widetilde{\mathcal{C}}_{ext}^{(\tau)} = C_0^{1-\tau} C_1^{\tau}$. □

We can now turn attention to the spherical setting and we begin by defining the appropriate fractional subspace.

Definition 5.19. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points in $G(z,\theta) \subset S^{d-1}$, then we define the local fractional order Sobolev subspace as*

$$\widetilde{W}_2^{k+\tau}(G(z,\theta)) = \Big\{ f \in W_2^{k+\tau}(G(z,\theta)) : f(\xi) = 0,\ \xi \in \Xi \cap G(z,\theta) \Big\}. \quad (5.29)$$

Using Proposition 5.17, we can recast Theorem 4.3 in terms of fractional Sobolev spaces, and its proof is completely analogous. In particular, following the proof through to inequality (4.7), we have

$$\|E_{G(z,\theta)} f\|^2_{W_2^{k+\tau}(S^{d-1})} \le 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \sum_{i=1}^{2} \|E_{B(x_i,r_i)} f_i\|^2_{W_2^{k+\tau}(\mathbb{R}^{d-1})}, \quad (5.30)$$

where,

$$f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \in W_2^{k+\tau}(B(x_i,r_i)), \quad \text{for } i \in \{1,2\}. \quad (5.31)$$

Let δ_k be as in Lemma 5.4. Let C_ϕ and c_ϕ be as in (2.16), and set

$$\mathcal{R}_0 = \max\Big\{3, \frac{4C_\phi}{c_\phi \delta_k}\Big\}. \quad (5.32)$$

Theorem 5.20. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let Ξ denote a set of distinct points on S^{d-1} whose mesh norm h satisfies*

$$(i)\ \ h \in (0,\pi/6), \quad (ii)\ \ \mathcal{R}_0 h < C_\mathcal{A}/3,$$

where $\mathcal{R}_0$ is given by (5.32). Let $z \in S^{d-1}$, $\theta \in (\mathcal{R}_0 h, C_\mathcal{A}/3)$, and let $E_{G(z,\theta)}$ denote the continuous extension operator given by (4.5). Then there exists a constant $\widetilde{\mathcal{K}}^{(\tau)} > 0$, independent of z and θ, such that

$$\|E_{G(z,\theta)} f\|_{W_2^{k+\tau}(S^{d-1})} \le \widetilde{\mathcal{K}}^{(\tau)} \|f\|_{W_2^{k+\tau}(G(z,\theta))} \quad f \in \widetilde{W}_2^{k+\tau}(G(z,\theta)). \quad (5.33)$$

Proof. The conditions of the theorem guarantee that the mesh norms $\{\rho_i\}_{i=1}^2$ of the transformed point sets $\{X_\theta^{(i)}\}_{i=1}^2$ satisfy (5.27). Since each f_i vanishes at $X_\theta^{(i)}$, $i \in \{1,2\}$, we can use Proposition 5.18, to continue inequality (5.30) as follows

$$\begin{aligned} \|E_{G(z,\theta)}f\|^2_{W_2^{k+\tau}(S^{d-1})} &\le 4\mathcal{K}_\chi\mathcal{K}_\mathcal{A}(\widetilde{\mathcal{C}}^{(\tau)}_{ext})^2\sum_{i=1}^{2}\|f_i\|^2_{W_2^{k+\tau}(B(x_i,r_i))} \\ &= (\widetilde{\mathcal{K}}^{(\tau)})^2\|f\|^2_{W_2^{k+\tau}(G(z,\theta))}. \end{aligned}$$

where $\widetilde{\mathcal{K}}^{(\tau)} = \sqrt{4\mathcal{K}_\chi\mathcal{K}_\mathcal{A}}\widetilde{\mathcal{C}}^{(\tau)}_{ext}$ is independent of both z and θ. □

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References

[1] J. P. Aubin: *Approximation of Elliptic Boundary-Value Problems,* Wiley-Interscience, (1972).

[2] J. Duchon: *Sur l'erreur d'interpolation des fonctions de plusieurs variable par les D^m -splines,* RAIRO Anal Numer. **12**, (1978) 325–334.

[3] S. Hubbert and T. M. Morton: *L_p-error estimates for radial basis function interpolation on the sphere,* preprint (2002).

[4] W. A. Light and H. Wayne: *Power functions and error estimates for radial basis function interpolation,* J. Approx. Theory **92**, (1992) 245–267. 111–130.

[5] J. L. Lions and E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications,* Vol. 1, Springer Verlag, Berlin-Heidelberg (1972).

[6] J. G. Ratcliffe: *Foundations of Hyperbolic Manifolds,* Springer-Verlag, New York (1994)

[7] E. M. Stein: *Singular integrals and differentiability properties of functions,* Princeton Univ. Press, (1971).

[8] H. Triebel: *Interpolation theory, function spaces and differential operators,* Amsterdam, North-Holland, (1978).

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