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Reflection and potentialism

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PhD Thesis

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Submitted October 2015

Abstract

It was widely thought that the paradoxes of Russell, Cantor, and Burali-Forti had been solved by the iterative conception of set. According to this conception, the sets occur in a well-ordered transfinite series of stages. On standard articulations – for example, those in Boolos (1971, 1989) – the sets are implicitly taken to constitute a plurality. Although sets may fail to exist at certain stages, they all exist *simpliciter*. But if they do constitute a plurality, what could stop them from forming a set? Without a satisfactory answer to this question, the paradoxes threaten to reemerge. In response, it has been argued that we should think of the sets as an inherently potential totality: whatever things there are, there could have been a set of them. In other words, any plurality could have formed a set. Call this *potentialism*. *Actualism*, in contrast, is the view that there could not have been more sets than there are: whatever sets there could have been, there are. This thesis explores a particular consideration in favour of actualism; namely, that certain desirable second-order resources are available to the acutalist but not the potentialist.

In the first part of chapter 1 I introduce the debate between potentialism and actualism and argue that some prominent considerations in favour of potentialism are inconclusive. In the second part I argue that potentialism is incompatible with

the potentialist version of the second-order comprehension schema and point out that this schema appears to be required by strong set-theoretic reflection principles. In chapters 2 and 3 I explore the possibilities for reflection principles which are compatible with potentialism. In particular, in chapter 2 I consider a recent suggestion by Geoffrey Hellman for a modal structural reflection principle, and in chapter 3 I consider some influential proposals by William Reinhardt for modal reflection principles.

I hereby declare that all the work presented in this thesis is my own:

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Sam Roberts

Acknowledgements

Intellectually, my greatest debt is to my supervisor Øystein Linnebo. He is a model of clarity and insight, and his continued advice and encouragement is more than I deserve. I was lucky to have written this thesis as a member of his Plurals, Predicates, and Paradox project in London and then later in Oslo. It was an extremely stimulating and supportive environment to have come up in, and I'm deeply grateful to the other members – Salvatore Florio, Jönne Kriener, Jon Litland, and Sean Walsh – for many fruitful discussions. I hope their positive influence is clear, but I fear it is not.

Personally, I couldn't have done very much of anything without my partner Siobhan. Also, my parents aren't bad.

Chapter 2: I want to thank Salvatore Florio, Geoffrey Hellman, Simon Hewitt, Leon Horsten, and an anonymous referee at the Review of Symbolic Logic for helpful comments on earlier drafts.

Chapter 3: I want to thank Leon Horsten, Pen Maddy, and Sean Walsh for helpful comments on earlier drafts (the first of which was written whilst on a visit to Irvine, which I am very grateful to Pen and Sean for supporting).

Contents

Chapter 1

Potentialism and reflection

It was widely thought that the paradoxes of Russell, Cantor, and Burali-Forti had been solved by the iterative conception of set. According to this conception, the sets occur in a well-ordered transfinite series of stages. On standard articulations – for example, those in Boolos (1971, 1989) – the sets are implicitly taken to constitute a plurality. Although sets may fail to exist at certain stages, they all exist *simpliciter*. But if they do constitute a plurality, what could stop them from forming a set? Without a satisfactory answer to this question, the paradoxes threaten to reemerge. In response, it has been argued that we should think of the sets as an inherently potential totality: whatever things there are, there could have been a set of them. In other words, any plurality could have formed a set. Call this *potentialism*. *Actualism*, in contrast, is the view that there could not have been more sets than there are: whatever sets there could have been, there are. The central question of this thesis is which of potentialism and actualism is true. This chapter serves as an introduction to the debate and to the central aim of the thesis; namely, to investigate the bearing of set-theoretic reflection principles on

the central question.

Here's the plan. In section 1 I introduce the debate between potentialism and actualism. I argue that some prominent considerations in that debate are inconclusive and that we should look for others. In sections 2 and 3 I develop a new consideration based on the idea that certain second-order resources are desirable but unavailable to the potentialist. In particular, in section 2 I argue that potentialism is incompatible with the potentialist version of the second-order comprehension schema and in section 3 I show that this schema follows from certain strong set-theoretic reflection principles. Assuming these principles are the best way we have to answer various questions left open by the standard axioms of ZFC, and given that they are compatible with actualism, this gives us a reason to prefer actualism over potentialism. The rest of the thesis will then consider some proposals for resisting this conclusion.

1.1 Potentialism and actualism

In this section I introduce the debate between potentialism and actualism, and argue that some prominent considerations in favour of potentialism are inconclusive.

1.1.1 The paradoxes

Russell's paradox

The set-theoretic paradoxes take a number of forms. Perhaps the simplest is Russell's. Suppose:

$$\exists x \forall y (y \in x \leftrightarrow y \notin y) \tag{1.1}$$

Instantiating y with x we then get:

$$\exists x (x \in x \leftrightarrow x \notin x) \tag{1.2}$$

which is a contradiction in classical logic. This shows that 1.1 is false. For some, the falsity of 1.1 demands explanation. As Dummett puts it:

> merely to say, 'If you persist in talking about the [Russell set], you will run into contradiction', is to wield the big stick, but not to offer an explanation. (p. 316, 1991)

Why is the purely logical argument for the falsity of 1.1 unexplanatory? It is unclear. Logic seems to be as capable of providing explanations as theories are in general, and the argument for the falsity of 1.1 is a prima facie example. Perhaps what is demanded is an explanation of why the theorems of classical logic are true. But explanation has to end somewhere, and logic seems as good a place as any.

Say that a condition φ determines a set x if $\forall y (y \in x \leftrightarrow \varphi)$. Assuming there are at least some sets, a more general moral to draw from the falsity of 1.1 is that the question:

(1) When does a condition φ determine a set?

requires a more sophisticated answer than "never" and "always". As Russell puts it:

> the complete solution of [Russell's paradox] would consist in the dis-
> covery of the precise conditions which [some φ] must fulfil in order to
> define [a set]. (p. 31, 1906)

What would count as an adequate answer to (1)? The preceding discussion sug-
gests at least two constraints. First, it should imply for various particular φ
whether or not they determine sets. For example, any answer couched in classical
logic will imply that "$x \notin x$" does not determine a set. Second, it should explain
why various particular φ do or do not determine sets. For example, any answer
couched in classical logic prima facie explains why "$x \notin x$" does not determine a
set. In general, an answer to (1) should be informative and explanatory.

Digression. It would be too much to expect an answer to be informative in every
case. For any sentence ψ, the reasoning of Russell's paradox shows that the condi-
tion "$x \notin x \wedge \neg\psi$" will determine a set just in case ψ (assuming that the empty set
exists). Any theory which implied for all conditions whether or not they determine
sets would thus be complete.

Requiring an answer to (1) to be informative and explanatory is an instance of
the more general abductive methodology. According to this methodology, theories
are judged relative to criteria such as generality, informativeness, explanatoriness,
unification, simplicity, and strength.[1] We are most justified in believing the theory
which best meets these criteria. I will adopt this methodology in what follows.

The most well-known answers to (1) are the Iterative (IC) and Limitation of
Size (LOS) conceptions of set. Roughly, IC says that the universe of sets is divided
into an unbounded well-ordered series of stages. A condition φ determines a set at

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[1]See Lipton (2004) for discussion and Williamson (2015) for a nice application of the method-
ology to the semantic paradoxes.

a stage s just in case the φ's all exist at some stage prior to s, and it determines a set *simpliciter* if there is some stage or other at which the φ's all exist.[2] As an answer to (1) IC is quite informative. Suitably formalised it implies the axioms of pairing, union, powerset, separation, and foundation.[3] It also seems to be explanatory in many cases too. Consider, for example, its explanation for why the axiom of separation is true – that is, why the condition "$y \in x \wedge \varphi$" always determines a set. First, we note that if x exists at a stage s, its elements will exist at some stage, s', prior to s. Trivially, the φ's in x will also exist at s' and it then follows that they must determine a set at s.

Roughly, LOS says that a condition φ determines a set just in case there are fewer φ's than sets.[4] As an answer to (1) LOS is also quite informative. Suitably formalised it implies the axioms of union, separation, replacement, and choice. In the presence of the axiom of infinity, it also implies the axiom of pairing.[5] Furthermore, it seems to be explanatory in many cases too. For example, its explanation for why the axiom of separation is true is that since x is a set, it will have fewer elements than the sets; the φ's in x will then be fewer than the sets and thus determine a set.

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[2]See Boolos (1971, 1989) for details. In standard set theory, the stages of the IC can be represented by the V_α's (which are, as usual, defined by recursion on the von Neumann ordinals: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and $V_\lambda = \bigcup_{\alpha < \lambda}$). For present purposes, little will be lost if IC is thought of in these terms.

[3]Again, see Boolos (1971, 1989) for details and see Tait (1998) for the observation that the axiom of infinity, pace Boolos, is no more part of IC than the axiom of replacement.

[4]For now, I will leave the locution "fewer than" primitive.

[5]See Levy (1968) and the references therein for details.

Plural Russell's paradox

Now consider the following version of Russell's paradox which arises in context of plural logic. It rests on two principles. First, there is a principle of plural comprehension. This says that every condition φ determines a plurality.[6] Formally:

(P-COMP) $\exists X \forall x (x \in X \leftrightarrow \varphi)$

where X is not free in φ. Second, there is a principle of plural collapse. This says that every plurality determines a set. Formally:

(COLLAPSE) $\forall X \exists x (X \equiv x)$

where $X \equiv x$ abbreviates $\forall y (y \in X \leftrightarrow y \in x)$. Since P-COMP and COLLAPSE jointly entail 1.1, they are jointly inconsistent in classical logic.

As with the falsity of 1.1, and assuming some pluralities determine sets, the falsity of either P-COMP or COLLAPSE shows that either the question:

(2) When does a condition φ determine a plurality?

or the question:

(3) When does a plurality determine a set?

requires a more sophisticated answer than "never" and "always".

It is easy to see that question (3) is subsumed by question (1). An answer to the latter is, *a fortiori*, an answer to the former since "$x \in X$" is a condition.

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[6] For simplicity I will frequently use the singular "plurality", though nothing I say will depend on misspeaking in this way.

Thus IC and LOS already provide answers to question (3). According to IC, some things determine a set just in case there is a stage at which they all exist; and according to LOS, some things determine a set just in case they are fewer than the sets. Moreover, in the presence of P-COMP, these answers to (3) seem to inherit the informativeness and explanatoriness of the corresponding answers to (1). Whenever IC or LOS implies that φ determines a set, it will imply that the plurality of φ's determines a set. Similarly, whenever IC or LOS explains why φ determines a set, it will explain why the plurality of φ's determines a set. A proponent of IC or LOS can thus afford to answer "always" to (2) and adopt P-COMP. This is particularly satisfying because P-COMP follows from the compelling thought that a plurality is nothing over and above the things which comprise it.[7] Since each individual φ exists trivially,[8] nothing more is needed for the plurality of φ's to exist. In so far as IC and LOS provide attractive answers to question (1), then, IC + P-COMP and LOS + P-COMP seem to provide attractive answers to questions (2) and (3).

Are there plausible alternatives to IC + P-COMP and LOS + P-COMP? In particular, could we instead have answered "always" to question (3) and adopted COLLAPSE? Assume every set determines a plurality. Formally: $\forall x \exists X (X \equiv x)$. Then COLLAPSE will imply that a condition determines a plurality just in case it determines a set, and questions (1) and (2) will be equivalent. But neither question is answered by COLLAPSE alone, since it does not tell us in general when conditions determine pluralities nor when they determine sets. It needs to be supplemented. The most obvious way to do this is by adopting IC or LOS.

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[7] See (Boolos, 1984, p. 448) for discussion.
[8] The claim that each φ exists has the form: $\forall x(\varphi \rightarrow \exists y(y = x))$.

How do IC + COLLAPSE and LOS + COLLAPSE compare to IC + P-COMP and LOS + P-COMP? In their answers to question (1) IC + COLLAPSE and IC + P-COMP are essentially identical. But whereas IC + COLLAPSE extends that answer to question (2) and gives a maximally liberal answer to question (3), IC + P-COMP extends it to question (3) and gives a maximally liberal answer to question (2). Similarly for LOS + COLLAPSE and LOS + P-COMP. The primary difference comes in explaining these maximally liberal stances. If COLLAPSE is true, it is a striking fact and prima facie it is unclear how it might be explained. But as I noted above, the truth of P-COMP can be explained by the thought that a plurality is nothing over and above the things which comprise it.[9]

Absent other ways to supplement COLLAPSE, IC + P-COMP and LOS + P-COMP appear to be preferable. Unfortunately, matters are not so simple. As we will see in the next section, there seem to be powerful arguments in favour of COLLAPSE.

1.1.2 Arguments for COLLAPSE

Since the clearest arguments for COLLAPSE are given in Linnebo (2010), I will focus on those.

The argument from specification

Sets are completely characterised by their membership conditions. First, a set is distinguished from all other sets by the elements it has. This is partly expressed by the axiom of extensionality which says that two sets are identical if they have the same elements. Second, there seems to be nothing more to a set than its

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[9]This does not merely push the explanation back one step, since that thought implies a number of other important principles concerning the behaviour of pluralities, for example in modal contexts. See sections 1.1.3 and 2.3.2 for discussion.

having the elements it has.[10] To completely specify a set it thus suffices to specify which elements it has. But that is exactly what a plurality does. A plurality X specifies some elements, namely the X's, and thus provides us with a complete characterisation of the set x such that:

$$X \equiv x$$

As Linnebo puts it:

> The semantics of plural quantification ensures that it is determinate which things are among $[X]$, and a set is completely characterized by specifying its elements. We can thus give a complete and precise characterization of the set that $[X]$ would form if they did form a set. What more could be needed for such a set to exist? (Linnebo, p. 146, 2010)

Given a plurality, nothing more seems to be needed for the existence of the corresponding set.[11] Therefore, we should accept COLLAPSE.

The argument from arbitrariness

As Linnebo (p. 152, 2010) essentially notes, relative to some natural set-theoretic assumptions, any view on which there is a plurality of all sets (call it V) will imply that a condition φ determines a set just in case the φ's are fewer than the V's. Call

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[10]As Linnebo puts it: "Once you specify the elements of a set, you have specified everything that is essential to it." (p. 146, 2010). It is worth pointing out that *being a set* is also essential to a set and does not in general depend on which elements it has. For example, the empty set has the same elements as any non-set. The claim is thus that once you have specified the elements of a set, you have specified everything that is essential to it *over and above its being a set.*

[11]See (Glanzberg, 2004, p. 553-554) for a somewhat similar argument.

this claim LOS_V. It tells us that the plurality V marks the cut off point between conditions which do and do not determine sets. Since V is the plurality of all sets, this cut off point appears to be non-arbitrary. The problem, as Linnebo sees it, is that:

> ...it is not clear that the plurality $[V]$ is special and non-arbitrary. No doubt there is something special and non-arbitrary about the concept of [a set]. But why should this non-arbitrariness be inherited by the plurality $[V]$ of objects falling under the concept? (p. 153, 2010)

For example, suppose there had been more sets than there are. In particular, suppose there had been a set v of the V's. In that case, there would have been a set – vis. v – with as many elements as the V's and LOS_V would have been false. At best LOS_V is only contingently true and is thus an arbitrary answer to question (1). Since any plausible alternative to COLLAPSE will adopt P-COMP, it will be committed to this arbitrary answer. Therefore, we should accept COLLAPSE.

The argument from circularity

Whereas the argument from arbitrariness targeted P-COMP, the final argument targets IC and LOS. Since it is clearest in the case of LOS, I will focus on that. It is based on the claim that LOS is unexplanatory in certain crucial cases. For example, it seems that LOS's explanation for why "x is a set" fails to determine a set is that the sets are not fewer than the sets. But, clearly, this is circular and thus unexplanatory.[12] Similarly, it seems that its explanation for why "x is an ordinal" fails to determine a set is that the ordinals are not fewer than the sets.

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[12]See Linnebo (p. 153-4, 2010).

But in the context of LOS the usual way of establishing that the ordinals are not fewer than the sets is by noting that "x is an ordinal" fails to determine a set and concluding via LOS that they are not fewer than the sets. Again, this is circular and thus unexplanatory.

Digression. This is not yet an argument in favour of COLLAPSE, since as we saw in the previous section, COLLAPSE is not an alternative to IC or LOS. Indeed, it seemed clear that in order to answer questions (1) and (2) COLLAPSE needed to be supplemented with something like IC or LOS. In so far as the argument from circularity is successful, then, it targets all the views we have considered so far.

1.1.3   Going modal

Until now I have implicitly assumed that ordinary first-order quantifiers range over absolutely all sets. For example, I implicitly took:

$$\forall y(y \in x \leftrightarrow \varphi)$$

to formalise the claim that x is the set of absolutely all φ's. The *potentialist* rejects this assumption.[13] According to them, there could have been sets other than there are. In order to quantify over absolutely all sets, then, we need also to quantify over the sets there could have been. Formally, we can do this using a modal operator $\Diamond$.

Definition 1. Let the *modalisation* (or $\varphi^{\Diamond}$) of a formula φ be the result of prefixing all of its universal quantifiers with $\Box$ and all of its existential quantifiers with $\Diamond$.

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[13]Potentialism in one form or another has been endorsed by Zermelo (1930), Parsons (1977), Putnam (1967), Hellman (1989), and recently Fine (2006), Linnebo (2010, 2013), and Studd (2013, forthcoming). My presentation closely follows that in Linnebo (2010, 2013).

When φ is a non-modal formula, $\varphi^{\Diamond}$ essentially says about what there could have been what φ says about what there is.[14]

The distinction between a formula and its modalisation affords the potentialist a robust response to the arguments from the previous section. Consider first the argument from specification. It concluded that given a plurality, nothing more is needed for the existence of the corresponding set. This in turn seemed to imply COLLAPSE. But the potentialist can now block this implication. In particular, they can claim that the argument only shows that there could have been a corresponding set and not that there is. In other words, they can claim that the argument implies COLLAPSE$^{\Diamond}$ rather than COLLAPSE.

Moreover, unlike COLLAPSE, COLLAPSE$^{\Diamond}$ is consistent with P-COMP.[15] It is inconsistent with the modalisation of P-COMP. Formally:

$$(\text{P-COMP}^{\Diamond}) \qquad\qquad \Diamond\exists X\Box\forall x(x \in X \leftrightarrow \varphi^{\Diamond})$$

In particular, it follows from COLLAPSE$^{\Diamond}$ that there could not have been a plurality of all possible non-self-membered sets.[16,17] However, in contrast to P-COMP, the

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[14]As long as the background logic for $\Diamond$ extends the modal logic T, $\varphi^{\Diamond}$ will also quantify over what there is. Similarly, as long as the background logic for $\Diamond$ extends the modal logic S4, $\varphi^{\Diamond}$ will quantify over what there could have could have been and so on. See below for discussion of the correct modal logic for $\Diamond$.

[15]*Proof sketch:* Let $K = \langle V_{\omega}, \subseteq \cap V_{\omega} \times V_{\omega} \rangle$ be a Kripke model where first-order quantifiers at x range over x, plural quantifiers range over $\mathcal{P}(x)$, and the interpretation of $\in$ is $\in \cap V_{\omega} \times V_{\omega}$. It is then easy to see that COLLAPSE$^{\Diamond}$ and every instance of P-COMP hold at every $x \in V_{\omega}$.

[16]*Proof:* I will assume that the background modal logic governing $\Diamond$ extends the modal logic T (see below for discussion). For contradiction, suppose $\Diamond\exists X\Box\forall x(x \in X \leftrightarrow x \notin x)$. From COLLAPSE$^{\Diamond}$ it follows that $\Diamond\exists X\Diamond\exists x[\Box(X \equiv x) \wedge \forall x(x \in X \leftrightarrow x \notin x)]$ and thus $\Diamond\exists X\Diamond\exists x[(X \equiv x) \wedge \forall x(x \in X \leftrightarrow x \notin x)]$ by T, which is impossible.

[17]To simplify discussion I will often use locutions like "possible plurality", "possible set", "possible world" etc. Nothing hangs on misspeaking in this way, and they can always be eliminated with the use of the primitive modal operator.

conception of pluralities as nothing over and above the things which comprise them provides no support for P-COMP$^\lozenge$. It tells us that a plurality of all possible φ's will exist at a world w just in case the possible φ's all co-exist at w, but it gives us no reason to think that in general there is some world at which the possible φ's all co-exist. Of course, since the things which exist at w and are φ at w trivially co-exist at w, it does tell us that a plurality of the φ's at w will exist at w. But that is just what P-COMP says. In general, COLLAPSE$^\lozenge$ seems to be consistent with that conception.

Now consider the argument from arbitrariness. It rested on two claims. First, given natural set-theoretic assumptions, any view on which there is a plurality of all sets will imply LOS$_V$. Second, LOS$_V$ is an arbitrary answer to question (1). Clearly, question (1) was intended to be about absolutely all sets. So we should now replace it with its modalisation:

(1*) When could there have been a set of all possible φ's?

[*In other words:* Given any condition φ, when could there have been a set x such that $\Box\forall y(y \in x \leftrightarrow \varphi)$?][18]

Similarly, LOS$_V$ should be replaced by its modalisation LOS$_V^\lozenge$. The problem is then that LOS$_V^\lozenge$ is an arbitrary answer to question (1*). But the potentialist avoids this re-formulated problem since their view does not imply LOS$_V^\lozenge$. Indeed, it follows from COLLAPSE$^\lozenge$ that LOS$_V^\lozenge$ is false; the V's could have formed a set even though they are trivially not fewer than the V's.

Finally, consider the argument from circularity. It rested on the claim that LOS is unexplanatory in some of its answers to question (1). In particular, it seems

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[18]Technically, the modalisation of question (1) would have $\varphi = \psi^\lozenge$ for some formula ψ. But the more general (1*) is clearly the question of interest.

that its explanation for why there is no set of all sets is that the sets are not fewer than the sets. Replacing (1) with (1*), the claim now becomes that LOS$^\Diamond$ is unexplanatory in some of its answers to question (1*). The potentialist appears to avoid this problem. From COLLAPSE$^\Diamond$ and P-COMP it follows that there could have been set of all possible φ's just in case there could have been a plurality of all possible φ's.[19] The potentialist can thus explain why conditions could or could not have determined sets in terms of whether they could or could not have determined pluralities. For example, the explanation for why there could not have been a set of all possible sets is that there could not have been a plurality of all possible sets.[20] Similarly for why there could not have been a set of absolutely all ordinals.

How informative is the potentialist's answer to question (1*)? This will depend on the interpretation of the modal operator $\Diamond$. Since it is standard to assume that there could not metaphysically have been sets other than there are, potentialists tend to deny that it expresses metaphysical possibility and a number of alternative interpretations have been proposed. For example, Linnebo (2009, 2013) takes it to concern a well-founded process of extending the mathematical ontology, Studd (forthcoming) takes it to concern permissible reinterpretations of an underlying language, and Fine (2006) takes it to concern mathematical postulation. For concreteness, I will follow Linnebo; but most of what I say will apply equally well to these other ways of thinking about the modality.

This interpretation immediately motivates a modal logic for $\Diamond$. The idea is that one world w' is possible from the perspective of another w just in case the math-

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[19]See theorem 2 for a proof.
[20]See footnote 16 for a proof.

emathical ontology of w' extends that of w. Restricting attention to mathematical ontology, then, this just comes to the requirement that the domain of w' extends that of w. Formally, w' is accessible from w just in case $\text{dom}(w) \subseteq \text{dom}(w')$. So the converse Barcan formula – which corresponds to the frame condition on Kripke models that domains be increasing along the accessibility relation – should hold. Formally:

$$\text{(CBF)} \qquad \exists \mathbf{x} \Diamond \varphi \rightarrow \Diamond \exists \mathbf{x} \varphi$$

where $\mathbf{x}$ is either a first-order or plural variable. Since $\text{dom}(w) \subseteq \text{dom}(w)$, the T axiom – which corresponds to the frame condition on Kripke models that the accessibility relation be reflexive – should hold; that is, $\varphi \rightarrow \Diamond \varphi$. Similarly, since $\text{dom}(w) \subseteq \text{dom}(w') \subseteq \text{dom}(w'')$ only if $\text{dom}(w) \subseteq \text{dom}(w'')$, the 4 axiom – which corresponds to the frame condition on Kripke models that the accessibility relation be transitive – should hold; that is, $\Diamond \Diamond \varphi \rightarrow \Diamond \varphi$. Furthermore, it is natural to think that mathematical ontologies can always be combined. So the directedness axiom G – which corresponds to the frame condition on Kripke models that the accessibility relation be directed – should also hold; that is, $\Diamond \Box \varphi \rightarrow \Box \Diamond \varphi$. The modal logic K + T + 4 + G is referred to as S4.2.

The interpretation also suggests axioms governing the modal behaviour of sets and pluralities. It is natural to think that a set's elements are prior to the set in the sense that it mathematically could not have existed unless its elements had. Similarly, if a plurality is nothing over and above the things which comprise it, then it could not have existed unless those things had. So the following principle of inextensibility should hold.

(InExt) $\Diamond \exists x \in \mathbf{y} \varphi \rightarrow \exists x \in \mathbf{y} \Diamond \varphi$

Moreover, it is natural to think that sets mathematically cannot change their elements. If x is an element of y, then x mathematically could not have failed to be an element of y; and if x is not an element of y, then it mathematically could not have been an element of y. Similarly for pluralities. So the following principle of stability should hold for $\in$.

(Stab$_\in$) $\Box(x \in \mathbf{y}) \vee \Box(x \notin \mathbf{y})$

In general, we say that φ is *stable* (in $\vec{\mathbf{x}}$) if $\forall \vec{\mathbf{x}}(\Box\varphi(\vec{\mathbf{x}}) \vee \Box\neg\varphi(\vec{\mathbf{x}}))$. Finally, the necessity of distinctness – i.e. ($\Box$-$\neq$) $\mathbf{x} \neq \mathbf{y} \rightarrow \Box(\mathbf{x} \neq \mathbf{y})$ – the axiom of extensionality – i.e. (Ext) $\forall z(z \in \mathbf{x} \leftrightarrow z \in \mathbf{y}) \rightarrow \mathbf{x} = \mathbf{y}$ – and standard classical quantificationl logic should hold. For definiteness, we can take the latter to be axiomatised by the following axioms.

(A1) $\forall \mathbf{x} \varphi \rightarrow \varphi[\mathbf{y}/\ \mathbf{x}]$, where $\mathbf{y}$ is free for $\mathbf{x}$ in φ

(A2) $\forall \mathbf{x}(\varphi \rightarrow \psi) \rightarrow (\forall \mathbf{x} \varphi \rightarrow \forall \mathbf{x} \psi)$

(A3) $\varphi \rightarrow \forall \mathbf{x} \varphi$, where $\mathbf{x}$ is not free in φ

(A4) $\mathbf{x} = \mathbf{x}$

(A5) $\mathbf{x} = \mathbf{y} \rightarrow (\varphi[\mathbf{x}/\mathbf{z}] \leftrightarrow \varphi[\mathbf{y}/\mathbf{z}])$, where $\mathbf{x}$ and $\mathbf{y}$ are free for $\mathbf{z}$ in φ

The rules of inference are GEN, from φ infer $\forall \mathbf{x} \varphi$; MP, from φ and $\varphi \rightarrow \psi$ infer

ψ; and NEC, if φ is provable, then so is $\square\varphi$.

As the next theorem shows, these principles jointly imply the modalisations of many axioms of standard set theory.

Definition 2. Let $\mathcal{L}_\in$ be the language of first-order set theory with variables $x_0, ..., x_n, ...$; $\mathcal{L}_\in^P$ the language of plural set theory, extending $\mathcal{L}_\in$ with variables $X_0, ..., X_n, ...$; and $\mathcal{L}_\Diamond$ the language extending $\mathcal{L}_\in^P$ with the modal operator $\Diamond$. $x \in X$ is considered well-formed and read "it_x is one of $them_X$".

Definition 3. Let ZF be the $\mathcal{L}_\in$ theory consisting of Extensionality, Infinity, Pairing, Union, Powerset, Foundation, Separation, and Replacement; let Z be ZF minus Replacement and Foundation; and let Z^- be Z minus Powerset.

Definition 4. Let PMST (for *Potentialist Modal Set Theory*) be the $\mathcal{L}_\Diamond$ theory consisting of A1-5, S4.2, InExt, $\text{Stab}_\in$, $\square\text{-}\neq$, Ext, COLLAPSE$^\Diamond$, and P-COMP.

Theorem 1. *PMST interprets Z^- - Infinity via modalisation*

Proof. The proof is by induction on the length of proofs, and is given in Linnebo (2013). As it will be useful later on, I repeat here the cases for the logical axioms A1-5 and the rules of inference. That PMST proves A1$^\Diamond$ is trivial given T and A1; similarly for A2$^\Diamond$ and K and A2. A4-5$^\Diamond$ are just instances of A4-5, since $\varphi^\Diamond[x/y] = \varphi[x/y]^\Diamond$ (as can be shown by a simple induction on the complexity of φ). Given A1 and A3, it is easy to see that A3$^\Diamond$ is equivalent to:

$$\varphi^\Diamond \to \square\varphi^\Diamond$$

and thus equivalent to the claim that each $\varphi^\Diamond$ is stable:

$$\Box\varphi^\Diamond \vee \Box\neg\varphi^\Diamond$$

We prove this by induction on the complexity of φ. The base cases are just the stability axioms for $\in$, $\Box$-$\neq$, and $x = y \to \Box(x = y)$ (which follows from A4, NEC, and A5). The conjunction case follow from K, and the negation case from double negation elimination. Now suppose $\Diamond\Diamond\exists x\varphi^\Diamond$. It follows from 4 that $\Diamond\exists x\varphi^\Diamond$ and thus $\Diamond\exists x\Box\varphi^\Diamond$ by the induction hypothesis and T. By CFB we then have $\Diamond\Box\exists x\varphi^\Diamond$ and thus $\Box\Diamond\exists x\varphi^\Diamond$ by G.

Finally, if PMST proves $\varphi^\Diamond$ and $(\varphi \to \psi)^\Diamond$, then it proves $\psi^\Diamond$ by MP since $(\varphi \to \psi)^\Diamond = \varphi^\Diamond \to \psi^\Diamond$; and if it proves $\varphi^\Diamond$, then it proves $\Box\forall x\varphi^\Diamond$ by GEN and NEC. $\square$

Above, I stated the potentialist's answer to question (1^*). We are now in a position to prove it. But first we need a lemma.

Definition 5. Say that $\varphi \in \mathcal{L}_\in^P$ is *bounded* if its quantifiers are all of the form $\exists x \in \mathbf{y}$.

Lemma 1 (PMST)**.** *If $\varphi \in \mathcal{L}_\in^P$ is bounded, then:*

$$\varphi \leftrightarrow \varphi^\Diamond$$

Proof. The proof is by induction on the complexity of φ, and is essentially given in Linnebo (2013). I will repeat it here for clarity. The base, conjunction, and negation cases are trivial. From the induction hypothesis it follows that $\exists x \in \mathbf{y}\varphi$

is equivalent to $\exists x \in \mathbf{y}\varphi^{\Diamond}$. Since $\varphi^{\Diamond} \leftrightarrow \Diamond\varphi^{\Diamond}$ by the proof of theorem 1 and the T axiom, that is equivalent to $\exists x \in \mathbf{y}\Diamond\varphi^{\Diamond}$. Finally, by CBF and InExt that is equivalent to $\Diamond\exists x \in \mathbf{y}\varphi^{\Diamond}$. □

Theorem 2 (PMST)**.**

$$\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \varphi) \quad \leftrightarrow \quad \Diamond\exists X\Box\forall y(y \in X \leftrightarrow \varphi)$$

Proof. Suppose x is such that $\Box\forall y(y \in x \leftrightarrow \varphi)$. By P-COMP, let X be co-extensive with x; i.e. $X \equiv x$. Then by lemma 1 we have $\Box(X \equiv x)$ and thus $\Box\forall y(y \in X \leftrightarrow \varphi)$. Conversely, suppose X is such that $\Box\forall y(y \in X \leftrightarrow \varphi)$. Then by COLLAPSE$^{\Diamond}$ $\Diamond\exists x\Box(X \equiv x)$. By axiom 4, it follows that $\Diamond\exists x\Box[X \equiv x \land \forall y(y \in X \leftrightarrow \varphi)]$ and thus $\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \varphi)$ as required. □

Potentialism is thus similar in many respects to the non-modal view IC + COLLAPSE. By theorem 1, it is similarly informative.[21] It also offers structurally similar explanations, with possible existence of the plurality of all possible φ's taking the place of existence at a stage of all φ's. But it seems to be significantly better than IC + COLLAPSE in at least two ways. First, it is consistent with the conception of pluralities as nothing over and above the things which comprise them. Second, it has responses to all of the arguments outlined in section 1.1.2.

Potentialism contrasts with *actualism*, which is the view that there could not have been anything other than there is. Formally, it can be expressed by the theory AMST (for *Actualist Modal Set Theory*) which consists of PMST minus COLLAPSE$^{\Diamond}$

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[21]PMST can also be naturally extended with other principles so as to imply Powerset and Foundation, bringing it closer to IC. See Studd (2013).

plus the Barcan formula:

$$(BF) \qquad\qquad \Diamond \exists \mathbf{x} \varphi \to \exists \mathbf{x} \Diamond \varphi$$

The Barcan formula corresponds to the frame condition on Kripke models that domains be decreasing along the accessibility relation. Together, CBF and BF effectively require that all possible worlds have the same domain. In particular, they imply that modalisation for formulas in $\mathcal{L}_{\in}^{P}$ is inert.

Lemma 2 (AMST). *If $\varphi \in \mathcal{L}_{\in}^{P}$, then:*

$$\varphi \leftrightarrow \varphi^{\Diamond}$$

Proof. The proof is essentially the same as for lemma 1 except that we use BF instead of InExt. $\qquad\square$

Without COLLAPSE$^{\Diamond}$, PMST loses almost all of its strength. To see this, note that any one world Kripke model for $\mathcal{L}_{\Diamond}$ in which plural quantifiers range over the powerset of the domain and Ext is true will model AMST. In what follows, I will thus take actualism to have been supplemented with one of the non-modal views IC or LOS. Since actualism effectively gives up the distinction between a formula and its modalisation, it gives up on the potentialist's responses to the arguments of section 1.1.2. If those arguments are sound, potentialism seems to be preferable. As I will now argue, however, it unclear whether they are.

1.1.4 Assessing the arguments for COLLAPSE

In this section I will assess the arguments for COLLAPSE outlined in section 1.1.2. I will argue that they are inconclusive as they stand.

The argument from arbitrariness

The argument from arbitrariness is based on two premises. First, that in the presence of some natural set-theoretic assumptions, P-COMP implies LOS_V. Second, that LOS_V is an arbitrary answer to question (1). This in turn depended on the claim that there could have been more sets than there are. For in that case, LOS_V would have been false. The argument then concludes that we should reject P-COMP.

Let me consider two replies. The first accepts its premises, but denies that the conclusion follows. Suppose we add to potentialism the empirical premise that Tim likes all possible pluralities. Since every possibly plurality could have determined a set by COLLAPSE$^\lozenge$, it will follow that a possible plurality could have determined a set just in case it is liked by Tim. Clearly, this would be an arbitrary answer to the modalisation of question (3):

> (3*) When could there have been a set of all possible X's?
>
> [*In other words:* Given any possible plurality X, when could there have been a set x such that $\square(X \equiv x)$?]

But that is no reason to reject potentialism or the empirical premise. What matters is that potentialism has some non-arbitrary answer to question (3*). In particular, that it includes COLLAPSE$^\lozenge$. In general, what matters is whether a theory implies

a good answer to a particular question not whether it implies a bad one. But both IC + P-COMP and LOS + P-COMP do seem to imply non-arbitrary answers to question (1). For example, LOS says that a condition φ determines a set just in case the φ's are fewer than the sets. Although this could be formulated as LOS_V, it need not be. To see this, let $\text{fun}_{1-1}(X)$ abbreviate the claim that X is a plurality of set-theoretic ordered pairs coding a one-one function, $\text{dom}_X(x)$ that x is in X's domain, and $\text{rng}_X(x)$ that x is in its range. Then we can formulate LOS as:

$$\exists x \forall y(y \in x \leftrightarrow \varphi) \quad \text{iff} \quad \neg \exists X(\text{fun}_{1-1}(X) \wedge \forall y(\text{dom}_X(y) \leftrightarrow \varphi) \wedge \forall x(\text{rng}_X(x)))$$

In contrast to LOS_V, this way of formulating LOS does not seem to be arbitrary. Even if there could have been a set v at a world w of all the sets there are, it need not follow that there is one-one function at w from v to the sets which exist at w. In general, it does not follow that this way of formulating LOS could have been false. Similar remarks apply to IC.

Digression. As I have characterised the position, actualism will include IC or LOS and thus inherit one of their answers to question (1). As lemma 2 shows, for $\varphi \in \mathcal{L}_{\in}^P$, there could have been a set of all possible φ's just in case there is a set of all φ's. So in that limited range of cases, either answer to question (1) extends to question (1*). What about arbitrary φ? Although the actualist thinks there could not have been anything other than there is, they may think that things could have been otherwise. For example, suppose we have in our language a predicate P which applies to nothing but could have applied to all possible non-self-membered sets. Formally: $\forall x \neg Px$ and $\lozenge \square \forall x(Px \leftrightarrow x \notin x)$. Then there is a set of all P's

according to the actualist, but there could not have been a set of all possible P's.[22]

Nonetheless, the actualist's answer to question (1) does extend to (1*) in a less direct way. In particular, the next lemma shows that AMST proves there could have been a set of all possible φ's just in case φ could have been stable and there is a set of all $\Diamond\Box\varphi$'s.

Lemma 3 (AMST).

$$\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \varphi) \quad \leftrightarrow \quad [\Diamond\forall y(\Box\varphi \vee \Box\neg\varphi) \wedge \exists x\forall y(y \in x \leftrightarrow \Diamond\Box\varphi)]$$

Proof. Left-to-right. Suppose $\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \varphi)$. From BF and CBF it follows that $\exists x\Diamond\forall y\Box(y \in x \leftrightarrow \varphi)$. Since $\in$ is stable, we have $\Box\forall y(\Box y \in x \vee \Box y \notin x)$ and thus $\Diamond\forall y[\Box(y \in x \leftrightarrow \varphi) \wedge (\Box y \in x \vee \Box y \notin x)]$. Thus, $\Diamond\forall y(\Box\varphi \vee \Box\neg\varphi)$. If $y \in x$, then $\Box\Box(Ey \wedge y \in x)$ by CBF and stability of $\in$ and so $\Diamond\Box\varphi$. Conversely, if $\Diamond\Box\varphi$, then $\Diamond\Box(Ey \wedge \varphi)$ by CBF and so $\Box\Diamond(Ey \wedge \varphi)$ by G. It follows that $\Diamond\Diamond x \in y$ and thus $x \in y$ by stability of $\in$. Right-to-left. Suppose $\Diamond\forall y(\Box\varphi \vee \Box\neg\varphi)$ and let x be such that $\forall y(y \in x \leftrightarrow \Diamond\Box\varphi)$. If $\Diamond\Box\varphi$, then $\Diamond\Box\Box\varphi$ by 4 and thus $\Box\Diamond\Box\varphi$ by G; and if $\neg\Diamond\Box\varphi$, then $\Box\neg\Diamond\Box\varphi$ by 4. So $\forall y(\Box\Diamond\Box\varphi \vee \Box\neg\Diamond\Box\varphi)$. A simple induction then shows that $\forall y(y \in x \leftrightarrow \Diamond\Box\varphi)$ implies $\Box\forall y(y \in x \leftrightarrow \Diamond\Box\varphi)$ and thus $\Box\Box\forall y(y \in x \leftrightarrow \Diamond\Box\varphi)$ by 4. If $\forall y(\Box\varphi \vee \Box\neg\varphi)$, then it similarly follows that $\Box\forall y(\varphi \leftrightarrow \Diamond\Box\varphi)$. So it follows from $\Diamond\forall y(\Box\varphi \vee \Box\neg\varphi)$ that $\Diamond\Box\forall y(\varphi \leftrightarrow \Diamond\Box\varphi)$ and thus $\Diamond\Box\forall y(y \in x \leftrightarrow \varphi)$ as required. $\qquad\square$

The converse is not true for the potentialist. That is, their answer to (1*) does not extend to (1). Indeed, PMST says nothing about which conditions determine sets.

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[22] *Proof:* Suppose there could have been. That is, suppose $\Diamond\exists y\Box\forall x(x \in y \leftrightarrow Px)$. From G and $\Diamond\Box\forall x(Px \leftrightarrow x \notin x)$ it follows that $\Box\Diamond\forall x(Px \leftrightarrow x \notin x)$ and thus that $\Diamond\exists y\Diamond[\forall x(Px \leftrightarrow x \notin x) \wedge \forall x(x \in y \leftrightarrow Px)]$. From CBF we then have $\Diamond\Diamond[\forall x(Px \leftrightarrow x \notin x) \wedge \exists y\forall x(x \in y \leftrightarrow Px)]$ which is impossible.

It is easy to see, for example, that PMST is satisfied in pointed Kripke models of the form $\langle V_\lambda, \subseteq \cap\{y \in V_\lambda : x \subseteq y\} \times \{y \in V_\lambda : x \subseteq y\}, x\rangle$. Nonetheless, (1) is a perfectly legitimate question. A natural way to supplement potentialism to answer it is with a principle of priority – which says if $x \in \text{dom}(w)$, then there is some $w' < w$ such that $x \subseteq \text{dom}(w')$ – and a principle of maximality – which says that if $x \subseteq \text{dom}(w)$ and $w < w'$, then $x \in \text{dom}(w')$. Together, these principles imply that a condition φ determines a set at a world w just in case the φ's at w all exist at some $w' < w$. Expressing these principles, though, requires resources which go beyond PMST. For example, Studd (2013) adopts two modal operators; one which 'looks back' to prior worlds and one like $\Diamond$ which 'looks forward' to subsequent worlds.

A somewhat similar point applies to question (2). For the actualist, there could have been a plurality of all possible φ's just in case φ could have been stable and there is a plurality of all $\Diamond\Box\varphi$'s. That is:

Lemma 4 (AMST).

$$\Diamond\exists X\Box\forall y(y \in X \leftrightarrow \varphi) \quad \leftrightarrow \quad [\Diamond\forall y(\Box\varphi \vee \Box\neg\varphi) \wedge \exists X\forall y(y \in X \leftrightarrow \Diamond\Box\varphi)]$$

Proof. Exactly analogous to the proof of lemma 3. □

So the actualist's answer to (2), namely "always", extends to its modalisation:

(2*) When could there have been a plurality of all possible X's?

[*In other words:* Given any condition φ, when could there have been a plurality X such that $\Box\forall y(y \in X \leftrightarrow \varphi)$?]

In particular, they will think that there could have been a plurality of all possible

φ's just in case φ could have been stable. The same is not true for the potentialist. That is, their answer to (2) does not extend to (2*). In PMST, it is hard to see what their answer could be. From the conception of pluralities as nothing over and above the things which comprise them it follows that there could have been a plurality of all possible φ's just in case the φ's could have all co-existed. But expressing this again requires new resources. In particular, it requires the ability to cross-reference worlds. In effect, we need to be able to say that there is a world w such that given any $x \in \text{dom}(w')$ for $w' \geq w$ which is φ, $x \in \text{dom}(w)$. This can be done using backtracking operators as discussed in Hodes (1984), but not in PMST.

The second reply denies the second premise. By lemma 2 the actualist will think that V necessarily contains everything; that is, $\Box \forall x(x \in V)$. The V's could not have determined a set according to them, and LOS_V could not have been false. In other words, the argument from arbitrariness begs the question against the actualist.

The argument from circularity

The argument from circularity rested on the claim that IC and LOS are unexplanatory in certain crucial cases. In particular, that LOS does not explain why there is no set of all sets.

Let me consider two responses. First, it is no accident that the argument was directed at LOS rather than IC. It is hard to see in what cases IC is unexplanatory in a similar way. In general, given the structural similarity between potentialism's answer to (1*) and IC's answer to (1), it is hard to see in which cases the former but not the latter is explanatory.

The second response highlights a hidden premise of the argument. In particular, the argument assumed that the explanation for why the sets do not determine a set must track the bi-conditional used in the statement of LOS. In other words, the argument assumed that the explanation for why φ does or does not determine a set must rely on the fact that the φ's are or are not fewer than the sets. But there is no reason to accept this premise. For example, the proponent of LOS could first use a purely logical argument to explain why there is no of all non-self-membered sets and then explain why each instance of the axiom of separation is true as outlined in section 1.1.1. In general, explanations may draw on a number of interconnected considerations and be no less explanatory for that.

The argument from specification

There are two ways to read the argument from specification. The first is as an explanatory demand. If we deny COLLAPSE, we need to explain why pluralities fail to determine sets. As Linnebo (p. 146, 2010) puts it: "what more could be needed [over and above the existence of some things] for [a set of them] to exist?".[23] Read this way, it is hard to see its force. Both IC and LOS tell us exactly what more is needed for them to determine a set. On IC, they have to all co-exist at some stage; and on LOS, they have to be fewer than the sets. Perhaps the real question is: what more could be needed for some things to possibly determine a set? That is, question (3^*). Here, of course, the non-modal views IC and LOS are on their own silent. But as components of actualism, they are not. Since their answers to question (1) extend to question (1^*) in AMST, they also extend *a fortiori* to (3^*) in AMST.

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[23]See also (Studd, 2013, p. 699-700) and (Fine, 2006, p. 23).

The second way of reading the argument is as a positive argument for COL-LAPSE; or rather, for COLLAPSE$^\diamond$. The problem with this reading is that it is unclear what the premises of the argument are supposed to be. They will at least include the claim that a plurality suffices to completely specify a corresponding set. But that hardly entails such a set exists or could have existed. Whatever premise we add, it will have to be sensitive to the distinction between COLLAPSE and COLLAPSE$^\diamond$. After all, COLLAPSE is just as inconsistent with potentialism as with actualism. For example, some conceivability-implies-mathematical-possibility premise would have the required sensitivity. Since a plurality gives us the resources to conceive what the world would be like if it determined a set, it would then follow that it could have determined a set. There would then be no pressure to think it did determine a set. Of course, such a premise would be highly contentious.[24] But it is unclear what could be used instead.

Summing up. Neither the argument from arbitrariness nor the argument from circularity has force, and the argument from specification needs to be supplemented. Moreover, potentialism is both similarly informative and explanatory to actualism on questions 1-3 and 1*-3* (at least, when the latter is supplemented with IC). Of course, there may be other positive arguments for potentialism. Or it may be that on closer analysis, potentialism gives better answers to 1-3 and 1*-3*.[25,26] But until such arguments or analysis is given, I hope to have shown that the most prominent considerations in favour of potentialism are inconclusive and thus that we should

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[24]See the papers in Gendler and Hawthorne (2002) for discussion.

[25](Linnebo, 2013, p.207), for example, claims that the difference between conditions which could have determined pluralities and those which could not is 'intrinsic', presumably in a way that the difference between conditions which determine pluralities at some stage and those which do not is not.

[26]Similarly, there may be other questions which bear on the debate. See, for example, Uzquiano (2006) for questions concerning the interaction of mereology and set theory.

look for new ones. In the next section, I do just that.

1.2 Potentialism and second-order logic

In the last section I showed that some prominent arguments for potentialism are inconclusive. In this section and the next I develop a new consideration in favour of actualism. Roughly, the idea is that certain resources are desirable but unavailable to the potentialist. In particular, in this section I will argue that the modalised comprehension schema of second-order logic is incompatible with potentialism. Then in the next section I will show that this schema follows from certain attractive set-theoretic reflection principles.

1.2.1 Modalised plural logic

Potentialism and actualism differ centrally on the modalised plural logics they entail. For the potentialist, there could have been a plurality of all possible $\varphi^\diamond$'s just in case there could have been a set of all possible $\varphi^\diamond$'s (lemma 1). They will thus think that many instances of P-COMP$^\diamond$ are false. For the actualist, on the other hand, there could have been a plurality of all possible $\varphi^\diamond$'s just in case there is a plurality of all φ's (lemma 2). They will thus think that all instances of P-COMP$^\diamond$ are true since each is equivalent to an instance of P-COMP.

In other words, potentialism, but not actualism, is incompatible with the modalised comprehension schema for second-order logic on its plural interpretation. It turns out that this fact is fairly robust. As I will now argue, relative to some natural assumptions, potentialism is incompatible with the modalised comprehension schema for second-order logic on any interpretation.

1.2.2   Modalised second-order logic

Suppose the potentialist expands their language with second-order variables F_0,...,
F_n,... and takes $x \in F$ to be well-formed. To help readability we can refer to F's
as *concepts* and say that F *applies* to x whenever $x \in F$. A natural initial question
about concepts is whether they determine sets. That is, whether:

$$\Box \forall F \Diamond \exists x \Box (F \equiv x)$$

where $F \equiv x$ abbreviates $\forall y (y \in F \leftrightarrow y \in x)$. Clearly, this depends on what
concepts there could have been. If there could have been a concept which neces-
sarily applied to all and only the non-self-membered sets – a concept F for which
$\Box \forall x (x \in F \leftrightarrow x \notin x)$ – for example, then they could not have determined sets in
general.

But whatever concepts there could have been, it is plausible that they de-
termine another kind of object; namely, what I will call *properties*. The notion
of property can be understood in terms of the notions of concept and set. Like
concepts, properties bear a relation of application to objects; and just as sets
are completely characterised by their membership conditions, properties are com-
pletely characterised by their application conditions. First, a property is distin-
guished from all other properties by the objects it could have applied to. This is
expressed by the following principle of intensionality:

$$p = q \quad \leftrightarrow \quad \Box \forall x (x \eta p \leftrightarrow x \eta q)$$

where p, q are properties and $x \eta p$ formalises the claim that p applies to x. Second,

there is nothing more to a property than its having the application conditions it has. To completely specify a property, it thus suffices to specify which objects it could have applied to. If the argument from specification outlined in section 1.1.2 can be made good, it would seem to generalise to show that every possible concept could have determined a property. Formally:

$$(\text{C-COLLAPSE}^\diamond) \qquad\qquad \Box\forall F\diamond\exists p\Box(F \equiv p)$$

where $F \equiv p$ abbreviates $\forall x(x \in F \leftrightarrow x\eta p)$.

Independently of the argument from specification, a potentialist who denied C-COLLAPSE$^\diamond$ would in any case incur an explanatory burden. They would have to explain why COLLAPSE$^\diamond$ holds even though C-COLLAPSE$^\diamond$ does not. I can see only two ways they might try to do this. First, they might try to exploit the fact that concepts can 'reach out' beyond the world where they exist and apply to new objects whereas pluralities cannot. In the terminology of section 1.1.1, they might try to exploit the fact that concepts can be extensible whereas pluralities are inextensible. For example, suppose there is a universal concept; that is, a concept F for which $\Box\forall x(x \in F)$. Let U be the plurality of everything; that is, $\forall x(x \in U)$. By COLLAPSE$^\diamond$, U could have determined a set which assuming Foundation$^\diamond$ will not be in U. So:

$$\diamond\exists x \in F(x \notin U)$$

But, since pluralities are stable, nothing could have failed to be in U. That is:

$$\neg\exists x \in F\Diamond(x \notin U)$$

Thus F is extensible. If concepts are extensible, then this may well explain why they fail to determine sets in general. But it is unclear how it could be used to explain why concepts fail to determine properties. Since the application relation for properties is modelled on the application relation for concepts, properties are just as capable of being extensible as concepts. So if the extensibility of F is a reason to think there could not have been a corresponding property, then it is prima facie a reason to think that F could not have existed in the first place.

Second, they might try to exploit the fact that properties can be self-applicable whereas concepts cannot (assuming concepts are not objects). For example, a property corresponding to a universal concept would apply to itself. More worryingly, a property corresponding to a concept which necessarily applied to all and only the non-self-applicable properties – a concept F for which $\Box\forall p(p \in F \leftrightarrow \neg[p\eta p])$ – would apply to itself just in case it did not. However, there is a broad class of concepts for which the corresponding properties need not be self-applicable. In particular, this is the case for concepts which only apply to pure sets; that is, sets whose elements are sets, whose elements' elements are sets and so on. Call these *ps-concepts*. Assuming properties are not sets, properties corresponding to ps-concepts will not apply to properties, to sets containing properties, to sets containing sets containing properties and so on. In a sense, such properties will be strongly non-self-applicable. This suggests the following restriction of C-COLLAPSE$^\Diamond$:

(PSC-COLLAPSE$^\Diamond$) $\qquad\qquad \Box\forall F(\Box\forall x(x \in F \to \mathrm{Ps}^\Diamond(x)) \to \Diamond\exists p\Box(F \equiv p))$

where $\mathrm{Ps}(x)$ formalises the claim that x is a pure set.[27]

I will now show that just as COLLAPSE$^\Diamond$ is inconsistent with P-COMP$^\Diamond$, given plausible assumptions PSC-COLLAPSE$^\Diamond$ is inconsistent with the modalised comprehension schema for concepts:

(C-COMP$^\Diamond$) $\qquad\qquad\qquad \Diamond\exists F\Box\forall x(x \in F \leftrightarrow \varphi^\Diamond)$

where F is not free in φ.

1.2.3 PSC-COLLAPSE$^\Diamond$ is inconsistent with C-COMP$^\Diamond$

First, we need to extend definitions 2, 3, and 4 to account for concepts and properties.

Definition 6. Let $\mathcal{L}^2_\in$ be the language of second-order set theory, extending $\mathcal{L}_\in$ with variables $F_0, ..., F_n, ...$; $\mathcal{L}^p_\in$ the language extending $\mathcal{L}^2_\in$ with the predicates $\mathrm{Set}(x)$ and η; and $\mathcal{L}^p_\Diamond$ the language extending $\mathcal{L}^p_\in$ with the modal operator $\Diamond$. $x \in F$ and $F = G$ are taken to be well-formed in $\mathcal{L}^2_\in$. For $\varphi \in \mathcal{L}_\in$, let φ^{ps} be the result of restricting all its quantifiers to Ps.

Definition 7. Let ZFU be ZF in the language of $\mathcal{L}^2_\in$ modified in the usual way to allow for urelements. The result of adorning a name of an axiom with "U" denotes

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[27]For instance, we could let $\mathrm{Ps}(x) =_{df} \forall f, n([f$ is a function $\land \mathrm{dom}(f) = n + 1 \land f(0) = x \land \forall m < n(f(m+1) \in f(m))] \to \mathrm{Set}(f(n)))$.

the modified axiom and unadorned names denote the unmodified axioms restricted to pure sets. For example, UPairing denotes $\forall x, y \exists z(\text{Set}(z) \land z = \{x, y\})$ and Pairing denotes $\forall x, y(\text{Ps}(x) \land \text{Ps}(y) \to \exists z[\text{Ps}(z) \land z = \{x, y\}^{ps}])$. I do not assume that C-COMP is included in ZFU's logic and thus UReplacement and USeparation are taken to be schemas rather than single, universally quantified, sentences. ZU is ZFU minus UReplacement and UFoundation and ZU^- is ZU minus Powerset.

Definition 8. Let PMSTU (for *Potentialist Modal Set Theory with Urelements*) be the $\mathcal{L}_{\Diamond}^p \cup \mathcal{L}_{\Diamond}$ theory extending PMST with stability axioms for $x \in F$, $\text{Set}(x)$, and $x\eta y$; COLLAPSE$^{\Diamond}$ replaced by its explicit restriction to sets: $\Box \forall X \Diamond \exists x (\text{Set}(x) \land \Box(X \equiv x))$; and PSC-COLLAPSE$^{\Diamond}$.

It is easy to see that the proof of theorem 1 generalises to show:

Theorem 3. *PMSTU interprets ZU^- - UInfinity via modalisation.*

Given the deduction theorem, it is a trivial corollary of theorem 3 that $\varphi_0, ..., \varphi_n$ imply φ in ZU^- - Infinity only if $\varphi_0^{\Diamond}, ..., \varphi_n^{\Diamond}$ imply $\varphi^{\Diamond}$ in PMSTU (whenever the φ_i are sentences). To show that PSC-COLLAPSE$^{\Diamond}$ is inconsistent with C-COMP$^{\Diamond}$ given $\varphi_0^{\Diamond}, ..., \varphi_n^{\Diamond}$ it thus suffices to show that PSC-COLLAPSE is inconsistent with C-COMP given $\varphi_0, ..., \varphi_n$. For suitable $\varphi_0, ..., \varphi_n$, I will now do that.

PSC-COLLAPSE **is inconsistent with** C-COMP

The argument that PSC-COLLAPSE is inconsistent with C-COMP relies on three assumptions. First, there is a principle of global choice. Let $\text{Lin}(F)$ abbreviate:

$$\forall x, y(\langle x, y \rangle \in F \lor x = y \lor \langle y, x \rangle \in F)$$

and Wf(F) abbreviate:

$$\forall x(\text{Set}(x) \wedge \exists y(y \in x) \rightarrow \exists y \in x \forall z \in x(\langle z, y \rangle \notin F))$$

Then the principle can be formulated as:

(GC) $$\exists F(\text{Lin}(F) \wedge \text{Wf}(F))$$

Second, there is a principle asserting the existence of a concept applying to sets well-ordered by $\subset$ such that every object is in some set to which the concept applies. In the case of pure set theory, for example, a concept applying to all and only the V_α's would be paradigm example. Formally:

(WO) $$\exists F(\text{Lin}(\subset \cap F \times F) \wedge \text{Wf}(\subset \cap F \times F) \wedge \forall x \exists y \in F(x \in y))$$

where $\text{Lin}(\subset \cap F \times F)$ and $\text{Wf}(\subset \cap F \times F)$ have the obvious meanings. Finally, there are the UReplacement and UPowerset axioms. Then we have:

Theorem 4 (ZFU - Infinity). *If* GC, WO, *and* PSC-COLLAPSE, *then some instance of* C-COMP *is false.*

Proof. The strategy will be to construct a one-one function from the objects into the pure sets. By PSC-COLLAPSE we will then effectively have a one-one function from the ps-concepts into the pure sets. Given C-COMP, a simple diagonalisation argument will show that this is impossible.

Let F and G witness GC and WO respectively, and let G_x denote the least

element of F containing x.[28] By C-COMP there is a H such that:

$$\langle x, y \rangle \in H \quad \leftrightarrow \quad G_x \subset G_y \vee (G_x = G_y \wedge \langle x, y \rangle \in F)$$

Clearly, H is a set-like well-order of the objects. Using UReplacement, the Mostowski collapse lemma will thus give us a one-one function $\varphi(x, y)$ from the objects into the (pure set) ordinals. Now, consider the following instance of C-COMP:

$$\exists F \forall y (y \in F \quad \leftrightarrow \quad \exists x (\varphi(x, y) \wedge \neg[y \eta x]))$$

It is easy to see that F only applies to pure sets (since it only applies to ordinals). But by the usual Russell-style reasoning, F cannot determine a property, contradicting PSC-COLLAPSE. $\qquad \square$

It follows that in PMSTU, GC$^\diamond$, WO$^\diamond$, UReplacement$^\diamond$, and UPowerset$^\diamond$ imply that some instance of C-COMP$^\diamond$ is false. Could the potentialist resist this conclusion? That is, could they accept C-COMP$^\diamond$ while denying one of GC$^\diamond$, WO$^\diamond$, UReplacement$^\diamond$, and UPowerset$^\diamond$? I will now argue that given C-COMP$^\diamond$ there is pressure on the potentialist to accept each of these principles.

UReplacement$^\diamond$

If the potentialist wants to provide an interpretation of ZFC via modalisation, then Replacement$^\diamond$ will have to be true. In the presence of COLLAPSE$^\diamond$ and InExt, Replacement$^\diamond$ effectively tells us that given any co-existent pure sets, the pure sets

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[28]G_x is well-defined because the $\subset$-predecessors of any $y \in G$ are all contained in $\mathcal{P}(y)$, which exists by UPowerset.

to which they are φ^{ps}-related can co-exist (when $\varphi \in \mathcal{L}_\in$ and φ^{ps} is functional$^\diamondsuit$). Underlying this, and similarly underlying UReplacement$^\diamondsuit$, is the general principle that given any co-existent objects, the objects to which they are φ-related can co-exist (when φ is functional$^\diamondsuit$). A potentialist who accepted Replacement$^\diamondsuit$ but denied that general principle would thus posit a substantial modal difference between the functionally related co-existences of pure sets in a specific range of cases and the functionally related co-existences of objects in general. But it is unclear how they might explain this difference. For example, the rationale (Linnebo, 2013, p. 17) gives for Replacement$^\diamondsuit$ is that co-existence is a matter of size. If the φ's co-exist and the ψ's are as many or fewer than the φ's, then the ψ's can co-exist. But this rationale is entirely general. Any functional relation χ from the φ's to the ψ's witnesses the ψ's being as many or fewer than the φ's irrespective of what formulas φ, ψ, and χ happen to be and, indeed, what they happen to apply to.

GC$^\diamondsuit$

As (Bernays, 1935, p. 260) argued, Choice seems to follow from the quasi-combinatorial conception of set. Heuristically, the idea is that there is a set corresponding to every arbitrary combination of elements of any given set. For example, since some arbitrary combination of elements of $x \times x$ is a well-ordering of x, there is a set which well-orders x. Extrapolated to concepts, the idea would be that there is a concept corresponding to every arbitrary combination of objects. For example, since some arbitrary combination of set-theoretic ordered pairs is a well-ordering of the objects, there is a concept which well-orders them. Another way to put the point is that if the sets are plenitudinous, then Choice$^\diamondsuit$ is true; and if the concepts are plenitudinous, then GC$^\diamondsuit$ is true. But C-COMP$^\diamondsuit$ requires the concepts

to be plenitudinous. At least, it requires them to be plenitudinous enough to make all of its instances true. A potentialist who accepted C-COMP$^\diamond$ but rejected GC$^\diamond$ would thus have to explain why the concepts are plenitudinous enough to make all instances of C-COMP$^\diamond$ true but not plenitudinous enough to make GC$^\diamond$ true. But it is unclear how they might do this.

WO$^\diamond$ **and UPowerset**$^\diamond$

A similar point would apply to WO$^\diamond$ if the potentialist had reason to think some arbitrary combination of sets witnessed it. I will now argue that they do. Since I only want to establish that some arbitrary combination of sets witnesses WO$^\diamond$, I will argue relatively informally and allow myself resources which go beyond those of PMSTU. In particular, I will make use of quantification over possible worlds equipped with a well-founded accessibility relation $<$ along which domains are increasing and third-order quantification over collections of objects and concepts (for which I will use calligraphic variables $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ etc). Boldface variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ etc will be used for concepts and objects, and $\mathrm{dom}(w)$ will denote the collection of concepts and objects which exist at w. For simplicity, I will restrict attention to ps-concepts. Thus, by PSC-COLLAPSE$^\diamond$, every concept determines a property.

We can define a notion of *immediate possibility*. In particular, we can say that $\mathbf{x}$ is immediately possible for a world w if there is a world $w' \geq w$ with $\mathbf{x} \in \mathrm{dom}(w')$ such that there are no worlds strictly prior to w' but strictly after w. In other words, $\mathbf{x}$ is immediately possible for w if nothing more is needed for the possible existence of $\mathbf{x}$ over and above w's having been the case. The notion naturally generalises to arbitrary collections of concepts and objects. Just as $\mathbf{x}$ may be immediately possible for w, it may be immediately possible for a collection

of concepts and objects $\mathcal{X}$. In that case, nothing more is needed for the possible existence of $\mathbf{x}$ over and above the existence of the $\mathcal{X}$'s. I claim that the potentialist can make sense of this generalised notion. Indeed, recall that something very much like it appeared in the argument from specification. It was because a set of some things is immediately possible for them that we were led to believe they could have determined a set.

Let $\mathrm{Ip}(\mathcal{X}, \mathbf{x})$ formalise the claim that $\mathbf{x}$ is immediately possible for $\mathcal{X}$. What principles does $\mathrm{Ip}(\mathcal{X}, \mathbf{x})$ obey? On the intended reading, $\mathrm{Ip}(\mathcal{X}, \mathbf{x})$ is not sensitive to the world of evaluation. That nothing is needed for the possible existence of $\mathbf{x}$ over and above the existence of the $\mathcal{X}$'s is an entirely intrinsic property of $\mathbf{x}$ and $\mathcal{X}$. Thus, $\mathrm{Ip}(\mathcal{X}, \mathbf{x})$ is a stable relation. For simplicity, we can thus assume $\mathrm{Ip}(\mathcal{X}, \mathbf{x})$ has the same extension at all worlds and we can unambiguously introduce an operator Γ taking collections of concepts and objects to collections of concepts and objects such that $\Gamma(\mathcal{X}) = \{\mathbf{x} : \mathrm{Ip}(\mathcal{X}, \mathbf{x})\}$. Clearly, Γ is sound and monotonic. Formally:

(Soundness) $$\mathcal{X} \subseteq \Gamma(\mathcal{X}).$$

and:

(Monotonicity) $$\text{If } \mathcal{X} \subseteq \mathcal{Y}, \text{ then } \Gamma(\mathcal{X}) \subseteq \Gamma(\mathcal{Y}).$$

Trivially, nothing is needed for the possible existence of $\mathbf{x}$ over and above the existence of $\mathbf{x}$; and if nothing is needed for the possible existence of $\mathbf{x}$ over and above the existence of the $\mathcal{X}$'s, then nothing is needed for the possible existence

of $\mathbf{x}$ over and above the existence of the $\mathcal{Y}$'s whenever $\mathcal{X} \subseteq \mathcal{Y}$.

The next principle is slightly less obvious. It says that any $\mathbf{x} \in \text{dom}(w)$ is immediately possible for the union of the domains of worlds prior to w. Formally:

(Dependence) If $\mathbf{x} \in \text{dom}(w)$, then $\mathbf{x} \in \Gamma(\bigcup_{w' < w} \text{dom}(w'))$.

If $\mathbf{x} \in \text{dom}(w)$, then trivially nothing is needed for the possible existence of $\mathbf{x}$ over and above the worlds prior to w having been the case. Dependence claims that when it comes to immediately possibility, a world is just as good as its domain. For example, one feature worlds have which might be relevant to the immediate possibility of $\mathbf{x}$ is the extensions of various predicates over their domains. But those extensions will be subcollections of their domains, and thus available whenever the domains are. In general, it is hard to see what else could be needed for the possible existence of $\mathbf{x} \in \text{dom}(w)$ over and above the existence of the concepts and objects which exist at worlds prior to w.

The final principle says that the $\Gamma(\mathcal{X})$'s can co-exist if the $\mathcal{X}$'s can; if nothing is needed for the possible existence of each $\Gamma(\mathcal{X})$ over and above the existence of the $\mathcal{X}$'s, then nothing more is needed for the possible co-existence of the $\Gamma(\mathcal{X})$'s. Formally:

(Compossibility) If $\mathcal{X} \subseteq \text{dom}(w)$, then $\Gamma(\mathcal{X}) \subseteq \text{dom}(w')$ for some $w' \geq w$.

Compossibility is plausible, but much less obvious than Soundness, Monotonicity, and Dependence. So why should the potentialist believe it? If they want to provide an interpretation of ZFC via modalisation, then Powerset$^{\Diamond}$ will have to be true.

In the presence of COLLAPSE$^\Diamond$ and InExt, it effectively tells us that the subsets of any co-existent pure sets can co-exist. Formally:

$$\text{(*) If } X \subseteq \text{dom}(w) \cap \text{Ps, then } \mathcal{P}(X) \subseteq \text{dom}(w') \text{ for some } w' \geq w$$

where $\mathcal{P}(X) = \{x : \exists Y \subseteq X \Box(Y \equiv x)\}$. Generalising this, UPowereset$^\Diamond$ tells us that the subsets of any co-existent objects can co-exist. Formally:

$$\text{(**) If } X \subseteq \text{dom}(w), \text{ then } \mathcal{P}(X) \subseteq \text{dom}(w') \text{ for some } w' \geq w$$

Compossibility can be taken to underlie both of these claims. It is natural to think the set of the X's is immediately possible for the X's. Suppose it is. Then Monotonicity implies:

$$(X\text{-}x) \qquad\qquad\qquad \mathcal{P}(X) \subseteq \Gamma(X)$$

and (*) and (**) are trivial consequences of Compossibility. A potentialist who accepted Powerset$^\Diamond$ but denied Compossibility would thus posit a substantial modal difference between the possible co-existence of certain immediately possible pure sets and the possible co-existence of immediately possible concepts and objects in general. But it is unclear how they might explain this difference.

A similar point applies to properties. That is, it is natural to think the property determined by a concept F is immediately possible for F. Then Monotonicity implies:

(F-p) $$\mathcal{C}(\mathcal{X}) \subseteq \Gamma(\mathcal{X})$$

where $\mathcal{C}(\mathcal{X}) = \{p : \exists F \in \mathcal{X} \Box (F \equiv p)\}$. Indeed, it is natural to think the converse holds; that a concept F is immediately possible for p whenever $\Box(F \equiv p)$. Since every concept determines a property, that implies:

(p-F) $$\text{If } F \in \mathcal{X}, \text{ then } F \in \Gamma(\mathcal{C}(\mathcal{X}))$$

These principles allow us to represent collections of concepts and objects as collections of objects. In particular, we can define $\mathcal{X}_1 = \mathcal{C}(\mathcal{X}) \cup [\mathcal{X} \cap \mathcal{O}]$ where $\mathcal{X} \cap \mathcal{O}$ is the collection of objects in $\mathcal{X}$. From F-p and Soundness it follows that:

$$\mathcal{X}_1 \subseteq \Gamma(\mathcal{X})$$

and from p-F it follows that $\mathcal{X} \subseteq \Gamma(\mathcal{C}(\mathcal{X})) \cup [\mathcal{X} \cap \mathcal{O}]$ and thus that:

$$\mathcal{X} \subseteq \Gamma(\mathcal{X}_1)$$

by Monotonicity and Soundness. Compossibility then implies that we can move between $\mathcal{X}$ and $\mathcal{X}_1$. Whenever the $\mathcal{X}$'s exist, the $\mathcal{X}_1$'s could have co-existed and vice versa. Indeed, by COLLAPSE$^\Diamond$ and InExt, the $\mathcal{X}_1$'s could have co-existed just in case they could have determined a set. In that case I will identify the $\mathcal{X}_1$'s with their set.

I will now use Soundness, Monotonicity, Dependence, Compossibility, X-x, F-p, and p-F to construct a witness for WO$^\Diamond$. By transfinite recursion over the

ordinals define:

$$\Gamma_0 = \emptyset$$

$$\Gamma_{\alpha+1} = \Gamma(\Gamma_\alpha)$$

$$\Gamma_\lambda = \bigcup_{\alpha < \lambda} \Gamma_\alpha$$

A simple induction establishes that the Γ_α's could have co-existed for each α. The successor case is immediate given Compossibility. For the limit case, we first map each $\alpha < \lambda$ to $\Gamma_{\alpha,1}$. Then by UReplacement$^\Diamond$ and InExt, there will be a world w with $\Gamma_{\alpha,1} \subseteq \mathrm{dom}(w)$ for $\alpha < \lambda$. Since $\Gamma_\alpha \subseteq \Gamma(\Gamma_{\alpha,1})$, it follows by Monotonicity that $\Gamma_\lambda \subseteq \Gamma(\mathrm{dom}(w))$ and thus that the Γ_λ's could have co-existed by Compossibility. So every $\Gamma_{\alpha,1}$ determines a set. Now, another induction establishes that every concept and object is in some Γ_α. For suppose not, and let w be a least world with $\mathbf{x} \in \mathrm{dom}(w)$ but $\mathbf{x} \notin \Gamma_\alpha$ for every α. Then for each $w' < w$, $\mathrm{dom}(w') \subseteq \Gamma_\alpha$ for some α and since $\mathrm{dom}(w')_1 \subseteq \Gamma(\mathrm{dom}(w'))$ we have $\mathrm{dom}(w')_1 \subseteq \Gamma_{\alpha+1}$ by Monotonicity. Since domains are increasing along the accessibility relation, $\mathrm{dom}(w') \subseteq \mathrm{dom}(w)$ and so $\mathrm{dom}(w')_1 \subseteq \mathrm{dom}(w)_1$ for all $w' < w$. By Compossibility, X-x, COLLAPSE$^\Diamond$, and the fact that $\mathrm{dom}(w)_1 \subseteq \Gamma(\mathrm{dom}(w))$, $\mathcal{P}(\mathrm{dom}(w)_1)$ will determine a set in some $w'' \geq w$. Thus, if we map each $\mathrm{dom}(w')_1$ for $w' < w$ to the least α with $\mathrm{dom}(w')_1 \subseteq \Gamma_\alpha$, UReplacement$^\Diamond$ will entail that such α have a least upper bound λ. Since $\mathrm{dom}(w') \subseteq \Gamma(\mathrm{dom}(w')_1)$ and $\Gamma_{\alpha+1} \subseteq \Gamma_{\lambda+1}$ by Soundness and Monotonicity,

it follows that $\bigcup_{w' < w} \mathrm{dom}(w') \subseteq \Gamma_{\lambda+1}$ and thus that $\mathbf{x} \in \Gamma_{\lambda+2}$ after all. So every object is in some $\Gamma_{\alpha,1}$. Finally, it is clear from Soundness and Monotonicity that $\subset$ is a well-order on the Γ_α's and thus on the $\Gamma_{\alpha,1}$'s. The latter therefore provide a suitable witness for $\mathrm{WO}^\diamond$.

Summing up. By theorem 4, $\mathrm{C\text{-}COMP}^\diamond$ is inconsistent in PMSTU with UReplacement$^\diamond$, $\mathrm{GC}^\diamond$, $\mathrm{WO}^\diamond$, and UPowerset$^\diamond$. But given $\mathrm{C\text{-}COMP}^\diamond$, it is hard for the potentialist to deny those principles. In this sense, potentialism is incompatible with $\mathrm{C\text{-}COMP}^\diamond$. Any reason to accept it is thus a reason to reject potentialism. In the next section I will look at one such reason.

1.3   Potentialism and reflection

Set-theoretic reflection principles say, roughly, that the universe of sets V is mirrored by or reflected in its initial segments V_α. They come in two flavours. First, there are indescribability principles. These say that whatever is true in V is true in some V_α. Formally:

(PR) $$\varphi \to \exists \alpha \varphi^{V_\alpha}$$

where φ^{V_α} is the result of restricting φ to V_α. Specific principles can be obtained from PR by specifying (i) a class of formulas for which it is to hold and (ii) how formulas of that class are to be restricted to V_α. For formulas in $\mathcal{L}^2_\in$, φ^{V_α} is usually taken to be the result of replacing all occurrences of $\exists x$ and $\exists F$ in φ with $\exists x \in V_\alpha$ and $\exists F \subseteq V_\alpha$ respectively, and replacing all occurrences of free second-order variables F in φ with $F \cap V_\alpha$. Let PR_1 denote the restriction of PR

to formulas in $\mathcal{L}_\in$ and PR_2 the restriction of PR to formulas in $\mathcal{L}^2_\in$.[29]

Second, there are indistinguishability principles. These say that the universe is indistinguishable from its initial segments relative to particular formulas. For first-order formulas, the idea can be expressed formally as:

$$(\mathrm{CR}_1) \qquad \exists \alpha [\emptyset \in V_\alpha \wedge \forall \vec{x} \in V_\alpha (\varphi \leftrightarrow \varphi^{V_\alpha})]$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$. For second-order formulas, though, things are not so straightforward. A natural first attempt to express the idea would be:

$$\exists \alpha [\emptyset \in V_\alpha \wedge \forall \vec{F} \subseteq V_\alpha \forall \vec{x} \in V_\alpha (\varphi \leftrightarrow \varphi^{V_\alpha})]$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{x}, \vec{F}$. But this principle is inconsistent. For consider the claim $\forall x (x \in X)$. Trivially, $\forall x \in V_\alpha (x \in X)$ when $X = V_\alpha$. But it cannot be the case that $\forall x (x \in X)$ when $X = V_\alpha$.

One way to avoid inconsistency is by moving the second-order quantifiers outside the scope of the initial existential quantifier. Formally:

$$(\mathrm{CR}_2) \qquad \forall \vec{F} \exists \alpha [\emptyset \in V_\alpha \wedge \forall \vec{x} \in V_\alpha (\varphi \leftrightarrow \varphi^{V_\alpha})]$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{x}, \vec{F}$. Another way is by allowing subconcepts of V_α to be systematically reinterpreted to concepts over V. The idea is then that there are (non-empty) V_α which are indistinguishable from V with respect to

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[29]PR_1 was first introduced by Lévy (1960b) and PR_2 by Bernays (1961).

φ up to a reinterpretation of its parameters over V_α. Formally, we say that there is concept j which codes a one-one function from subconcepts of V_α to concepts and is such that:

(SCR) $\qquad \exists \alpha[\emptyset \in V_\alpha \wedge \forall \vec{F} \subseteq V_\alpha \forall \vec{x} \in V_\alpha(\varphi(\vec{x}, j(\vec{F})) \leftrightarrow \varphi^{V_\alpha}(\vec{x}, \vec{F}))]^{30}$

where $\varphi \in \mathcal{L}_\in^2$ with free variables among $\vec{x}, \vec{F}$.

Why should we be interested in reflection principles? Perhaps the most prominent reason is that they might be used to effect reductions in incompleteness – that is, to answer various interesting questions left open by the standard axioms of ZFC.[31] Since Montague (1961) it has been known that CR_1 and PR_1 are provable in ZFC and thus that they cannot do this. Indeed, Lévy (1960a) showed that CR_1 is equivalent to Replacement and Infinity over the remaining axioms.[32] Furthermore, Silver and Reinhardt showed that although PR_2 implies the existence of many so-called small large cardinals, it is also relatively weak.[33] In particular, it fails to imply that there are measurable cardinals. Since CR_2 is equivalent to PR_2,[34] it is similarly weak. SCR, however, is much stronger. In particular, it implies the existence of measurable cardinals and thus that $V \neq L$ and the exis-

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[30]This principle is essentially present in Reinhardt (1974a, 1980) (see chapter 3), but the current formulation is based on the principle GRP in Welch (ms). The "S" stands for "structural". See **?** (ms) for discussion.

[31]See, for instance, Gödel's reported remarks in (Wang, 1997, p. 285). See Gödel (1964) for a classic statement of the project of supplementing the axioms of ZFC to answer these questions, and Koellner (2006) for an illuminating discussion in light of recent developments in set theory.

[32]Lévy and Vaught (1961) further showed that PR_1 is strictly weaker than CR_1 over Z. See lemma 14 in chapter 2 for a related result.

[33]See (Kanamori, 2003, p. 109). See also Koellner (2009) for an extension of this result to similar principles suggested by Tait (2005).

[34]*Proof sketch:* Clearly CR_2 implies PR_2. For the converse direction, let F be such that $\forall x(x \in F \leftrightarrow \varphi)$. Applying PR_2 we get a V_α for which $\forall x \in V_\alpha(x \in F \leftrightarrow \varphi^{V_\alpha})$. Putting these two facts together gives us $\forall x \in V_\alpha(\varphi \leftrightarrow \varphi^{V_\alpha})$ as required.

tence of a proper class of measurable Woodin cardinals and thus that the axiom of determinacy holds in $L(\mathbb{R})$.[35]

Reflection principles face a number of problems. The most difficult is a form of bad company. Certain formulas uniquely describe V and thus distinguish it from its initial segments. For example, absolutely infinitary formulas like:

$$\exists x_0, ..., x_\alpha, ...(x_0 \neq x_1 \wedge, ..., \wedge x_0 \neq x_\alpha, ..., x_\alpha \neq x_{\alpha+1}, ...)$$

are true in V but not in any V_α.[36] It is thus incumbent on a proponent of a reflection principle to say for which class of formulas it holds and why. I think these problems can be overcome and that reflection principles like SCR are the best way we have of effecting the above mentioned reductions in incompleteness, but I will not argue that here. Rather, I will assume that they are and see what consequences this has for the debate between potentialism and actualism.

In the last section I argued that C-COMP$^\diamond$ is incompatible with potentialism. But, assuming that every set is co-extensive with a concept, it is easy to see that PR_2 and SCR imply C-COMP.[37] It then follows from theorem 3 that SCR$^\diamond$ and $PR_2^\diamond$ imply C-COMP$^\diamond$. Potentialism thus appears to be incompatible with strong reflection principles. The actualist, on the other hand, can interpret these principles plurally since P-COMP$^\diamond$ is already a consequence of their view. We thus have a prima facie reason to prefer actualism over potentialism.

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[35]See Welch (ms) for discussion.

[36]See Koellner (2009) and chapter 3 for discussion.

[37]*Proof sketch:* For any V_α, Separation implies that there is a set $x \subseteq V_\alpha$ such that $(\forall y(y \in x \leftrightarrow \varphi))^{V_\alpha}$ and thus a concept $G \subseteq V_\alpha$ such that $(\forall y(y \in G \leftrightarrow \varphi))^{V_\alpha}$. Thus, $(\forall \vec{F}, \vec{x} \exists G \forall y(y \in F \leftrightarrow \varphi))^{V_\alpha}$. By the contrapositive of PR_2 or by SCR it then follows that $\forall \vec{G}, \vec{x} \exists F \forall y(y \in G \leftrightarrow \varphi)$ as required.

Can the potentialist resist this conclusion? In particular, can they formulate a principle similar in motivation and strength to PR_2 or SCR which avoids commitment to C-COMP$^\diamond$? In the next two chapters, I will look at two ways they might do this. In chapter 2 I look at a proposal by Geoffrey Hellman for implementing the indescribability idea, and in chapter 3 I look at a proposal by William Reinhardt for implementing the indistinguishability idea.

Chapter 2

Modal structural reflection

In this chapter I investigate a suggestion by Geoffrey Hellman that promises to yield a restricted plural R_2-like principle which is both consistent with COLLAPSE$^\diamond$ and strong. Many of the issues which arise for Hellman's suggestion are unique to the modal structural setting. For this reason, I have made the chapter self-contained.

2.1 Modal structuralism

By employing a modal operator, plural quantification, and mereology, the modal structuralist attempts to give an interpretation of set theory which eschews quantification over abstracta. A claim about the sets is translated as a claim about what would be the case in various (appropriately related) structures satisfying the axioms of second-order ZFC (ZFC2). For example:

$$\forall\alpha\exists\beta(\alpha < \beta) \tag{2.1}$$

is translated as:

$$\Box \forall M \forall \alpha \in M \Diamond \exists M' \sqsupseteq M \exists \beta \in M'(M' \vDash \alpha < \beta) \tag{2.2}$$

where M, M' range over pluralities coding ZFC2 structures, and where $M' \sqsupseteq M$ means that M' is an end-extension of M (see section 3 for more details). Informally, (2) says that for any possible ZFC2 structure and any ordinal in that structure, there is a possible end-extension also satisfying ZFC2 which contains a larger ordinal.[1,2]

In addition to extending the modal structural (ms-)translation of the language of arithmetic to the language of set theory,[3] this framework allows the modal structuralist to give a Zermelo (1930) inspired solution to the set-theoretic paradoxes. It would take us too far afield to consider this solution in detail, but the following brief sketch should help to fix ideas.

The set-theoretic paradoxes can be seen to arise from a tension between two plausible claims – namely, that any plurality forms a set (Collapse) and that for any condition φ there is a plurality of all and only the φ's (Plural Comprehension). As usual, once we consider the plurality of all and only the non-self-membered sets, we are quickly led into inconsistency. The modal structuralist proposes to resolve

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[1] For ease of exposition, I will frequently talk of possible objects, pluralities, structures, and worlds. Nothing I say will depend on misspeaking in this way, and can always be reformulated using the modal operator, plural quantification, and mereology.

[2] The locus classicus for modal structuralism is Hellman (1989), with Hellman (1996) adding plural quantification and mereology.

[3] Since there are non-isomorphic models of ZFC2, the simple translation used in the arithmetical case would give the wrong results. For example, the modal structural translation of the language of arithmetic is $\varphi^{tr} = \Box \forall M(M \vDash PA2 \rightarrow M \vDash \varphi)$ and its analogue for set theory $\varphi^{tr} = \Box \forall M(M \vDash ZFC2 \rightarrow M \vDash \varphi)$. If $\kappa < \kappa'$ are the first two inaccessibles, $V_\kappa \vDash ZFC2 +$ "there are no inaccessibles" and $V_{\kappa'} \vDash ZFC2 +$ "there are inaccessibles". So neither (there are no inaccessibles)^{tr} nor (there are inaccessibles)^{tr} would come out true.

this tension by making two moves. First, they point out that there is a version of Collapse which is consistent with Plural Comprehension. In particular, it is consistent with Plural Comprehension that, necessarily, every plurality *possibly* forms a set in some ZFC2 structure (the Extendibility Principle).[4] Second, they claim that the reasons we have to accept Collapse are at most reasons to accept the Extendibility Principle.[5]

2.2   Reflection and modal structuralism

Set-theoretic reflection is usually motivated by the thought that since the universe of sets V is absolutely infinite, it is indescribable – whatever is true in V is also true in some initial segment V_α.[6] Formally:

(R) $$\varphi \to \exists\alpha\varphi^{V_\alpha}$$

For second-order φ with parameters, R is quite strong. Over second-order Zermelo set theory (Z2), it implies Foundation, second-order Replacement, inaccessible, Mahlo, weakly compact, and the various Π^1_n-indescribable cardinals.[7] Let "R_2" denote this restriction of R.

R_2 can be pushed into the service of two central projects in the foundations and philosophy of set theory. The first is the project of effecting a reduction

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[4]See section 3.2.3 for a precise statement and discussion of the Extendability Principle.

[5]The formulation of the paradoxes used here is indebted to Linnebo (2010), but the resolution is at least implicit in Hellman (1989, 2002). See the former for an extended argument in favour of Collapse and a similar resolution in the modal non-structural setting.

[6]See, for instance, Koellner (2009) and the references therein.

[7]Indeed, over Extensionality, second-order Separation, and Choice, R_2 also implies the other axioms of Z2. See Bernays (1976) and Kanamori (2003) §6 for details.

in incompleteness – that is, of finding and justifying new axioms which settle interesting questions left open by the axioms of ZFC.[8] The second is the project of developing an account of sets which (i) provides a solution to the set-theoretic paradoxes, and (ii) explains why the various axioms, and theorems, of ZFC are nonetheless true. Adding a suitable version of R_2 to a standard formalisation of the iterative conception would arguably yield a unified account which fully meets the second project[9] and goes some non-trivial way toward meeting the first.

There is reason to think that R_2-like principles are unavailable to the modal structuralist, however. First, the ms-translation of R_2 is outright inconsistent with the Extendability Principle (see section 4). So it is unclear whether the modal structuralist can even state an R_2-like principle consistently. Second, supposing they could, it is unclear whether they would be able to motivate such a principle in terms of indescribability. After all, there is no absolutely infinite universe of sets according to them. Rather, there are various ZFC2 structures, each of which (by the Extendability Principle) forms a set in some larger ZFC2 structure.

These prima facie problems suggest an argument against the modal structuralist. Suppose they can't adopt a consistent principle similar in motivation and strength to R_2 (over Z2). Further, suppose they can't pursue the above projects in some other equally attractive way; at least, that they can't pursue them to the same extent. Then all other things being equal, we should reject modal structuralism. Of course, this is only a sketch of the argument, and I don't intend to give a full defence of it here.[10] Instead, I will focus on its first premise. The

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[8]See Gödel (1964) for a classic statement of the project, and Koellner (2006) for an illuminating discussion in light of recent developments in set theory.

[9]See Paseau (2007) for discussion.

[10]Though see section 6 for objections to the second premise (objection 2) and the conclusion (objection 1).

central question of this chapter is thus whether the modal structuralist can adopt a consistent principle similar in motivation and strength to R_2.

Here's the plan. In section 3 I do some ground clearing. In particular, I provide an axiomatisation of modal structuralism (MSST), define the ms-translation in full generality, and show that ms-translation is a faithful interpretation of Z^* in MSST (theorem 7). In section 4 I consider a novel suggestion due to Geoffrey Hellman which appears to yield a strong and consistent R_2-like principle motivated in terms of indescribability. Unfortunately, the principle is inconsistent (theorem 8). Nonetheless, I argue in section 5 that Hellman's suggestion contains the most promising way to implement the indescribability motivation for R_2 in the modal structural setting. This motivation is formalised as a principle I call $R^{\diamond}$ and I show that ms-translation is a faithful interpretation of $Z^* + R_1$[11] in MSST $+ R^{\diamond}$ (theorem 11). Since $Z^* + R_1$ is significantly weaker than $Z2 + R_2$, I claim that this is good evidence for a negative answer to the central question.

2.3   Modal structural set theory

Above, the notion of strength relevant to the central question was left implicit. It will be helpful to make it explicit before we more on. Our interest in the central question, recall, stems from an interest in the extent to which the modal structuralist can pursue the two projects outlined in section 2. Each project suggests a measure of strength. The first suggests measuring strength in terms of large cardi-

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[11]R_1 is just R restricted to first-order φ with parameters.

nal hypotheses.[12] Roughly, the more ms-translations of large cardinal hypotheses a theory implies, the stronger it is. The second suggests measuring strength in terms of theorems of ZFC. Roughly, the more ms-translations of theorems of ZFC a theory implies, the stronger it is. The notion relevant to the central question combines these measures. In particular, given a suitable modal structural base theory B, I will say that a principle P is weaker than R_2 if B + P proves fewer (ms-translations of) large cardinal hypotheses and theorems of ZFC than Z2 + R_2. In the rest of this section I develop the technical machinery necessary to show that various principles are weaker than R_2 in this sense.

2.3.1 Formal preliminaries

Let $\mathcal{L}_\in$ be the language of first-order set theory with variables $x_0, ..., x_n, ...$; $\mathcal{L}_\in^2$ the language of second-order set theory, extending $\mathcal{L}_\in$ with variables $Y_0, ..., Y_n, ...$; and $\mathcal{L}_\Diamond$ the language of pure second-order logic extended with the modal operator $\Diamond$ and first-order terms $\langle x, y \rangle$. $x \in y$ is thus well-formed in $\mathcal{L}_\in$ and $\mathcal{L}_\in^2$ but not in $\mathcal{L}_\Diamond$, and $x \in X$ is well-formed in $\mathcal{L}_\in^2$ and $\mathcal{L}_\Diamond$ but not in $\mathcal{L}_\in$.[13] Ex abbreviates $\exists y(y = x)$; similarly for $E\langle x, y \rangle$. EX abbreviates $\exists Y \subseteq X(Y = X)$. This formulation of EX – which is equivalent to $\exists Y(Y = X)$ – is used so that EX is bounded in the sense of definition 3 (section 3.3).

I will use the Levy hierarchy to measure the complexity of formulas in $\mathcal{L}_\in$. In particular, if φ's quantifiers are all of the form $\exists x \in y$ or $\forall x \in y$, then it is Π_0, Σ_0, and Δ_0. In general, if φ is Π_n, then $\exists \vec{x} \varphi$ is Σ_{n+1}, and if φ is Σ_n, then $\forall \vec{x} \varphi$ is

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[12]On using the large cardinal hierarchy to measure reduction in incompleteness see, for instance, (Koellner, 2009, p.208).

[13]Since the second-order variables in $\mathcal{L}_\Diamond$ are taken to range over pluralities (see sections 1 and 3.2), $x \in X$ should be be read "it$_x$ is one of them$_X$".

Π_{n+1}. A formula is Π_n^T, Σ_n^T, or Δ_n^T if it is equivalent in the theory T to a Π_n, Σ_n, or to both a Π_n and a Σ_n, formula respectively.

Let ZFC be the $\mathcal{L}_\in$ theory consisting of Extensionality, Infinity, Pairing, Union, Powerset, Foundation, Separation, Choice, and:

(Collection) $\qquad\qquad \forall x \exists y \varphi(x, y, \vec{z}) \rightarrow \forall u \exists v (\forall x \in u)(\exists y \in v) \varphi(x, y, \vec{z})$

where φ's free variables are among $x, y, \vec{z}$ and where $x, y, \vec{z}, u, v$ are all distinct. Π_n-Col and Σ_n-Col denote the restriction of Collection to Π_n and Σ_n formulas respectively. Zermelo set theory (Z) is ZFC minus Collection and Foundation. Let Z^* be Z plus $\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)$ (where $\text{trans}(y)$ abbreviates $\forall x \in y \forall z \in x (z \in y)$). Over Z, this axiom is equivalent to the claim that every set is in some inaccessible rank (i.e. $\exists \alpha (In(\alpha) \wedge x \in V_\alpha)$, where $In(\alpha)$ abbreviates "α is inaccessible"). The first formulation is used because it is logically equivalent to a Σ_2 formula (since "$\text{trans}(x)$" is Δ_0 and "$ZFC2^y$" is a conjunction of Π_1 formulas). In what follows, I will assume that "$\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)$" has been formulated so as to be explicitly Σ_2 and that Infinity and Choice (with its initial universal quantifier omitted) have been formulated so as to be explicitly Σ_1. Let κ_0 denote the least inaccessible, $\kappa_{\alpha+1}$ the least inaccessible greater than κ_α, and $\kappa_\lambda = \bigcup_{\alpha < \lambda} \kappa_\alpha$ for λ a limit.

2.3.2 MSST

In this section I formulate a modal structural base theory (which I call MSST for *Modal Structural Set Theory*) following the outline in Hellman's (1989) and

(1996). It consists of three packages of principles. There is the underlying logic, axioms governing the behaviour of ordered pairs, and axioms (like the Extendability Principle) that tell us which structures there are and how they relate to one another.

Logic

The underlying logic has two groups of axioms. First, there are the instances in $\mathcal{L}_\Diamond$ of the truth-functional tautologies, the S5 axioms, and the following quantificational and identity axioms (where $\mathbf{x}, \mathbf{y}$ are either both first-order or both second-order variables):

(A1) $\forall \mathbf{y}(\forall \mathbf{x}\varphi \to \varphi[\mathbf{y}/\mathbf{x}])$, where $\mathbf{y}$ is free for $\mathbf{x}$ in φ

(A2) $\forall \mathbf{x}(\varphi \to \psi) \to (\forall \mathbf{x}\varphi \to \forall \mathbf{x}\psi)$

(A3) $\varphi \leftrightarrow \forall \mathbf{x}\varphi$, where $\mathbf{x}$ is not free in φ

(A4) $\mathbf{x} = \mathbf{x}$

(A5) $\mathbf{x} = \mathbf{y} \to (\varphi[\mathbf{x}/\mathbf{z}] \leftrightarrow \varphi[\mathbf{y}/\mathbf{z}])$, where $\mathbf{x}$ and $\mathbf{y}$ are free for $\mathbf{z}$ in φ

Second, there are the following axioms governing pluralities.

(A6) $\exists X \forall x(x \in X \leftrightarrow \varphi)$

(A7) $x \in X \to \Box(EX \to Ex \wedge x \in X)$

(A8) $EX, Y \wedge X \subseteq Y \to \Box(EY \to EX \wedge X \subseteq Y)$

(A9) $\Box \forall x[\Diamond(x \in X) \leftrightarrow \Diamond(x \in Y)] \to X = Y$

The rules of inference are GEN, from φ infer $\forall \mathbf{x}\varphi$; MP, from φ and $\varphi \rightarrow \psi$ infer ψ; and NEC, if φ is a theorem of MSST, then so is $\square\varphi$.

Remarks. Let A1* be the result of dropping the initial quantifier $\forall \mathbf{y}$ in A1. Then – over the truth-functional tautologies – A1*-A5 constitutes an axiomatisation of classical quantificational logic,[14] and A1-A5 an axiomatisation of free quantificational logic. The choice of a free logic is essentially forced by the Extendability Principle, which implies that any plurality possibly exists but fails to contain something (i.e. $\forall X \Diamond(EX \wedge \exists x(x \notin X)))$.[15] Given A6, A7, and the S5 axioms this contradicts the claim, provable using A1*, that necessarily everything exists necessarily.[16]

The difference between A1*-A5 and A1-A5 essentially only shows up in arguments involving NEC. More precisely, a simple induction on the length of proofs shows that if φ is provable from some premises Γ using the truth-functional tautologies, A1*-A5, S5, MP, and GEN, then $E\vec{\mathbf{x}} \rightarrow \varphi$ is provable from Γ using the truth-functional tautologies, A1-A5, S5, MP, and GEN (where φ's free variables are among $\vec{\mathbf{x}}$). So in many cases we can reason classically in MSST.

A6-A9 axiomatise the compelling thought that a plurality is nothing over and above the things which comprise it. To see this, note that this thought can be glossed as a conjunction of the following four claims: an object continues to be in a plurality whenever that plurality exists (A7); when a plurality exists, so do the things which comprise it (A7); when the things which comprise a plurality exist,

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[14]See, for instance, Mendelson (1997).

[15]That the Extendibility Principle implies this depends on how we formulate it. In the terminology of section 3.2.3, it depends on formulating it as EP*.

[16]*Proof:* Using A6, let U be the plurality of everything. By the above claim, there is a possible world w where U exists and $x \notin U$, for some x. If necessarily everything exists necessarily, then, by the modal axioms 5 and T, x exists simpliciter. So $x \in U$. But since U exists at w, A7 implies that $x \in U$ at w, after all.

so does the plurality (A6 and A8); and pluralities comprising the same things are identical (A9). (A8) follows from the other axioms, but is added for ease of exposition.[17]

Ordered pairs

In order to formulate the Extendibility Principle or define the ms-translation, we first need to define the notion of a structure for the language $\mathcal{L}_\in^2$. This raises a problem. The usual way to interpret $\in$ is by some set-theoretic ordered pairs. On the modal structural view, however, set-theoretic pairing is only available within structures, and thus can't be used for the definition of structure itself. Hellman's (1996) response to this problem is to add enough mereology to code pairs as in Burgess, Hazen, and Lewis (1991).

For the sake of generality and simplicity, I will deviate from this response. In particular, I will take ordered pairing as a primitive governed by axioms A10-A15 below. Given a suitable mereology, it will then be straightforward to modify the proofs in the chapter by replacing each use of A10-A15 with a mereological analogue. Thus, up to a change in the definition of structure, the main results of the chapter will hold for various mereological approaches to pairing.

(A10) $\forall x, y, x', y'(\langle x, y \rangle = \langle x', y' \rangle \to x = x' \wedge y = y')$

(A11) $\forall x, y(E\langle x, y \rangle)$

(A12) $E\langle x, y \rangle \to Ex \wedge Ey$

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[17]See (Linnebo, 2013, section 4) and Uzquiano (2011) for further discussion of the interaction between plural and modal logic.

(A13) $\tau = \tau' \rightarrow (\varphi[\tau/z] \leftrightarrow \varphi[\tau'/z])$, where τ, τ' are first-order terms free for z in φ

and where φ contains no modal operators

(A14) $\tau = \tau' \rightarrow \Box(\tau = \tau')$

(A15) $\langle x, y \rangle \in X \rightarrow \Box(EX \rightarrow E\langle x, y \rangle \wedge \langle x, y \rangle \in X)$

The Extendability Principle, Existence, and Stability

Let a *structure* be pair of pluralities X, Y such that $\forall x \in Y \exists y, z \in X(x = \langle y, z \rangle)$. For ease of exposition and where it won't cause confusion, I will identify X, Y with Y and write $dom(Y)$ for X. For $\varphi \in \mathcal{L}^2_\in$, let $Y \vDash \varphi$ be the result of replacing each occurrence of $x \in y$ in φ with $\langle x, y \rangle \in Y$, each occurrence of $\exists x$ with $\exists x \in dom(Y)$, and each occurrence of $\exists X$ with $\exists X \subseteq dom(Y)$ (re-lettering to avoid clashes of variables). I will say that Y' is an *end-extension* of Y, in symbols $Y \sqsubseteq Y'$, if (i) EY, Y'; (ii) $dom(Y) \subseteq dom(Y')$; and (iii) for any $x \in dom(Y)$ and $y \in dom(Y')$, $\langle y, x \rangle \in Y$ iff $\langle y, x \rangle \in Y'$. Y' is a *proper* end-extension of Y, in symbols $Y \sqsubset Y'$, if $Y \sqsubseteq Y'$ and $Y \neq Y'$. Metavariables M, M', M'' etc will be used for structures satisfying ZFC2, and where it won't cause confusion I will write $x \in M$ for $x \in dom(M)$, $X \subseteq M$ for $X \subseteq dom(M)$, and $\mathbf{x} \in M$ for both.

Using these notions, we can now give a precise definition of the ms-translation.

Definition 9. Let $^{pt}_Y$ be the following translation from $\mathcal{L}^2_\in$ to $\mathcal{L}_\Diamond$.[18]

- $(x = y)^{pt}_Y = x = y$

- $(x \in X)^{pt}_Y = x \in X$

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[18]This translation closely follows semantics given in Hellman (1989, p.76). The "*pt*" stands for "Putnam translation", since it was first outlined in Putnam (1967) (with structures satisfying Z2 replacing those satisfying ZFC2).

- $(x \in y)^{pt}_Y = Y \vDash x \in y$

- $^{pt}_Y$ commutes with the connectives.

- If φ contains free variables other than $\mathbf{x}$, then:

$$(\exists \mathbf{x} \varphi)^{pt}_Y = \Diamond \exists M \sqsupseteq Y \exists \mathbf{x} \in M \varphi^{pt}_M$$

- If φ contains at most $\mathbf{x}$ free, then:

$$(\exists \mathbf{x} \varphi)^{pt}_Y = \Diamond \exists M \exists \mathbf{x} \in M \varphi^{pt}_M$$

making sure to avoid clashes of variables.

If φ is a sentence, $\varphi^{pt}_Y = \varphi^{pt}_{Y'}$, and I will denote it by φ^{pt}.

The final three axioms can now be stated.

THE EXTENDABILITY PRINCIPLE (EP)

$$\Box \forall M \Diamond \exists M'(M \sqsubset M')$$

EXISTENCE (E)

$$\Diamond \exists M(M = M)$$

STABILITY (S)

$$[\forall \vec{\mathbf{x}}(\exists \mathbf{y}\varphi \rightarrow \forall z \exists \mathbf{y}(z = z \wedge \varphi))]^{pt}$$

where $\exists \mathbf{y}\varphi \in \mathcal{L}_{\in}^2$'s free variables are among $\vec{\mathbf{x}}$ and z does not occur in $\exists \mathbf{y}\varphi$.

Remarks. Axioms EP and E are taken from Hellman (1989, p.71-2). EP is equivalent to the claim that necessarily any subplurality of a ZFC2 structure forms a set in some possible end-extension also satisfying ZFC2, i.e.:

$$\Box \forall M \forall X \subseteq M \Diamond \exists M' \sqsupseteq M \exists x \in M'(X =^{M'} X) \tag{2.3}$$

where $X =^{M'} x$ means that X is the plurality of elements of x according to M'.[19] This differs from the gloss I gave of the Extendability Principle in section 2, which was that any plurality whatsoever possibly forms a set in some ZFC2 structure, i.e.:

(EP*) $\qquad\qquad\qquad \Box \forall X \Diamond \exists M \exists x \in M(X =^{M} x)$

EP* implies EP, but not conversely.[20] Given that the Extendibility Principle is supposed to provide a response to the set-theoretic paradoxes, it seems clear that the modal structuralist is committed to the stronger EP* (and thus to a free logic, as noted in section 3.2.1). After all, our reasons for thinking that a plurality X

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[19]*Proof sketch:* EP $\Rightarrow$ (3). Suppose $X \subseteq M$. From EP it follows that $\Diamond \exists M'(M' \sqsupseteq M)$. By corollary 1, $M = V_\kappa^{M'}$, and so by second-order Separation in M', there is some $x \in M'$ such that $X =^{M'} x$. (3) $\Rightarrow$ EP. Take $X = dom(M)$.

[20]*Proof sketch:* EP* $\Rightarrow$ EP. Taking $X = dom(M)$, it follows from EP* that $\Diamond \exists M' \exists x \in M'(dom(M) =^{M'} x)$. By the quasi-categoricity theorem (see (Hellman, 1989, pp. 68-9) for details), it follows that M will be isomorphic to a proper initial segment of M'. Using the isomorphism, we can then build a proper end-extension of M. EP $\not\Rightarrow$ EP*. Let K be a Kripke model with exactly one world $w = dom(w) = V_{\kappa_\omega}$ (where w sees itself). Letting plural quantifiers range over $\mathcal{P}(w)$ and $\langle x, y \rangle$ denote the ordered pair of x and y, it is easy to check that $w \vDash MSST - S$. The proofs of lemmas 10-12 and theorem 6 can then be adapted to show that $w \vDash S$. But, taking $X = V_{\kappa_\omega}$, it is clear that $w \not\vDash EP^*$.

should form a set don't appear to be sensitive to whether X happens to be a subplurality of some ZFC2 structure. Since it won't affect the main results of this chapter,[21] however, I will work with the simpler EP.

Axiom S really encodes two claims, depending on the free variables in φ. If φ contains free variables other than $\mathbf{y}$, it says that $\exists \mathbf{y} \varphi$ holds from the perspective of some M only if it holds from the perspective of any end-extension of M (i.e. $(\exists \mathbf{y} \varphi)^{pt}_M \rightarrow \Box \forall M' \sqsupseteq M (\exists \mathbf{y} \varphi)^{pt}_{M'}$, where $\vec{\mathbf{x}} \in M$). This essentially just the existential case of Hellman's Stability Theorem (1989, p.76). If φ contains at most $\mathbf{y}$ free, on the other hand, it says that some M contains a φ^{pt} only if every M can be end-extended to contain a φ^{pt} (i.e. $(\exists \mathbf{y} \varphi)^{pt} \rightarrow \Box \forall M \Diamond \exists M' \sqsupseteq M \exists \mathbf{y} \in M' \varphi^{pt}_{M'}$). Hellman does not consider this latter claim and thus not axiom S, but axiom S is key to proving the ms-translations of the logical truths in the following sense. It suffices with the rest of MSST to prove them (theorem 5); and since it is the ms-translation of a logical truth but not provable from the other axioms of MSST,[22]

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[21]In particular, theorems 6 and 10 are easily seen to extend to MSST + EP* and MSST + EP* + R$^\Diamond$ respectively.

[22]*Proof sketch:* Let K be the Kripke model with set of worlds $W = \{\langle \alpha, n \rangle : (\alpha < \omega \wedge n = 0) \vee (\alpha < \omega + \omega \wedge n = 1)\}$, where $dom(\langle \alpha, n \rangle) = \{\langle x, n \rangle : x \in V_{\kappa_\alpha}\}$, plural quantifiers at w range over $\mathcal{P}(dom(w))$, and $\langle \langle x, m \rangle, \langle y, n \rangle \rangle$ denotes $\langle \langle x, y \rangle, m \rangle$ if $m = n$ and otherwise 0 (where 0 is in the 'outer domain'). Every world sees every world. It is easy to verify that MSST-S holds at all worlds. Now, note that there is $\lambda \in V_{\kappa_{\omega+1}}$ such that $V_{\kappa_{\omega+1}} \vDash (\kappa_\omega \text{ exists})^{V_\lambda}$. Let $M \subseteq dom(\langle \omega + 1, 1 \rangle)$ be an isomorphic copy of $V_{\kappa_{\omega+1}}$. So $M \vDash (\kappa_\omega \text{ exists})^{V_\lambda}$, for some $\lambda \in M$. Since "$(\kappa_\omega \text{ exists})^{V_\lambda}$" is Π_1^{ZF}, it follows from the proof of lemma 7 (see footnote 28) that $[(\kappa_\omega \text{ exists})^{V_\lambda}]^{pt}_M$ is true at $\langle \omega + 1, 1 \rangle$ and thus that $[\exists \lambda (\kappa_\omega \text{ exists})^{V_\lambda}]^{pt}$ is true at all worlds. So the antecedent, $[\exists \lambda (\kappa_\omega \text{ exists})^{V_\lambda}]^{pt}$, of an instance of S is true at all worlds in W. Now we want to show that its consequent is false in all worlds in W. For contradiction, suppose it is not; that is, suppose $[\forall x \exists \lambda (x = x \wedge \kappa_\omega \text{ exists})^{V_\lambda}]^{pt}$ holds at some world in W. Let M be any structure in any 0-world (that is, $M \subseteq dom(\langle \alpha, 0 \rangle)$ for some α). It follows that there is world w containing an $M' \sqsupseteq M$ such that $[(\kappa_\omega \text{ exists})^{V_\lambda}]^{pt}_{M'}$, for some $\lambda \in M'$. By lemma 7 again, $M' \vDash (\kappa_\omega \text{ exists})^{V_\lambda}$ and so $M' \vDash \kappa_\omega$ exists. But since M is only end-extended by structures in 0-worlds, w must be of the form $\langle \beta, 0 \rangle$ and thus M' must be isomorphic to some V_{κ_n} (for $n \leq \beta$), which is impossible.

It is worth noting that by adapting the proof of lemma 12, we can also show that K satisfies the first claim encoded by S (i.e. $(\exists \mathbf{y} \varphi)^{pt}_M \rightarrow \Box \forall M' \sqsupseteq M (\exists \mathbf{y} \varphi)^{pt}_{M'}$, where $\vec{\mathbf{x}} \in M$). Furthermore, it is easy to see that it satisfies EP* and Hellman's Accumulation Principle for ZFC2 structures (i.e. $\Diamond \exists M (M \vDash \varphi) \wedge \Diamond \exists M (M \vDash \psi) \rightarrow \Diamond \exists M, M'(M \vDash \varphi \wedge M' \vDash \psi)$, for sentences $\varphi, \psi \in \mathcal{L}^2_\in$

it is necessary to prove them.

2.3.3 MSST and Z*

In this section I determine the strength of MSST. The main result is that ms-translation is a faithful interpretation of Z* in MSST (theorem 7).[23] In other words, MSST and Z* have exactly the same (first-order) set-theoretic content. It follows that MSST proves the ms-translation of the claim that there is an unbounded class of inaccessibles but not the ms-translation of the claim that κ_ω exists. It also follows that MSST proves the ms-translations of the theorems of Z* + Π_0-Col but not the ms-translations of all instances of Π_1-Col (lemmas 14 and 15). Without supplementation, then, MSST is significantly weaker than Z2 + R_2.

A wide variety of questions in MSST turn on whether notions like $Y \vDash \varphi$, "Y is a structure", $Y \sqsubseteq Y'$, and φ_Y^{pt}, are invariant between possible worlds. For example, we might want to know whether ZFC2 structures continue to be ZFC2 structures in any world where they exist. As the following lemma shows, a broad class of notions are indeed invariant between possible worlds.

Definition 10. Say that $\varphi \in \mathcal{L}_\Diamond$ with parameters among $\vec{\mathbf{x}}$ is *stable* if:

$$E\vec{\mathbf{x}} \wedge \varphi \rightarrow \Box(E\vec{\mathbf{x}} \rightarrow \varphi)$$

Definition 11. Say that $\varphi \in \mathcal{L}_\Diamond$ is *bounded* if its quantifiers are of the form $\exists x \in X$ or $\exists X \subseteq Y$.

(1989, p. 43)). Axiom S thus goes significantly beyond these claims.

[23]A translation π from the language of a theory T to the language of another theory T' is a *faithful* interpretation of T in T' just in case $T \vdash \varphi$ iff $T' \vdash \pi(\varphi)$.

Lemma 5 (MSST). *All bounded formulas in $\mathcal{L}_\Diamond$ are stable.*

Proof. By induction on the complexity of φ. For $\in$, we use A7 and A15; for $=$, A5 and A14; the case for $\wedge$ is trivial; for $\Diamond$ and $\neg$, we use the modal axioms 5 and T; and for $\exists x$ and $\exists X$, we use A7 and A8 respectively. $\square$

Since $Y \vDash \varphi$, "Y is a structure", and $Y \sqsubseteq Y'$ are bounded, it follows from lemma 5 that they are stable. And although "φ_Y^{pt}" is not bounded, a similar induction shows that it is stable too.

The next lemma brings out a simple but useful consequence of the definition of end-extension; namely, that whenever $M \sqsubset M'$, the domain of M is just V_κ, for some κ inaccessible in M'.

Definition 12. Let $X =^M x$ abbreviate $EX \wedge X \subseteq M \wedge \forall y \in M(\langle y, x \rangle \in M \leftrightarrow y \in X)$ and V_α^M denote the unique $x \in M$ such that $M \vDash x = V_\alpha$.

Note that "$X =^M x$" and "$X =^M V_\alpha^M$" are bounded.

Lemma 6 (MSST). *If $M \sqsubseteq M'$, then for all $\alpha \in M$:*

$$V_\alpha^M = V_\alpha^{M'}$$

Proof. $\subseteq$: if $M \vDash x \in V_\alpha$, then since M' is an end-extension of M and "$rank(x) = y$" is Δ_1^{ZF}, it follows that $M' \vDash x \in V_\alpha$. $\supseteq$: for contradiction, suppose that $M' \vDash x \in V_\alpha$ but $M \vDash x \notin V_\alpha$, for some $\alpha \in M$. If α is the least ordinal where this happens, α is a successor (say $\beta + 1$). By the induction hypothesis, it follows that $\forall y \in M'(M' \vDash y \in x \rightarrow M \vDash y \in V_\beta)$. But then, since M satisfies second-order Separation, we have $x \in M$ and thus $M \vDash x \in V_\alpha$ after all. $\square$

Corollary 1 (MSST). *If $M \sqsubset M'$, then:*

$$dom(M) =^{M'} V_\kappa^{M'}$$

for some κ inaccessible in M'.

Proof. Suppose $M \sqsubset M'$. Since "x is an ordinal" is Δ_0, it follows from lemma 6 that there must be some $\alpha \in M' \setminus M$. Let κ be the least such ordinal. Then $Ord(M) =^{M'} \kappa$, and so $dom(M) =^{M'} V_\kappa^{M'}$ by lemma 6. Since $M \vDash ZFC2$, $M' \vDash ZFC2^{V_\kappa}$, and so κ is inaccessible in M'. □

The proofs of the last two lemmas implicitly appealed to the fact that whenever M' is an end-extension of M and φ is Δ_0 with parameters in M, $M \vDash \varphi$ if and only if $M' \vDash \varphi$. This is just a slight modification of a standard result in ZFC; namely, that Δ_0 formulas are absolute for transitive sets.[24] It is well-known that this extends to Σ_1 formulas when the transitive set in question is an inaccessible rank.[25] Given corollary 1, it follows that whenever M' is an end-extension of M and φ is Σ_1 with parameters in M, $M \vDash \varphi$ if and only if $M' \vDash \varphi$. The next lemma uses this fact to establish an absoluteness result for ms-translation. In particular, it shows that if φ is Σ_1 with parameters in M, then $M \vDash \varphi$ if and only if φ_M^{pt}.

Lemma 7 (MSST). *Suppose EM and $\vec{x} \in M$. Then:*

$$(M \vDash \varphi) \leftrightarrow \varphi_M^{pt}$$

where $\varphi \in \mathcal{L}_\in$ is Σ_1 with free variables among $\vec{x}$.

Proof. By induction on the complexity of φ. The only difficult cases are the right-to-left directions for $\exists x \in y$ and $\exists x$. For the latter, suppose EM, $\vec{x} \in M$, and

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[24] A formula φ with parameters among $\vec{x}$ is *absolute* for a set y just in case $\forall \vec{x} \in y(\varphi^y \leftrightarrow \varphi)$.

[25] See, for instance, (Kanamori, 2003, p. 299).

$(\exists x \varphi)_M^{pt}$. By S,[26] there is a world w at which:

$$\exists M' \sqsupseteq M \exists x \in M' \varphi_{M'}^{pt} \tag{2.4}$$

A7 implies that $\vec{x} \in M$ at w, and the induction hypothesis yields $M' \vDash \exists x \varphi$. From the result mentioned above it follows that $M \vDash \exists x \varphi$ at w. We can then apply lemma 5 to "$M \vDash \exists x \varphi$" to get $M \vDash \exists x \varphi$, as required. The case for $\exists x \in y$ is similar. $\square$

The next lemma shows that ms-translation is not sensitive to our choice of "base" structure. As long as φ's parameters are in M, φ_M^{pt} is equivalent to $\varphi_{M'}^{pt}$ for any $M' \sqsupseteq M$.[27]

Lemma 8. *Suppose $M \sqsubseteq M'$ and $\vec{\mathbf{x}} \in M$. Then:*

$$\varphi_M^{pt} \leftrightarrow \varphi_{M'}^{pt}$$

where $\varphi \in \mathcal{L}_\in^2$ with free variables among $\vec{\mathbf{x}}$.

Proof. By induction on the complexity of φ. The only difficult case is that for $\exists \mathbf{x}$. The left-to-right direction is a trivial consequence of S. For the right-to-left direction, suppose $(\exists \mathbf{x} \varphi)_{M'}^{pt}$. Either there are free variables other than $\mathbf{x}$ in φ or not. If not, $(\exists \mathbf{x} \varphi)_{M'}^{pt} = (\exists \mathbf{x} \varphi)_M^{pt}$ and we are done. If there are, $(\exists \mathbf{x} \varphi)_{M'}^{pt} = \Diamond \exists M'' \sqsupseteq M' \exists \mathbf{x} \in M'' \varphi_{M''}^{pt}$. So suppose $\Diamond \exists M'' \sqsupseteq M' \exists \mathbf{x} \in M'' \varphi_{M''}^{pt}$. By A8 and lemma 5 applied to "$M \sqsubseteq M'$", it follows that $\Diamond \exists M'' \sqsupseteq M \exists \mathbf{x} \in M'' \varphi_{M''}^{pt}$, which is to say $(\exists \mathbf{x} \varphi)_M^{pt}$. $\square$

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[26]Notice that S is not needed if there are free variables other than x in φ. In that case, (4) follows from $(\exists x \varphi)_M^{pt}$ alone. It is for such φ that lemma 7 is used in footnote 24, which is why the argument there does not require S.

[27]The lemma is essentially just an object language statement of Hellman's Stability Theorem (1989, p.76).

Theorem 5. *MSST interprets Z^* via ms-translation.*

Proof. By induction on the length of proof we show for every theorem φ of Z^* that MSST proves:

$$EM \wedge \vec{x} \in M \to \varphi_M^{pt}$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$. Since E can be used to discharge EM when φ is a sentence, it will follow that MSST proves the ms-translation of every sentence provable in Z^*.

For the axioms of Z^* there are four cases to consider.

CASE 1: The logical axioms of Z^* (i.e. the truth-functional tautologies and A1*-A5). The only difficult cases are A2 and A3. For A2, the only tricky case is where φ contains a free variable other than x but ψ does not. Then, from $(\forall x(\varphi \to \psi))_M^{pt}$ it will follow that:

$$(\forall x \varphi)_M^{pt} \to \Box \forall M' \sqsupseteq M \forall x \in M' \varphi_{M'}^{pt} \tag{2.5}$$

By (the contrapositive of) S, the right-hand side of (5) will entail $(\forall x \psi)^{pt}$, as required. For A3, the only tricky case is the left-to-right direction. So assume EM, $\vec{x} \in M$, and φ_M^{pt}. Since "φ_M^{pt}" is stable and x is not free in φ, it follows from A7 that $\Box \forall M' \sqsupseteq M \forall x \in M' (\vec{x} \in M \wedge \varphi_M^{pt})$. By lemma 8 that entails $\Box \forall M' \sqsupseteq M \forall x \in M' \varphi_{M'}^{pt}$, which is to say $(\forall x \varphi)_M^{pt}$.

CASE 2: The non-logical axioms of Z^* - Separation - $\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)$. Omitting initial universal quantifiers, every such axiom φ with free variables among $\vec{x}$ has complexity at most Σ_2. So if EM and $\vec{x} \in M$, then since $M \vDash \varphi$ it follows from lemma 7 that φ_M^{pt}.

CASE 3: Separation. Suppose EM and $\vec{x}, z \in M$. Since M satisfies second-order Separation, there is some $x \in M$ such that:

$$\forall y \in M(\langle y, x \rangle \in M \leftrightarrow \langle y, z \rangle \in M \wedge \varphi_M^{pt}) \tag{2.6}$$

It is easy to see that (6) is stable. It follows from A7, the definition of end-extension, and lemma 8 that:

$$\Box \forall M' \sqsupseteq M \forall y \in M'(\langle y, x \rangle \in M' \leftrightarrow \langle y, z \rangle \in M' \wedge \varphi_{M'}^{pt}) \tag{2.7}$$

CASE 4: $\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)$. Suppose EM and $x \in M$. By EP, corollary 1, and A7 $\Diamond \exists M' \sqsupseteq M[M' \vDash \exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)]$. Since "$\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)$" is Σ_2, it follows from lemma 7 that $\Diamond \exists M' \sqsupseteq M[\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)]_{M'}^{pt}$. Lemma 8 and the stability of "$[\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)]_M^{pt}$" then yield $[\exists y(\text{trans}(y) \wedge ZFC2^y \wedge x \in y)]_M^{pt}$, as required.

To finish the proof, we need to show that our translation is preserved by MP and GEN. The case for MP is trivial. So suppose $EM \wedge \vec{x}, y \in M \to \varphi_M^{pt}$ is provable in MSST. It follows by GEN and NEC that:

$$\Box \forall M' \sqsupseteq M(\vec{x} \in M' \to \forall y \in M' \varphi_{M'}^{pt}) \tag{2.8}$$

By A7, we have $\vec{x} \in M \to \Box \forall M' \sqsupseteq M(\vec{x} \in M')$. Together with (8) that yields:

$$EM \wedge \vec{x} \in M \to (\forall y \varphi)_M^{pt} \tag{2.9}$$

$\square$

To establish a converse of theorem 5, we need to find an interpretation of MSST in Z*. As the next theorem shows, the most obvious translation from $\mathcal{L}_\lozenge$ to $\mathcal{L}_\in$, which takes possible worlds to be ranks V_λ, plural quantification over V_λ to be first-order quantification over $V_{\lambda+1}$, and pairs $\langle x, y \rangle$ to be set-theoretic ordered pairs, is just such an interpretation.

Definition 13. Let Z* be definitionally expanded with the standard axioms for $\langle , \rangle$ and a new stock of first-order variables $\alpha, y_0, ..., y_n, ...$ Then, let ^{tr} be the following translation from $\mathcal{L}_\lozenge$ to the expanded language.[28]

- $\tau^{tr} = \tau$

- $Y_n^{tr} = y_n$

- ^{tr} commutes with the atomic predicates and the connectives.

- $(\exists x \varphi)^{tr} = \exists x \in V_\alpha \varphi^{tr}$

- $(\exists Y_n \varphi)^{tr} = \exists y_n \subseteq V_\alpha \varphi^{tr}$

- $(\lozenge \varphi)^{tr} = \exists \alpha (lim(\alpha) \wedge \varphi^{tr})$

It is straightforward but tedious to verify that Z* proves $lim(\alpha) \to \varphi^{tr}$ whenever φ is an instance of a truth-functional tautology, an S5 axiom, or one of A1-15.[29] It is also easy to see using GEN that Z* proves $lim(\alpha) \to (\forall x \varphi)^{tr}$ and $lim(\alpha) \to (\square \varphi)^{tr}$ whenever it proves $lim(\alpha) \to \varphi^{tr}$. To show that Z* interprets MSST, therefore, it suffices to show that Z* proves $lim(\alpha) \to \varphi^{tr}$ when φ is EP, E, or an instance of S.

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[28]This translation closely follows Linnebo (2013, p. 20).

[29]The assumption that α is a limit is used for the pairing axioms.

To prove the translations of these axioms we need to know how the crucial notions on which they rely – namely, $(Y \text{ is a structure})^{tr}$, $(Y \vDash \varphi)^{tr}$, $(Y \sqsubseteq Y')^{tr}$, and $(\varphi_Y^{pt})^{tr}$ – behave in Z^*. Luckily, it turns out that the first three are equivalent to the usual notions of structure, satisfaction, and end-extension in $\mathcal{L}_\in$. To see this, first note that a simple induction establishes:

Lemma 9 (Z^*). *Suppose* $lim(\alpha)$, $\vec{y} \subseteq V_\alpha$, *and* $\vec{x} \in V_\alpha$. *Then:*

$$\varphi^{tr} \leftrightarrow \varphi^*$$

where $\varphi \in \mathcal{L}_\Diamond$ *with variables among* $\vec{Y}, \vec{x}$ *is bounded and does not contain* $\Diamond$, *and where* φ^* *is the result of replacing all second-order variables* X *in* φ *with* X^{tr}.

Now, it is easy to see that $(Y \text{ is a structure})^*$, $(Y \vDash \varphi)^*$, and $(Y \sqsubseteq Y')^*$ are, more or less, the usual notions of structure, satisfaction, and end-extension in $\mathcal{L}_\in$.[30] So since "Y is a structure", $Y \vDash \varphi$, and $Y \sqsubseteq Y'$ are bounded and do not contain $\Diamond$, it follows from lemma 9 that if $lim(\alpha)$ and $y, y' \subseteq V_\alpha$, then:

$$(Y \text{ is a structure})^{tr} \leftrightarrow y \text{ is a structure} \tag{2.10}$$

$$(Y \vDash \varphi)^{tr} \leftrightarrow y \vDash \varphi \tag{2.11}$$

$$(Y \sqsubseteq Y')^{tr} \leftrightarrow y \sqsubseteq y'^{[31]} \tag{2.12}$$

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[30]For simplicity, I will use the same notation introduced in section 3.2.3 for these notions, including metavariables M, M', M'' etc. for structures satisfying ZFC2.

[31]In accordance with the convention established in section 3.2.3, y and y' stand in for pairs of sets x, y and x', y' where x, x' act as domains and y, y' are subsets of $x \times x$ and $x' \times x'$ respectively.

Given (10), (11), and (12) it becomes clear that Z^* proves EP^{tr} and E^{tr}. Proving the translations of instances of S is trickier. But we can start by showing that $(\varphi_Y^{pt})^{tr}$ is also equivalent to a simpler notion in $\mathcal{L}_\in$.

Definition 14. For any $\varphi \in \mathcal{L}_\in^2$, let φ_y^p be the result of deleting the modal operators in φ_Y^{pt} and replacing each variable $\mathbf{x}$ with $\mathbf{x}^{tr}$.

φ_y^p is just a non-modal structural version of ms-translation. It is what becomes of ms-translation when we consider sets coding ZFC2 structures instead of possible pluralities coding ZFC2 structures. For example, $(\forall z \exists w (z \in w))^p$ says that for any sets x and $y \subseteq x \times x$ which satisfy ZFC2 and any $z \in x$, there are sets x' and $y' \subseteq x' \times x'$ end-extending x, y which satisfy ZFC2 and are such that there is $w \in x'$ with $\langle z, w \rangle \in y'$. In the notation we have adopted for these $\mathcal{L}_\in$ notions, $(\forall z \exists w (z \in w))^p$ is just:

$$\forall M \forall z \in M \exists M' \sqsupseteq M \exists w \in M'(M' \vDash z \in w)$$

The next lemma shows that this simple non-modal structural version of ms-translation is equivalent to the ^{tr}-translation of the modal structural version.

Lemma 10 (Z^*). $(\varphi_Y^{pt})^{tr} \leftrightarrow \varphi_y^p$

Proof. By induction on the complexity of φ. The only difficult case is that for $\exists \mathbf{x}$, which follows easily from lemma 9. $\square$

We now use this simplification of $(\varphi_Y^{pt})^{tr}$ to prove (lemma 12) the ^{tr}-translation of lemma 8.

Lemma 11 (Z^*). *Let j be an isomorphism between M and M', with $\vec{y} \subseteq M$ and $\vec{x} \in M$. Then:*

$$\varphi^p_M(\vec{y}, \vec{x}) \leftrightarrow \varphi^p_{M'}(j[\vec{y}], j(\vec{x}))$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{Y}, \vec{x}$.

Proof. By induction on the complexity of φ. The only difficult case is that for $\exists \mathbf{x}$. For the left-to-right direction, suppose there is some $M'' \sqsupseteq M$ and $\mathbf{z} \in M''$ such that $\varphi^p_{M''}(\mathbf{z}, \vec{y}, \vec{x})$. We can use M'' to build an end-extension M''' of M' and an isomorphism $i \supseteq j$ between M'' and M'''. It follows from the induction hypothesis that $\varphi^p_{M'''}(i(\mathbf{z}), j[\vec{y}], j(\vec{x}))$ and thus that $(\exists \mathbf{z}\varphi)^p_{M'}(j[\vec{y}], j(\vec{x}))$. The right-to-left direction is similar. $\qquad\square$

Lemma 12 (Z*). *Suppose $M \sqsubseteq M'$, $\vec{y} \subseteq M$, and $\vec{x} \in M$. Then:*

$$\varphi^p_M \leftrightarrow \varphi^p_{M'}$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{Y}, \vec{x}$.

Proof. By induction on the complexity of φ. The only difficult case is the left-to-right direction for $\exists \mathbf{x}$. So, suppose there is some $M'' \sqsupseteq M$ and $\mathbf{z} \in M''$ such that $\varphi^p_{M''}(\mathbf{z}, \vec{y}, \vec{x})$. By quasi-categoricity, M'' is isomorphic to some M''' such that either $M''' \sqsubseteq M'$ or $M''' \sqsupseteq M'$. Let j be such an isomorphism. By lemma 11, $\varphi^p_{M'''}(j(\mathbf{z}), j[\vec{y}], j(\vec{x}))$. If $M''' \sqsubseteq M'$, then by the induction hypothesis $\varphi^p_{M'}(j(\mathbf{z}), j[\vec{y}], j(\vec{x}))$ and thus $(\exists \mathbf{z}\varphi)^p_{M'}(j[\vec{y}], j(\vec{x}))$. If $M''' \sqsupseteq M'$, then $(\exists \mathbf{z}\varphi)^p_{M'}(j[\vec{y}], j(\vec{x}))$ by definition of $^p_{M'}$. Now, $M \sqsubseteq M''$ and it is also easy to see that since $M \sqsubseteq M'$, $M \sqsubseteq M'''$. A simple induction on $\in$ in M then establishes that $j \restriction M = id \restriction M$. Thus since $\vec{y} \subseteq M$ and $\vec{x} \in M$, $(\exists \mathbf{z}\varphi)^p_{M'}(\vec{y}, \vec{x})$. $\qquad\square$

Theorem 6. *Z^* interprets MSST via ^{tr}.*

Proof. Given the preceding remarks, it suffices to show that Z^* proves $[\forall \vec{\mathbf{x}}(\exists \mathbf{z}\varphi \to$
$\forall w \exists \mathbf{z}(w = w \land \varphi))]^p$ for each $\varphi \in \mathcal{L}^2_\in$. So, working in Z^*, suppose $\vec{y} \subseteq M$, $\vec{x} \in M$,
and $(\exists \mathbf{z}\varphi)^p_M$ (where $\exists \mathbf{z}\varphi$'s free variables are among $\vec{Y}, \vec{x}$). Either there are free
variables in φ other than $\mathbf{z}$ or not. If there are, it follows immediately from lemma
12 that $[\forall w \exists \mathbf{z}(w = w \land \varphi)]^p_M$. If not, $(\exists \mathbf{z}\varphi)^p_M = (\exists \mathbf{z}\varphi)^p = \exists M \exists \mathbf{z} \in M \varphi^p_M$. So
suppose there is some M and $\mathbf{z} \in M$ such that φ^p_M. Let M' be arbitrary. By
quasi-categoricity, M' is isomorphic to some M'' such that either $M'' \sqsubseteq M$ or
$M'' \sqsupseteq M$. If $M'' \sqsubseteq M$, then we can build an end-extension, M''', of M' isomorphic
to M. It will follow from lemma 11 that $\exists \mathbf{z} \in M''' \varphi^p_{M'''}$. If $M'' \sqsupseteq M$, then lemmas
12 and 11 imply $\exists \mathbf{z} \in M' \varphi^p_{M'}$. In either case we have $\exists M''' \sqsupseteq M' \exists \mathbf{z} \in M''' \varphi^p_{M'''}$
and since M' was arbitrary, $[\forall w \exists \mathbf{z}(w = w \land \varphi)]^p$. $\square$

Theorems 1 and 2 already tell us a good deal about MSST's strength. For example,
theorem 1 tells us that MSST is at least as strong as Z^*. Indeed, with some effort
we can use them to provide a complete characterisation of MSST's strength. It
will be easier, however, to do this using the next theorem. By establishing that
ms-translation is a faithful interpretation of Z^* in MSST, it will allow us to read
off MSST's strength directly from the more familiar Z^*. For example, it is easy to
see that Z^* proves that there is an unbounded class of inaccessibles (since it thinks
that every set is contained in some inaccessible rank) but not that κ_ω exists (since
V_{κ_ω} models Z^*). So it will follow trivially from the theorem that MSST proves the
ms-translation of the former but not the ms-translation of the latter. Similarly,
lemmas 14 and 15 below show that Z^* proves all instances of Π_0-Col but not all
instances of Π_1-Col. So it will follow trivially from the theorem that MSST proves
the ms-translations of all instances of the former but not the ms-translations of

all instances of the latter.[32] The theorem will therefore allow us to conclude easily that MSST is significantly weaker than $Z2 + R_2$.

It is important to note that the interest of a faithful interpretation of Z^* in MSST is not limited to its providing a characterisation of MSST's strength. It also allows us to read off MSST's set-theoretic content from Z^* *in general.* For example, suppose we were wondering whether MSST proves the ms-translation of the continuum hypothesis (CH) or its negation. Perhaps, by moving to the modal structural setting, we could settle CH one way or the other. Since CH is independent of all known large cardinal hypotheses, however, MSST's strength tells us very little about whether it does. Nonetheless, a faithful interpretation of Z^* in MSST will allow us to use bog-standard inner and outer model theory on Z^* to show that, as we might have expected, the ms-translation of CH is independent of MSST.

Lemma 13 (Z^*). *If $\vec{x} \in V_\alpha$, then:*

$$\varphi \leftrightarrow \varphi^p_{V_\alpha}$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$.

Proof. By induction on the complexity of φ. The only difficult case is that for $\exists x$. For the left-to-right direction, suppose there is some y such that $\varphi(y, \vec{x})$ where $\vec{x} \in V_\alpha$. Then $y \in V_\kappa$, for some inaccessible $\kappa \geq \alpha$. By the induction hypothesis, $\varphi^p_{V_\kappa}(y, \vec{x})$ and thus $(\exists y \varphi)^p_{V_\alpha}(\vec{x})$. For the right-to-left direction, suppose there is some $M \sqsupseteq V_\alpha$ and $y \in M$ such that $\varphi^p_M(y, \vec{x})$. By quasi-categoricity, M is isomorphic to an inaccessible rank V_κ, with $\kappa \geq \alpha$. Let j be such an isomorphism. By lemma 11

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[32]Since Σ_{n+1}-Col is equivalent to Π_n-Col in Z^* (see Devlin (1984), lemma 11.3), this result is optimal.

and the induction hypothesis, $\exists y \varphi(y, j(\vec{x}))$. Since $V_\alpha \sqsubseteq M$ and $V_\alpha \sqsubseteq V_\kappa$, a simple induction establishes that $j \upharpoonright V_\alpha = id \upharpoonright V_\alpha$. Thus $\exists y \varphi(y, \vec{x})$. $\square$

Theorem 7. *MSST faithfully interprets Z* via ms-translation.*[33]

Proof. Trivial from theorems 5 and 6 and lemmas 10 and 13. $\square$

Lemma 14. $Z^* \vdash \Pi_0\text{-}Col.$

Proof. First, note that since ZFC proves that Σ_1 formulas are absolute for inaccessible ranks, and Z* thinks there are unbounded ranks satisfying ZFC, Z* will also prove that Σ_1 formulas are absolute for inaccessible ranks. Now, working in Z* suppose that $\forall x \exists y \varphi(x, y, \vec{z})$, where φ is Π_0. Let κ be an inaccessible such that $u, \vec{z} \in V_\kappa$. Then, $\forall x \in u \exists y \varphi(x, y, \vec{z})$ and thus since "$\exists y \varphi(x, y, \vec{z})$" is absolute for V_κ, $\forall x \in u \exists y \in V_\kappa \varphi(x, y, \vec{z})$. $\square$

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[33]It is worth noting that ^{tr} is not a faithful interpretation of MSST in Z*. To see this, consider the claim that any two objects can co-exist – that is, $\Box \forall x \Box \forall y \Diamond (Ex \wedge Ey)$. Call this Compossible. It is easy to see that Compossible^{tr} is (trivially) provable in Z*. But it is also straightforward to construct models of MSST where Compossible is false. For instance, we could modify the Kripke model in footnote 24 so that its set of worlds is $\{\langle \alpha, n \rangle : \alpha < \omega \wedge (n = 0 \vee n = 1)\}$ and then use analogues of lemmas 10-12 and theorem 6 to establish S.

This example also helps to explain where a natural strategy for proving S from the other axioms of MSST breaks down. In particular, by inspecting the proofs of lemmas 10-12 and theorem 6, we can see that S could be proved from the other axioms *if* they proved a version of the quasi-categoricity theorem for arbitrary ZFC2 structures. But in the presence of EP* such a theorem will imply Compossible. To see this, let x and y be two possible objects. By EP*, $\Diamond \exists M (x \in M)$ and $\Diamond \exists M'(y \in M')$, and by rearranging M and M', we can assume without loss of generality that x is the empty set in M (i.e. $x = 0^M$) and y is the empty set in M' (i.e. $y = 0^{M'}$). Now, if the quasi-categoricity theorem held for M and M', there would be a possible function j which mapped x to y. Since functions are just pluralities of pairs, that means there would be a possible plurality X such that $\langle x, y \rangle \in X$. But, clearly, if EX and $\langle x, y \rangle \in X$, then $Ex \wedge Ey$ (by A12 and A15).

Conversely, the claim that any two possible ZFC2 structures can co-exist – that is, $\Box \forall M \Box \forall M' \Diamond (EM \wedge EM')$ (S-Compossible) – will imply a version of the quasi-categoricity theorem for arbitrary ZFC2 structures. For if M and M' co-exist in a world w, the non-modal quasi-categoricity theorem can be used in w to get the required isomorphism. Whether Compossible or S-Compossible are compatible with modal structuralism will depend on the interpretation of $\Diamond$. For example, on a metaphysical interpretation it is plausible to think that there could have been objects x and y such that $\neg \Diamond (Ex \wedge Ey)$ contradicting Compossible and thus, by EP*, structures M and M' such that $\neg \Diamond (EM \wedge EM')$ contradicting S-Compossible. See Williamson (2010) for discussion.

Lemma 15. $Z^* \not\vdash \Pi_1\text{-}Col$

Proof. Working in $Z^* + \Pi_1$-Col, we show that κ_ω exists. First, note that for any n there are at least n inaccessibles. In particular, for any x there is an f such that if $x \in \omega$, then (i) f is a function; (ii) $dom(f) = x + 1$; (iii) $\forall i \leq x In(f(i))$; and (iv) $\forall i < x(f(i) < f(i+1))$. Since (i-iv) are all at most $\Pi_1{}^{34}$ and "$x \in \omega$" is Δ_0, we can apply Π_1-Col to get a V_α which contains such a function for each n. Thus $\alpha \geq \kappa_\omega$. $\qquad\square$

2.4 Modal structural reflection

In section 2 I raised two prima facie problems for motivating R_2-like principles in the modal structural setting. The first was that since the ms-translation of R_2 is inconsistent with EP, it is unclear whether the modal structuralist can state an R_2-like principle consistently. The second was that since they do not recognise an absolutely infinite universe of sets, it is unclear whether they can motivate an R_2-like principle in terms of indescribability. In this section I consider a suggestion by Geoffrey Hellman (forthcoming) which seems to overcome these problems whilst yielding a principle similar in strength to R_2. Unfortunately, Hellman's principle is inconsistent (theorem 8). In the next section I argue that Hellman's suggestion nonetheless contains the most promising way to implement the indescribability motivation for R_2 in the modal structural setting. The central question is then taken to turn on whether this new motivation implies a principle as strong as R_2.

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[34]In particular, "$In(x)$" can be written as "x is an ordinal $\wedge \forall f \forall y < x[(f$ is a function $\wedge \forall z \in dom(f)(z \subseteq y) \wedge rng(f) \subseteq x) \rightarrow \exists w < x(rng(f) \subseteq w)]$" and the bounded quantifier $\forall i \leq x$ can then be absorbed given Π_1-Col. That bounded quantifiers can be absorbed into Π_n and Σ_n formulas given Π_n-Col follows from the proof of lemma 11.6 in Devlin (1984).

I show that it does not (theorem 11).

Hellman's suggestion has two parts, each providing a response to one of the above problems. The first part is summed up in the following quote.

> The mathematical possibilities of ever larger structures are so vast as to be "indescribable": whatever condition we attempt to lay down to characterize that vastness fails in the following sense: if indeed it is accurate regarding the possibilities of mathematical structures, it is also accurate regarding a mere segment of them, where such a segment can be taken as the domain of a single structure. (p. 10, forthcoming)

There are two ideas here. First, there is an analogue of the indescribability motivation for R_2. Heuristically, we can think of the analogue of V as the collection $\mathcal{S}$ of all possible ZFC2 structures. Then the thought is that since $\mathcal{S}$ is absolutely infinite, it is indescribable – whatever is true in $\mathcal{S}$ is also true in some mere segment of ZFC2 structures $\mathcal{X}$. If we gloss "φ is true in $\mathcal{S}$" as φ^{pt} and "φ is true in $\mathcal{X}$" as $(\varphi^{pt})^{\mathcal{X}}$ – where $(\varphi^{pt})^{\mathcal{X}}$ is the result of replacing all occurrences of $\exists M$ in φ^{pt} with $\exists M \in \mathcal{X}$ – then we can formulate the claim that $\mathcal{S}$ is indescribable (in $\mathcal{L}_{\Diamond}$) as:

$$(\text{IR}^{\Diamond}) \qquad\qquad \varphi^{pt} \to \exists \mathcal{X}(\varphi^{pt})^{\mathcal{X}}$$

for sentences $\varphi \in \mathcal{L}_{\in}^2$. The "I" here stands for "informal", since the principle contains the heuristic variable $\mathcal{X}$ ranging over mere segments of ZFC2 structures. Eventually we will want to eliminate $\mathcal{X}$ in favour of a notion expressible in $\mathcal{L}_{\Diamond}$, but it is clear enough for present purposes.

The second idea in the quote is that a mere segment of ZFC2 structures $\mathcal{X}$ "can be taken as the domain of a single [ZFC2] structure". The thought is that as far as the truth of φ is concerned, a mere segment of ZFC2 structures is indistinguishable from some single ZFC2 structure. More precisely:

(Identification) $$(\varphi^{pt})^{\mathcal{X}} \rightarrow \Diamond \exists M(M \vDash \varphi)$$

Although these two ideas seem to provide an attractive response to the second problem, they run straight into the first. In particular, $\mathrm{IR}^{\Diamond}$ and Identification jointly entail the inconsistent:

(MSR) $$\varphi^{pt} \rightarrow \Diamond \exists M(M \vDash \varphi)$$

To see that MSR is inconsistent, it suffices to consider the claim that every plurality fails to contain something, i.e.:

(F) $$\forall X \exists x (x \notin X)$$

Trivially, F can't be true in any single ZFC2 structure M, since $X = dom(M)$ would provide a counterexample. But as the next lemma shows, the ms-translation of F is equivalent to EP. So F is true in $\mathcal{S}$, but not in any single ZFC2 structure, contradicting MSR. Similarly, since F can't be true in any V_α in any ZFC2 structure, the lemma also shows that R_2^{pt} is inconsistent.

Lemma 16 (MSST - EP). *EP is equivalent to* F^{pt}.

Proof. For the left-to-right direction, suppose $X \subseteq M$. From EP it follows that $\lozenge \exists M'(M' \sqsupset M)$ and so $\lozenge \exists M' \sqsupseteq M \exists x \in M'(x \notin X)$. For the right-to-left direction, simply take $X = dom(M)$.[35] $\square$

The second part of Hellman's suggestion tries to avoid inconsistency by restricting MSR "explicitly to sentences $[\varphi]$ that are consistent with the accepted set theory, [ZFC2]" (forthcoming, p.10). Since φ can be second order, this restriction bifurcates. For second-order languages we have a notion of semantic consistency and a notion of syntactic consistency.[36] Each notion yields a corresponding restriction of MSR. In particular, we have:

(MSR-sem)   If φ is semantically consistent with ZFC2, i.e. $\lozenge \exists M(M \vDash \varphi)$, then:

$$\varphi^{pt} \to \lozenge \exists M(M \vDash \varphi)$$

and:

(MSR-syn) If φ is syntactically consistent with ZFC2, i.e. $(ZFC2 \nvdash \neg\varphi)^{pt}$, then:

$$\varphi^{pt} \to \lozenge \exists M(M \vDash \varphi)$$

MSR-sem is trivially true and therefore cannot be what Hellman has in mind. But as the next theorem shows, MSR-syn is inconsistent. The second part of Hellman's suggestion thus fails to solve the first problem.

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[35]Since $\neg F$ is provable in ZFC2, the lemma also shows that ms-translation cannot be an interpretation of ZFC2.

[36]For any set of sentences Γ, we say that Γ is *semantically consistent* if it has a (possible) full second-order model, and we say that Γ is *syntactically consistent* if no contradiction can be derived from it in second-order logic.

Theorem 8. *MSST + MSR-syn is inconsistent.*

Proof. Let R be a Rosser sentence for ZFC2 (formulated so as to be Σ_1). Since Z^* proves that ZFC2 is consistent, it will prove $ZFC2 \nvdash \neg(R \to F)$.[37] From theorem 3 it follows that MSST proves $(ZFC2 \nvdash \neg(R \to F))^{pt}$. Now, working in MSST, MSR-syn implies:

$$(R \to F)^{pt} \to \Diamond \exists M(M \vDash R \to F) \tag{2.13}$$

Since F^{pt} is true and ^{pt} commutes with $\to$, the antecedent of (13) is true by lemma 16. It follows that $\Diamond \exists M(M \vDash \neg R)$. Since R is Σ_1, lemma 7 implies that $\neg R^{pt}$. We can then run the preceding argument with $\neg R$ replacing R to get R^{pt}. Thus, $R^{pt} \wedge \neg R^{pt}$. $\qquad\square$

2.5 Saving MSR from inconsistency?

It might be tempting at this point to look for other restrictions on MSR; restrictions which restore consistency while retaining as much of its strength as possible. But merely doing this would be a mistake. Any such restriction should be well-motivated. In particular, it should come equipped with an explanation of why MSR holds for the restricted class of cases and not for others. The real problem with MSR-syn, for example, is not inconsistency but rather that there seems to be no such explanation. It appears completely *ad hoc*. In general, it is unclear whether there are *any* restrictions of MSR which are consistent, strong, and well-motivated. That leaves us with Identification and IR$^{\Diamond}$.

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[37]Indeed, it won't prove $\neg(R \to \varphi)$ for any φ.

The arguments of section 4 can be modified to show that Identification is inconsistent with MSST and that it remains inconsistent under Hellman's restriction.[38] As with MSR, it is unclear whether there are any restrictions on Identification which are consistent, strong, and well-motivated. That leaves us with $\mathrm{IR}^\lozenge$. As I mentioned in the previous section, $\mathrm{IR}^\lozenge$ is a formulation of the claim that $\mathcal{S}$ is indescribable. The thought that since $\mathcal{S}$ is absolutely infinite, it is indescribable in this sense strikes me as an attractive way to transpose the indescribability motivation for R_2 into the modal structural setting. There is also reason to think it is the strongest, consistent, way to do this.

I will assume that the indescribability motivation has the general form: since $\mathcal{C}$ is absolutely infinite, it is indescribable – whatever is true in $\mathcal{C}$ is also true in some initial segment of $\mathcal{C}$ (where $\mathcal{C}$ is a collection of pluralities – like $\mathcal{S}$ – or of objects). I will also assume that initial segments of $\mathcal{C}$ are subcollections of $\mathcal{C}$ which are suitably small. As a measure of smallness, I will use possible co-existence. This is natural because whenever some pluralities or objects can co-exist, we can 'union' them into a plurality X which, by EP*, will be strictly smaller than some other possible plurality.[39] Now, in the modal structural setting there are two salient absolutely infinite collections. Heuristically, we can think of the first as the collection $\mathcal{P}$ of all possible objects. A sentence is true in $\mathcal{P}$ if it is (i) true simplicitier, and (ii) its quantifiers are all of the form $\lozenge \exists \mathbf{x}$. More precisely, let $\mathcal{L} = \mathcal{L}_\lozenge - \{\lozenge\}$. A sentence

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[38] *Proof sketch:* Let $\mathcal{X}$ be any segment of ZFC2 structures without a maximal element by end-extension. Then F will be true in $\mathcal{X}$ by a modification of the argument for lemma 16, and we can run the argument for theorem 8 with Identification replacing MSR-syn, noting that $((R \to F)^{pt})^{\mathcal{X}}$ holds.

[39] This is a relatively strict measure of smallness, since it deems not small pairs of objects which can't co-exist. We could liberalise it by allowing a collection to count as small if it is equinumerous with a small collection in my sense. A pair of objects which can't co-exist would then count as small because it could be put in one-one correspondence with a pair of objects which can. Nothing I say will depend on using the stricter notion.

$\varphi \in \mathcal{L}$ is true in $\mathcal{P}$ if the result, $\varphi^{\mathcal{P}}$, of replacing all occurrences of $\exists \mathbf{x}$ in φ with $\Diamond \exists \mathbf{x}$ is true. The claim that $\mathcal{P}$ is indescribable (in $\mathcal{L}_{\Diamond}$) then becomes:

$$(\text{Ind-}\mathcal{P}) \qquad\qquad \varphi^{\mathcal{P}} \rightarrow \Diamond \exists X \varphi^{X}$$

for sentences $\varphi \in \mathcal{L}$. Unfortunately, for reasons similar to those in case of MSR, Ind-$\mathcal{P}$ is inconsistent. In particular, by EP*, any possible plurality possibly fails to contain something; that is, $\text{F}^{\mathcal{P}}$. But F^{Y} can't be true for any possible plurality Y, since $X = Y$ would provide a counterexample.[40]

The second salient absolutely infinite collection is $\mathcal{S}$. In the previous section I glossed "φ is true in $\mathcal{S}$" as φ^{pt} for $\varphi \in \mathcal{L}_{\in}^{2}$. But we might want to be more liberal. In particular, we might want to take a sentence to be true in $\mathcal{S}$ if (i) it is true simpliciter, (ii) its plural quantifiers range over possible ZFC2 structures and subpluralities thereof, and (iii) its first-order quantifiers range over the elements of those structures. More precisely, suppose $\varphi \in \mathcal{L}$ is a sentence such that its quantifiers are all of the form $\exists M$ or $\exists \mathbf{x} \in M$. Then φ is true in $\mathcal{S}$ if the result, $\varphi^{\mathcal{S}}$, of replacing all occurrences of $\exists M$ in φ with $\Diamond \exists M$ is true. The claim that $\mathcal{S}$ is indescribable (in $\mathcal{L}_{\Diamond}$) then becomes:

$$(\text{Ind-}\mathcal{S}) \qquad\qquad \varphi^{\mathcal{S}} \rightarrow \exists \mathcal{X} \varphi^{\mathcal{X}}$$

for all such φ. It is easy to check that φ^{pt} is of the form $\psi^{\mathcal{S}}$ and thus that Ind-$\mathcal{S}$ is a generalisation of IR$^{\Diamond}$. Nonetheless, Ind-$\mathcal{S}$ is not stronger than IR$^{\Diamond}$. In particular,

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[40]It is worth noting that Ind-$\mathcal{P}$ is not inconsistent with EP (or even MSST), since it holds in the Kripke model in footnote 22. But, as I mentioned in section 3.2.3, the modal structuralist is committed to the stronger EP* and thus to the inconsistency of Ind-$\mathcal{P}$.

if we formalise Ind-$\mathcal{S}$ along the lines of $R^\Diamond$ below, theorems 9, 10, and 11 can be modified to show that ms-translation is a faithful interpretation of $Z^* + R_1$ in MSST + Ind-$\mathcal{P}$.

Of the two salient ways of implementing indescribability in the modal structural setting, then, one is inconsistent and the other is as strong as $IR^\Diamond$. So it is natural to take the central question to turn on $IR^\Diamond$. If $IR^\Diamond$ is weaker than R_2, that is good evidence that the modal structuralist can't adopt a consistent principle similar in motivation and strength to R_2. In the remainder of the chapter, I will show that $IR^\Diamond$ is indeed weaker than R_2. In particular, once $IR^\Diamond$ is formalised in $\mathcal{L}_\Diamond$ as a principle I call $R^\Diamond$, I will show that ms-translation is a faithful interpretation of $Z^* + R_1$ in MSST + $R^\Diamond$ (theorem 11). In other words, MSST + $R^\Diamond$ and $Z^* + R_1$ have exactly the same (first-order) set-theoretic content. It follows that MSST + $R^\Diamond$ proves the ms-translation of the claim that κ_α exists for all α (since the existence of κ_λ follows from the existence of κ_α for $\alpha < \lambda$ by R_1) but not the ms-translation of the claim that there is an inaccessible limit of inaccessibles (since V_κ will model $Z^* + R_1$ when κ is such a cardinal). It also follows that it proves the ms-translations of the theorems of $Z^* + \Pi_1$-Col but not the ms-translations of all instances of Π_2-Col (lemmas 17 and 18). So $R^\Diamond$ affords only a minor increase in strength over MSST and is thus significantly weaker than R_2.

2.5.1 $IR^\Diamond$ to $R^\Diamond$

Formalising $IR^\Diamond$ in $\mathcal{L}_\Diamond$ is just a matter of formalising $\mathcal{X}$ in $\mathcal{L}_\Diamond$. What constraints should we impose on such a formalisation? Above, I assumed that initial segments of $\mathcal{S}$ are collections of ZFC2 structures whose elements can all co-exist. And on

analogy with the transitivity of initial segments of V, we might also want to assume that initial segments of $\mathcal{S}$ are downward closed under end-extension (i.e. whenever $M \in \mathcal{X}$ and $M' \sqsubseteq M$, $M' \in \mathcal{X}$). It is hard to see what else we should assume about initial segments of $\mathcal{S}$. So I will take a *suitable* formalisation of $\mathcal{X}$ in $\mathcal{L}_\Diamond$ to be any notion expressible in $\mathcal{L}_\Diamond$ that collects together ZFC2 structures which can co-exist and which are downward closed.

Clearly, there will be many suitable formalisations of $\mathcal{X}$ in $\mathcal{L}_\Diamond$ and thus many suitable formalsiations of IR$^\Diamond$. Since none of them are privileged, each places an upper bound of the strength of IR$^\Diamond$.[41] In what follows, I will focus on one. The idea is simple. We take $\mathcal{X}$ to be interpreted as a set of ZFC2 structures. Specifically, $\mathcal{X}$ will be interpreted as a set of ZFC2 structures in some V_α. Relative to this interpretation, IR$^\Diamond$ will say that whatever is true in $\mathcal{S}$ it is also true in the set of ZFC2 structures of some V_α. Recalling definition 14, we can state that more precisely as:

$$\varphi^{pt} \to \Diamond \exists M \exists \alpha \in M (M \vDash (\varphi^p)^{V_\alpha}) \tag{2.14}$$

for sentences $\varphi \in \mathcal{L}^2_\in$.[42]

The principles we have considered so far – like IR$^\Diamond$, Ind-$\mathcal{P}$, Ind-$\mathcal{S}$, and (14) – have only been stated for sentences. It is well-known, however, that sentential versions of reflection can be weaker than those with parameters.[43] Since indescribability seems to imply the parameterised versions of these principles, we should

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[41]If we like, we could think of *the* formalisation of IR$^\Diamond$ in $\mathcal{L}_\Diamond$ as an infinite disjunction of schemas, one for each suitable formalisation of IR$^\Diamond$.

[42]Since φ^p will contain variables of the form y_i (which, strictly speaking, are not in $\mathcal{L}_\Diamond$), we assume it has been suitably re-lettered.

[43]See, for instance, Lévy (1960).

liberalise (14) to allow for them. We can do this by relativising (14) to a structure M, requiring that M exists and that the parameters in φ be contained in M. For simplicity, I will assume that the reflecting structure M' is a proper end-extension of M and thus that M and φ's parameters form sets in M'. More precisely, if EM, $\vec{Y} \subseteq M$, and $\vec{x} \in M$, then:

$$(\text{R}^{\Diamond}) \qquad \varphi_M^{pt} \to \Diamond \exists M' \sqsupseteq M \exists \alpha \in M'(M' \vDash (\varphi_{V_\kappa}^p)(\vec{z}, \vec{x})^{V_\alpha})$$

where $dom(M) = V_\kappa^{M'}$ and $\vec{z} =^{M'} \vec{Y}$ and where $\varphi \in \mathcal{L}_{\in}^2$ with free variables among $\vec{Y}, \vec{x}$.

Theorem 9. *MSST + $R^{\Diamond}$ interprets $Z^* + R_1$ via ms-translation.*

Proof. Given the proof of theorem 5, it suffices to show that MSST + $\text{R}^{\Diamond}$ proves:

$$EM \wedge \vec{x} \in M \wedge \varphi_M^{pt} \to (\exists \alpha \varphi^{V_\alpha})_M^{pt} \tag{2.15}$$

where $\varphi \in \mathcal{L}_{\in}$ with free variables among $\vec{x}$. So suppose EM, $\vec{x} \in M$, and φ_M^{pt}. By $\text{R}^{\Diamond}$ applied to $(\varphi \wedge F)_M^{pt}$ we have:

$$\Diamond \exists M' \sqsupseteq M \exists \alpha \in M'(M' \vDash (\varphi_{V_\kappa}^p \wedge F^p)^{V_\alpha}) \tag{2.16}$$

Since $V_\alpha \vDash (F^p)$, α is a limit of inaccessibles, and thus $V_\alpha \vDash Z^*$. So we can use lemma 13 on $\varphi_{V_\kappa}^p$ inside V_α to get that φ^{V_α}. Since "$x = V_y \wedge \varphi^x$" is $\Pi_1^{Z^*}$, lemma 7 and theorem 5 then imply that $(\exists \alpha \varphi^{V_\alpha})_M^{pt}$. $\square$

Theorem 10. *$Z^* + R_1$ interprets MSST + $R^{\Diamond}$ via ^{tr}.*

Proof. Given the proof of theorem 6, it suffices to show that $Z^* + R_1$ proves $lim(\alpha) \to \psi^{\Diamond tr}$ where ψ is an instance of $R^\Diamond$. So suppose that $\vec{y} \subseteq M$, $\vec{x} \in M$, and $\varphi^p_M(\vec{y}, \vec{x})$ (where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{Y}$ and $\vec{x}$). By quasi-categoricity, there is an isomorphism $j : M \to V_\kappa$, for some κ inaccessible. It follows from lemma 11 that $\varphi^p_{V_\kappa}(j[\vec{y}], j(\vec{x}))$ and thus by R_1 $\exists \alpha(\varphi^p_{V_\kappa}(j[\vec{y}], j(\vec{x})))^{V_\alpha}$. Picking an inaccessible $\kappa' > \kappa, \alpha$, it follows that:

$$V_{\kappa'} \vDash \exists \alpha(\varphi^p_{V_\kappa}(j[\vec{y}], j(\vec{x})))^{V_\alpha} \tag{2.17}$$

Now, let $i : V_{\kappa'} \to M' \sqsupset M$ be an isomorphism. Then:

$$M' \vDash \exists \alpha(\varphi^p_{V_{i(\kappa)}}(i(j[\vec{y}]), i(j(\vec{x}))))^{V_\alpha} \tag{2.18}$$

A simple induction on $\in$ in M establishes that $i \upharpoonright V_\kappa = j^{-1}$. It follows that $dom(M) = V^{M'}_{i(\kappa)}$, $i(j[\vec{y}]) =^{M'} \vec{y}$, and $i(j(\vec{x})) = \vec{x}$. Our result then follows from lemmas 9 and 10. $\square$

Theorem 11. *MSST + $R^\Diamond$ faithfully interprets $Z^* + R_1$ via ms-translation.*

Proof. Trivial from theorems 9 and 10 and lemmas 10 and 13. $\square$

Lemma 17. $Z^* + R_1 \vdash \Pi_1$-*Col.*

Proof. Recall from lemma 14 that Z^* proves that Σ_1 formulas are absolute for inaccessible ranks. It follows that Σ_1 formulas are also absolute for ranks V_α where α is a limit of inaccessibles. Now suppose $\forall x \exists y \varphi(x, y, \vec{z})$, where φ is Π_1. By R_1 on $\forall x \exists y \varphi(x, y, \vec{z}) \wedge E\vec{z} \wedge Eu \wedge Z^*$, we get a V_α for which φ is absolute and thus for which $\forall x \in u \exists y \in V_\alpha \varphi(x, y, \vec{z})$. $\square$

Lemma 18. $Z^* + R_1 \nvdash \Pi_2\text{-}Col.$

Proof. Working in $Z^* + \Pi_2$-Col, we build a model of $Z^* + R_1$. Let "$x \vDash y$" be a $\Delta_1^{Z^*}$ satisfaction relation (where x is a model and y a formula/finite variable assignment pair).[44] If we could find a limit of inaccessibles α such that:

$$\forall x \in V_\alpha(\exists\beta(V_\beta \vDash x) \to \exists\beta < \alpha(V_\beta \vDash x)) \tag{2.19}$$

then it would follow that $V_\alpha \vDash Z^* + R_1$ (since any formula and finite variable assignment over V_α will be in V_α).[45] We now show that the existence of such an α is provable in $Z^* + \Pi_2$-Col.

Let "$\exists\beta(V_\beta \vDash x) \to (V_\alpha \vDash x)$" $= \Phi(x, \alpha)$. Since "$x = V_y$" is $\Pi_1^{Z^*}$ and "$x \vDash y$" is $\Delta_1^{Z^*}$, $\Phi(x, \alpha)$ is $\Pi_2^{Z^*}$. Clearly, $\forall x \exists\alpha \Phi(x, \alpha)$ and so by Π_2-Col:

$$\forall\delta\exists\gamma[In(\gamma) \wedge \forall x \in V_\delta \exists\alpha < \gamma \Phi(x, \alpha)] \tag{2.20}$$

Let "$[In(\gamma) \wedge \forall x \in V_\delta \exists\alpha < \gamma \Phi(x, \alpha)]$" $= \Psi(\delta, \gamma)$. Since "$In(x)$" is $\Pi_1^{Z^*}$ and bounded quantifiers can be absorbed into Π_2 formulas (see footnote 36), $\Psi(\delta, \gamma)$ is $\Pi_2^{Z^*}$. From (20) it follows that for every x there is an f such that if $x \in \omega$, then (i) f is a function; (ii) $dom(f) = x + 1$; (iii) $\forall i < x(\Psi(f(i), f(i+1)))$; and (iv) $\forall i < x(f(i) < f(i+1))$. Since (i-iv) are all at most $\Pi_2^{Z^*}$, we can apply Π_2-Col to get a strictly increasing ω sequence f such that $\Psi(f(i), f(i+1))$ and $\cup rng(f)$ will be our required α. $\qquad\square$

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[44]See, for instance, Kunen (2011) Definition I.15.5.

[45]The idea of using such an α to get a model of $Z^* + R_1$ is due to Lévy and Vaught (1961).

2.6 Objections

In this section I consider some objections to my argument. The first two accept its conclusion – i.e. that the modal structuralist can't adopt a consistent principle similar in motivation and strength to R_2 – but question its import. The third rejects its conclusion, arguing that by stating $R^\diamondsuit$ in more expressive language than $\mathcal{L}_\diamondsuit$ we can obtain a principle as strong as R_2.

Objection 1: As Reinhardt and Silver have shown, $Z2 + R_2$ is consistent relative to the existence of the first Erdős cardinal $\kappa(\omega)$ and thus consistent with $V = L$.[46] So R_2 effects an 'insignificant' reduction in incompleteness in Koellner's (2009, p. 208) sense. The cost to the modal structuralist of failing to motivate principles approaching the strength of R_2 is therefore negligible.

Reply: It is of course true that there are many interesting questions which are not settled by R_2. But it would be wrong to conclude that the reduction in incompleteness afforded by R_2 is negligible, since there are also many interesting questions which *are* settled by R_2.[47] If we are primarily concerned with settling open questions in descriptive set theory, for example, then our standard of significance may well coincide with Koellner's. But if we are concerned with settling open questions in set theory more generally, that standard will look overly strict.

Objection 2: Instead of appealing to indescribability, we can motivate reflection

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[46]See Proposition 7.16 in Kanamori (2003).

[47]For a recent sample, see Krueger and Schimmerling (2011) and Hamkins and Woodin (2005) who make use of greatly Mahlo and weakly compact cardinals respectively.

principles by the thought that since the universe of sets is absolutely infinite, it is indistinguishable from its initial segments.[48] This arguably implies the schema of *complete* reflection:

$$(\text{CR}_1) \qquad\qquad \exists \alpha > \beta \forall \vec{x} \in V_\alpha(\varphi \leftrightarrow \varphi^{V_\alpha})$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$. It is well-known that in contrast to R_1, CR_1 does suffice over Z^* to derive all instances of Collection.[49] By applying this motivation to $\mathcal{S}$, the modal structuralist could obtain a version of $\text{R}^\diamond$ which faithfully interprets $\text{Z}^* + \text{CR}_1$. So even though they might not be able to motivate a principle as strong as R_2 using indescribability, they can do better using indistinguishability.

Reply: There are two things to note here. The first is that although this strategy will result in a stronger principle with respect to theorems of ZFC, it will not add any large cardinal strength to $\text{MSST} + \text{R}^\diamond$. To see this, note that whenever κ is an inaccessible limit of inaccessibles, V_κ satisfies $\text{Z}^* + \text{CR}_1$. The second thing to note is that the proposed motivation can arguably be used by the non-modal structuralist to obtain second-order reflection principles which are significantly stronger than R_2. For example, the principle GRP investigated by Welch (ms) is a natural generalisation of CR_1 to the second-order and it implies that the Axiom of Determinacy holds in $L(\mathbb{R})$. But the arguments of section 4 can be adapted to show that GRP^{pt} is inconsistent with the Extendability Principle.[50] By invoking

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[48]See, for instance, Lévy (1960b).

[49]See, for instance, Lévy and Vaught (1961). They call R_1 PR (for *partial* reflection).

[50]This observation also allows for an alternative reply to objection 1. In particular, if we

this new motivation, the modal structuralist might gain some strength in terms of theorems of ZFC but they will make significant losses in terms of reduction in incompleteness.

Objection 3: Instead of interpreting the second-order quantifiers in $\mathcal{L}^2_\in$ using pluralities, we can interpret them using some other suitable entity. For example, we could interpret them using Fregean concepts. As long as these entities do not obey an analogue of the Extendability Principle, there will be no pressure to think that the resulting ms-translation of F holds. Such entities could therefore be used to state consistent versions of R^{pt}_2 and MSR.

Reply: There are two constraints on any such entities which jointly suggest that this and similar strategies will not work for the modal structuralist. The first constraint, as mentioned in section 1, is that they should not be abstracta. The second constraint follows from R^{pt}_2 and MSR. In particular, since all instances of second-order comprehension are true in every possible ZFC2 structure, it will follow from MSR that their ms-translations are true. So, on the one hand, there has to be enough such entities to make the ms-translations of arbitrary instances of second-order comprehension come out true; and, on the other hand, they can't be abstracta. It is unclear whether there is any kind of entity which obeys these two constraints.[51]

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rerun the arguments of this chapter with indistinguishability replacing indescribability and GRP replacing R_2, we will obtain arguments to the effect that the modal structuralist can only motivate reflection principles weaker than CR_1 and thus significantly weaker in Koellner's sense than GRP.

[51]Though see Rayo and Yablo (2001) for a dissenting voice.

Chapter 3

Reinhardt and reflection

In this chapter I investigate some influential reflection principles due to William Reinhardt. I have two main aims. First, I want to provide clear and accessible versions of those principles.[1] Second, I want to see whether the potentialist should adopt them.

Here's the plan. In section 1 I outline a simplified version of the theory developed in Reinhardt (1980). I then use this to shed light on the two central theories of Reinhardt (1974a). In section 2 I provide a precise axiomatisation (RMST) of the simplified theory and determine its strength. I show that it proves exactly the same sentences in the language of second-order set theory as second-order ZF plus the principle of Structural Complete Reflection (SCR) introduced in chapter 1. Finally, in section 3 I look at a number of criticisms of Reinhardt's theories and argue that the potentialist should not adopt them.

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[1]Reinhardt's papers on reflection are notoriously dense and hard to decipher. For example, what Kanamori (2003, p. 314) calls "the most mature and sophisticated formalization of [his] ideas" – namely, the theory in his (1980) – employs a formidable 45 axioms.

3.1 Reinhardt and reflection

Starting with his dissertation (1967), William Reinhardt made a number of proposals for new axioms in set theory. This culminated in his (1974a) and (1980) which focus in particular on reflection principles. In this section I will outline simple versions of the theories developed in these papers. More precisely, in section 1.1 I outline a simplified version of the modal set theory in Reinhardt (1980); in section 1.2 I outline the modal realist theory most authors take Reinhardt (1974a) to be advocating; and in section 1.3 I outline the theory Reinhardt (1974a) actually advocates.

3.1.1 The modal account

The *modal account* takes potentialism as a background assumption. Accordingly, any sets mathematically could have determined a set. In particular, there could have been a set of the actually existing φ's. Formally:[2]

$$(\text{@-\textsc{collapse}}^{\Diamond}) \qquad\qquad \Diamond \exists x \forall y (y \in x \leftrightarrow @Ey \wedge \varphi)$$

As in chapter 1, I will follow Linnebo (2009, 2013) in taking the modality to concern a well-founded process of extending the mathematical ontology. The appropriate modal logic for $\Diamond$ is then S4.2 supplemented with the Converse Barcan Formula.[3]

Against this background, the most distinctive claim of the modal account is a modal reflection principle. It says that the sets contain every possible kind of set:

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[2]@-\textsc{collapse}$^{\Diamond}$ is a version of Reinhardt's axiom I1. See (Reinhardt, 1980, p.298).

[3]See chapter 1 section 3 for discussion and section 2.1 of this chapter for details of the logic of @ and of quantification for the modal account.

if there could have been φ, then there is a φ. Formally:[4]

(R) $$\forall \vec{\mathbf{x}}[\Diamond \exists y \varphi \to \exists y \varphi]$$

where φ's free variables are among $\vec{\mathbf{x}}$. In other words, R says that what there could have been is reflected in what there is.

Already, @-COLLAPSE$^\Diamond$ and R have significant set-theoretic consequences. For example, they imply that Pairing is actually true. To see this, suppose that x and y actually exist. By @-COLLAPSE$^\Diamond$, there could have been a set of them. So R then implies that there is, actually, a set of them. Similarly, @-COLLAPSE$^\Diamond$ and R imply that Union is actually true. To see this, suppose that x actually exists. By @-COLLAPSE$^\Diamond$, there could have been a set y of sets which actually exist and are elements of elements of x; in other worlds, a set of the actually existing sets in the union of x. Now, recall from chapter 1 that sets are *inextensible*; they could not have existed without their elements. Formally:[5]

(INEXT) $$Ex \wedge \Diamond(y \in x) \to Ey$$

So, since x actually exists, its elements actually exist and the elements of its elements also actually exist. Thus, y is just a union of x. R then implies that there is, actually, a union of x. It is a simple exercise to adapt these arguments to show that Infinity is also actually true.[6]

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[4]R is a consequence of Reinhardt's axiom I2, since on his theory there is a proposition corresponding to each φ for which I will take R to hold (see below). See (Reinhardt, 1980, p.298).

[5]INEXT is just Reinhardt's axiom S4. See (Reinhardt, 1980, p.275).

[6]See theorem 12 CASE 1 for details.

However, R faces an immediate problem. It appears to be inconsistent. To see this, first note that @-COLLAPSE$^\Diamond$ implies that there could have been a set of the actually existing non-self-membered sets. Formally:

$$\Diamond \exists x \forall y (y \in x \leftrightarrow @Ey \wedge y \notin y)$$

By the reasoning of Russell's paradox, it follows that there could have been a set which does not actually exist; that is, $\Diamond \exists x \neg @Ex$. But R implies that if there could have been a set which does not actually exist, then there is, actually, a set which does not actually exist. Contradiction.

Given potentialism, then, R does not hold for arbitrary formulas. A proponent of the modal account thus owes us a story about which formulas it does hold for and why. I will return to this challenge in section 3.2, but for now I will just assume that R holds for formulas in the language of second-order set theory, $\mathcal{L}^2_\in$. Recall that $\mathcal{L}^2_\in$ supplements the language of first-order set theory $\mathcal{L}_\in$ with variables $F_0, ..., F_i, ...$ and takes $x \in F$ but not $F = G$ to be well-formed. Following the convention of chapter 1, I will refer to the F's as *concepts* and say that F *applies* to x whenever $x \in F$.

Although neither @-COLLAPSE$^\Diamond$ nor R imply the existence of any possible concepts,[7] R does place substantial constraints on them. For example, it follows from R that the application relation for concepts is *stable* in the following sense.

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[7]*Proof sketch:* Consider a two world Kripke model with $w_0 = \text{dom}(w_0) = V_\alpha$ as the actual world and $w_1 = \text{dom}(w_1) = V_\beta$ where $\alpha < \beta$, w_0 accesses w_1, w_0, w_1 access themselves, and V_α is a first-order elementary substructure of V_β (that is, $\forall \vec{x} \in V_\alpha(V_\alpha \vDash \varphi \leftrightarrow V_\beta \vDash \varphi)$ whenever $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$). Since there are no concepts in the model, every second-order formula is equivalent to a first-order formula and it is then easy to see that R will hold at both worlds. Since $\alpha < \beta$, @-COLLAPSE$^\Diamond$ and the other principles of modal account so far discussed will also hold at both worlds.

Given any concept F and any set x, F will necessarily apply to x or necessarily fail to apply to x. Formally:

$$\forall F \forall x (\Box(x \in F) \vee \Box(x \notin F))$$

To see this, suppose that F could have applied to x and could have failed to apply to x. It would then follow from R that F applies to x and fails to apply to x, which is impossible.

In this respect, concepts are like sets. Recall from chapter 1 that the membership relation for sets is stable in the following sense. Either x necessarily contains y or necessarily fails to contain y. Formally:[8]

$(\text{STAB}_\in)$ $\qquad\qquad\qquad\qquad\qquad\qquad \Box(y \in x) \vee \Box(y \notin x)$

Nonetheless, concepts differ from sets in an important way. In particular, concepts can be extensible; they could have existed without the things they apply to. For example, suppose there is a concept F which actually exists and actually applies to all sets; that is, $@EF$ and $@\forall x(x \in F)$. It follows from R that F necessarily applies to all sets. Then, since $@\text{-}\textsc{collapse}^\Diamond$ implies that there could have been a set which does not actually exist, F could have applied to something which does not actually exist.

But what concepts are there? I will return to this question in section 3.4, but for now I will just assume that there is always at least one concept co-extensive with any condition on the sets. Formally:[9]

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[8]$\textsc{stab}_\in$ is essentially just Reinhardt's axiom S3. See (Reinhardt, 1980, p. 275).

[9]$\textsc{c-comp}$ is essentially just Reinhardt's axiom I3. See (Reinhardt, 1980, p.300).

(C-COMP) $\qquad\qquad\qquad\qquad \exists F \forall x (x \in F \leftrightarrow \varphi)$

So there will always be a concept applying to all sets. As it turns out, C-COMP increases the strength of the modal account considerably. Over STAB$_\in$, STAB$_=$ – i.e. $x \neq y \to \Box(x \neq y)$ – INEXT, Foundation, and Extensionality, any two of @-COLLAPSE$^\Diamond$, R, and C-COMP are consistent relative to ZFC.[10] But together they prove the same sentences of $\mathcal{L}^2_\in$ as second-order ZF plus the principle of Structural Complete Reflection (SCR) (see theorem 13 in section 2). Thus, they imply that there are inaccessible cardinals, measurable cardinals, Woodin cardinals, and that the axiom of determinacy holds in $L(\mathbb{R})$.[11]

And that's it. Over the background potentialism, the modal account comprises just three distinctive claims. First, there is the claim that there could have been a set of the the actually existing φ's. Second, there is a comprehension schema for concepts which says that there is always a concept co-extensive with φ. Finally, there is a modal reflection principle which says that if there could have been a $\varphi \in \mathcal{L}^2_\in$, then there is a φ.

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[10]*Proof sketch:* Any single world Kripke model will model R and C-COMP where the second-order quantifiers range over the powerset of the domain. @-COLLAPSE$^\Diamond$ and C-COMP will hold in any two world Kripke model where $w_0 = \mathrm{dom}(w_0) = V_\alpha$ is the actual world and $w_1 = \mathrm{dom}(w_1) = V_\beta$ with $\alpha < \beta$, the second-order quantifiers ranging over the powersets of the domains, w_0 accessing w_1, and w_0, w_1 accessing themselves. Finally, any finite set of instances of R and @-COLLAPSE$^\Diamond$ can be interpreted in a two world Kripke model like that in footnote 7 where V_α is an elementary substructure of V_β for the corresponding finite set of first-order formulas. And for any finite set of first-order formulas, ZFC proves that such a model exists by the reflection theorem. It is easy to see that these models also satisfy STAB$_\in$, STAB$_=$, INEXT, Foundation, and Extensionality.

[11]See Welch (ms) for details.

3.1.2 The modal realist account

The *modal realist* account is just an extensional version of the modal account. Much like the Lewisian metaphysical modal realist rejects a primitive metaphysical modal operator in favour of quantification over worlds, the modal realist account rejects the primitive mathematical modal operators of the modal account in favour of a distinction between actual and non-actual (or merely possible) sets and concepts. Thus, rather than supplementing $\mathcal{L}_\in^2$ with actuality and possibility operators, the modal realist account supplements it with an actuality predicate $\mathcal{V}$, where $x \in \mathcal{V}$ and $F \in \mathcal{V}$ mean that x and F actually exist.

The central claims of modal realist account are just the obvious analogues of @-COLLAPSE$^\lozenge$, R, and C-COMP in this language. So, corresponding to @-COLLAPSE$^\lozenge$, it includes the claim that there is a set of the actually existing φ's. Formally:

$$(\mathcal{V}\text{-COLLAPSE}) \qquad\qquad \exists x \forall y (y \in x \leftrightarrow y \in \mathcal{V} \wedge \varphi)$$

Corresponding to R, it includes a reflection principle which says that if there is a φ, then actually there is a φ. Formally:

$$(\mathcal{V}\text{-R}) \qquad\qquad \forall \vec{\mathbf{x}} \in \mathcal{V}[\exists y \varphi \to (\exists y \varphi)^{\mathcal{V}}]$$

where $\varphi^{\mathcal{V}}$ is the result of restricting all quantifiers in φ to $\mathcal{V}$ and φ's free variables are among $\vec{\mathbf{x}}$. Analogous to the arguments that @-COLLAPSE$^\lozenge$ and R imply that Pairing and Union are actually true are arguments that $\mathcal{V}$-COLLAPSE and $\mathcal{V}$-R imply that Pairing and Union are true when their quantifiers are restricted to the

actually existing sets. For example, suppose $x, y \in \mathcal{V}$. By $\mathcal{V}$-COLLAPSE, there is a set of them. So $\mathcal{V}$-R implies that there is, in $\mathcal{V}$ a set of them. A similar argument can be given for Union if we assume the analogue of the claim that sets are inextensible; namely, that sets do not actually exist without their elements. Formally:

$$(\mathcal{V}\text{-I{\scriptsize N}E{\scriptsize XT}}) \qquad\qquad x \in \mathcal{V} \wedge y \in x \rightarrow y \in \mathcal{V}$$

As with R, $\mathcal{V}$-R does not hold for arbitrary formulas. To see this, note that $\mathcal{V}$-COLLAPSE implies that there is a set which does not actually exist; that is, $\exists x (x \notin \mathcal{V})$. If $\mathcal{V}$-R held for that claim it would follow that $\exists x \in \mathcal{V}(x \notin \mathcal{V})$, which is impossible. A proponent of the modal realist account thus owes us a story about which formulas $\mathcal{V}$-R does hold for and why. Following the restriction of R to $\mathcal{L}^2_\in$, I will for now just assume that $\mathcal{V}$-R holds for formulas in $\mathcal{L}^2_\in$.

Finally, corresponding to C-COMP, the modal realist account includes the claim that there is an actually existing concept which actually applies to all and only the φ's. Formally:

$$(\mathcal{V}\text{-C-COMP}) \qquad\qquad \exists F \in \mathcal{V} \forall x \in \mathcal{V}(x \in F \leftrightarrow \varphi)$$

By following the proof of theorem 12 and making suitable changes where necessary, it can be shown that over Foundation and Extensionality, the modal realist account also proves the same sentences of $\mathcal{L}^2_\in$ as ZF2 + SCR. In that sense, the modal and modal realist accounts effect the same reduction in incompleteness.

Digression. Although Reinhardt more or less formulates $\mathcal{V}$-COLLAPSE, $\mathcal{V}$-I{\scriptsize N}E{\scriptsize XT},

and $\mathcal{V}$-c-comp,[12] he does not formulate $\mathcal{V}$-R. Instead, he considers a principle he calls S4.[13] Since it may well be unclear how that principle relates $\mathcal{V}$-R, let me now briefly discuss it.

The best way to understand S4 is by approaching it from $\mathcal{V}$-R in a few steps. First, note that since the first-order quantifiers in $\varphi^{\mathcal{V}}$ only range over sets in $\mathcal{V}$, the second-order quantifiers in $\varphi^{\mathcal{V}}$ are equivalent to first-order quantifiers over subsets of $\mathcal{V}$. More precisely:

Lemma 19. Suppose $\vec{x} \in \mathcal{V}$ and $\forall z \in \mathcal{V}(z \in \vec{y} \leftrightarrow z \in \vec{F})$. Then:

$$\varphi^{*\mathcal{V}}(\vec{x}, \vec{y}) \leftrightarrow \varphi^{\mathcal{V}}(\vec{x}, \vec{F})$$

where $\varphi^{*\mathcal{V}}$ is the result of replacing the first-order quantifiers $\exists x$ in φ with $\exists x \in \mathcal{V}$ and second-order quantifiers $\exists F$ with $\exists y \subseteq \mathcal{V}$ (making sure to avoid clashes of variables), and $\varphi \in \mathcal{L}^2_{\in}$ with free variables among $\vec{x}, \vec{F}$.

Proof. By induction on the complexity of φ. The base, conjunction, negation, and $\exists x$ cases are trivial. The case for $\exists F$ is immediate given $\mathcal{V}$-c-comp and $\mathcal{V}$-collapse. $\square$

Second, note that for any set there is, up to co-extensiveness, one actually existing concept co-extensive with it on the actually existing sets. More precisely:

Lemma 20. Suppose $F, G \in \mathcal{V}$, $\forall x \in \mathcal{V}(x \in y \leftrightarrow x \in F)$, and $\forall x \in \mathcal{V}(x \in y \leftrightarrow x \in G)$. Then:

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[12]See, for example, (Reinhardt, 1974b, p.15). Extensionality is just his principle S0, $\mathcal{V}$-InExt is the first conjunct of S1, $\mathcal{V}$-c-comp is equivalent to S3.3 in the presence of S2 (which is just Separation), and since he takes the sets in $\mathcal{V}$ to determine a set, $\mathcal{V}$-collapse follows from S2. Note that Reinhardt's "properties" are essentially just the actually existing concepts of the modal realist account.

[13]See, for example, (Reinhardt, 1974a, p.196).

$$\forall x(x \in F \leftrightarrow x \in G)$$

Proof. Clearly, $\forall x \in \mathcal{V}(x \in F \leftrightarrow x \in G)$. Since $F, G \in \mathcal{V}$, it follows from $\mathcal{V}$-R that $\forall x(x \in F \leftrightarrow x \in G)$. □

So we can definitionally expand our language with an operator $j(y)$ for that concept, governed by the following axioms:

$$j(y) \in \mathcal{V}$$

$$\forall x \in \mathcal{V}(x \in y \leftrightarrow x \in j(y))$$

$$\exists F \in \mathcal{V}\forall x(x \in F \leftrightarrow x \in j(y))$$

We allow $j(y)$ in all the logical axioms, $\mathcal{V}$-COLLAPSE, and $\mathcal{V}$-C-COMP, but not in $\mathcal{V}$-R.[14] Finally, note that the relation:

$$F \equiv G =_{df} (F \in \mathcal{V} \leftrightarrow G \in \mathcal{V}) \wedge \forall x(x \in F \leftrightarrow x \in G)$$

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[14]To see that this is a legitimate definitional expansion, consider a translation ^{tr} from the new language to the old which commutes with connectives and quantifiers and is such that $(x \in j(y))^{tr} = \exists F \in \mathcal{V}[\forall z(z \in F \leftrightarrow z \in y) \wedge x \in F]$ and $(j(y) \in \mathcal{V})^{tr} = \exists F \in \mathcal{V}[\forall z(z \in F \leftrightarrow z \in y) \wedge F \in \mathcal{V}]$. It is easy to see that ^{tr} takes new instances of old axioms to old instances of old axioms, that the translations of the new axioms governing $j(y)$ are provable from $\mathcal{V}$-C-COMP and lemma 20, and that ^{tr} is the identity on the old language.

obeys the identity axioms for the new language; that is, $F \equiv F$ and $F \equiv G \rightarrow$ ($\varphi[F/H] \leftrightarrow \varphi[G/H]$) where F and G are free for H. Since the third axiom governing $j(y)$ entails that there is an F with $F \equiv j(y)$, it will then follow immediately from lemma 19 and $\mathcal{V}$-R that:

(S4) $$\forall \vec{x} \in \mathcal{V} \forall \vec{y} \subseteq \mathcal{V}[\varphi^{*\mathcal{V}}(\vec{x}, \vec{y}) \leftrightarrow \varphi(\vec{x}, j(\vec{y}))]$$

3.1.3 The model account

Most commentators take Reinhardt (1974a) to be advocating the modal realist account, though formulated with S4 rather than $\mathcal{V}$-R.[15] However, what he actually advocates is what I will call the *model account*.[16]

The *model account* differs from the modal and modal realist accounts in two important ways. First, it liberalises the notion of concept. The modal account, recall, assumes that concepts apply to sets but remains agnostic on the question whether concepts could apply to other concepts. Indeed, in the language of second-order set theory it is strictly ungrammatical to say that there are such concepts. The model account allows for these claims. More precisely, rather than supplementing $\mathcal{L}_{\in}$ with second-order variables, it supplements it with a predicate $\mathcal{S}$, where $x \in \mathcal{S}$ means that x is a set and $x \notin \mathcal{S}$ means that x is a concept. We can then say that there are concepts which apply to other concepts using the first-order statement: $\exists x, y \notin \mathcal{S}(x \in y)$. Unrestricted first-order quantifiers then range over the sets and concepts taken together.

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[15]See, for example, Parsons (1977) and Maddy (1988).
[16]See the discussion in Reinhardt (1974a) section 6.

The axioms governing sets and concepts are very similar to those governing the actual and non-actual (or merely possible) sets of the modal realist account. In particular, corresponding to $\mathcal{V}$-COLLAPSE, there is an axiom which says that there is a set or concept of all of the sets which satisfy φ. Formally:[17]

$(\mathcal{S}\text{-COLLAPSE})$ $\qquad\qquad\qquad \exists x \forall y (y \in x \leftrightarrow y \in \mathcal{S} \wedge \varphi)$

Corresponding to $\mathcal{V}$-R, there is an axiom which says that if there is a set or concept which satisfies φ, then according to the sets there is a set which satisfies φ. Formally:

$(\mathcal{S}\text{-R})$ $\qquad\qquad\qquad \forall \vec{x} \in \mathcal{S}[\exists y \varphi \rightarrow (\exists y \varphi)^{\mathcal{S}}]$

where $\varphi^{\mathcal{S}}$ is the result of restricting the quantifiers in φ to $\mathcal{S}$ and $\varphi \in \mathcal{L}_{\in}$ with free variables among $\vec{x}$. Finally, there is an axiom which says that the sets are strongly inextensible in the concepts; that the elements of a set are again sets and that subcollections of sets are sets. Formally:

$(\mathcal{S}\text{-INEXT})$ $\qquad\qquad\qquad x \in \mathcal{S} \wedge (y \in x \vee y \subseteq x) \rightarrow y \in \mathcal{S}$

Similar arguments to those given in the proof of theorem 12 show that over Extensionality and Foundation, these axioms imply that ZF holds restricted to the sets and that it holds unrestrictedly. On the model account, then, concepts are

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[17]$\mathcal{S}$-COLLAPSE, $\mathcal{S}$-R, and $\mathcal{S}$-INEXT are simple consequences of Reinhardt's axiom 6.1. See (Reinhardt, 1974a, p.199).

extremely similar to sets and one might wonder how exactly they differ. I will return to this question in section 3.1.

It would be natural at this point to extend the main claims of the modal or modal realist accounts to the new language with $\mathcal{S}$. For example, since concepts obey Extensionality on the model account, we could use R to argue that concepts could not have failed to obey Extensionality. Or more interestingly, we could adapt the proof of theorem 12 to show that there are inaccessible, measurable, and various other cardinals. However, the model account demurs from such an extension. Indeed, it rejects the modal operators of the modal account and the distinction between actual and non-actual sets of the modal realist account which are used to formulate these claims.[18] In their place, the model account adopts an axiom asserting the existence of a standard model of the modal and modal realist accounts. More precisely, let Ω be the concept applying to all the set ordinals. Then the model account adopts the claim that there is an elementary embedding from $V_{\Omega+1}$ to $V_{\lambda+1}$ with critical point Ω; that is, Ω is 1-extendible. Formally:

$$(\Omega\text{-1EX}) \qquad \exists j : V_{\Omega+1} \prec V_{\lambda+1}, \text{ such that } \mathrm{crit}(j) = \Omega$$

It is straightforward to verify that the corresponding two world Kripke model – with w_0, w_1 the only worlds, w_0 representing the actual world, first-order quanti-

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[18]Reinhardt's motivation for doing this seems to be ideological parsimony. He says, for example:

> ...we could now introduce [merely possible] sets and [merely possible concepts]. However, we wish to avoid as much as possible an unending series of extensions of the types of objects allowed in our theories (especially [merely possible] objects). (1974a, p. 199)

fiers at w_0 ranging over V_Ω and second-order quantifiers over $j[V_{\Omega+1}]$, first-order quantifiers at w_1 ranging over V_λ and second-order quantifiers over $V_{\lambda+1}$, w_0 accessing w_1, and w_0, w_1 accessing themselves – satisfies the modal account.[19] It is similarly straightforward to verify that the model $\mathcal{M} = \langle V_\lambda, V_{\lambda+1}, V_\Omega, j[V_{\Omega+1}] \rangle$ – where V_λ interprets the possible sets, $V_{\lambda+1}$ the possible concepts, V_Ω the actually existing sets, and $j[V_{\Omega+1}]$ the actually existing concepts – satisfies the modal realist account.

3.2 The modal account formalised

In this section I provide a precise formulation of the modal account outlined in section 1 (which I call RMST for *Reinhardt's Modal Set Theory*) and determine its strength. The main result is that RMST proves exactly the same sentences in the language of second-order set theory as ZF2 + SCR.

3.2.1 RMST

RMST consists of three packages of principles. There is the underlying logic, basic principles governing the behaviour of sets, and the axioms @-COLLAPSE$^\Diamond$, C-COMP, and R we have already encountered.

Preliminaries

Let $\mathcal{L}_\in$ be the language of first-order set theory with variables $x_0, ..., x_i, ...$; $\mathcal{L}^2_\in$ the language of second-order set theory, extending $\mathcal{L}_\in$ with variables $F_0, ..., F_i, ...$; and $\mathcal{L}_\Diamond$ the language of modal second-order set theory with an actuality operator,

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[19]See section 3.3 for further discussion of this kind of model.

extending $\mathcal{L}^2_\in$ with $\Diamond$ and @. I will take $x \in y$, $x = y$, and $x \in F$ to be well-formed but not $F = G$. *Ex* abbreviates $\exists y(y = x)$, but since $F = G$ is not well-formed, $\exists G(G = F)$ cannot serve as an existence predicate for concepts. Below, I will define an alternative.

Let ZF2 be the $\mathcal{L}^2_\in$ theory consisting of Extensionality, Infinity, Pairing, Union, Powerset, Foundation, the second-order versions of Separation and Replacement, and the universal closures of instances of C-COMP.

Logic

The underlying logic has two groups of axioms. First, there are the instances in $\mathcal{L}_\Diamond$ of the truth-functional tautologies, the S4.2 axioms, and the following quantificational and identity axioms (where $\mathbf{x}, \mathbf{y}$ are either both first-order or both second-order variables):

(A1) $\forall \mathbf{y}(\forall \mathbf{x}\varphi \rightarrow \varphi[\mathbf{y}/\mathbf{x}])$, where $\mathbf{y}$ is free for $\mathbf{x}$ in φ

(A2) $\forall \mathbf{x}(\varphi \rightarrow \psi) \rightarrow (\forall \mathbf{x}\varphi \rightarrow \forall \mathbf{x}\psi)$

(A3) $\varphi \leftrightarrow \forall \mathbf{x}\varphi$, where $\mathbf{x}$ is not free in φ

(A4) $x = x$

(A5) $x = y \rightarrow (\varphi[x/z] \leftrightarrow \varphi[y/z])$, where x and y are free for z in φ

(CBF) $\exists \mathbf{x}\Diamond\varphi \rightarrow \Diamond\exists \mathbf{x}\varphi$

Second, there are axioms governing the actuality operator.

(A6) $@(\varphi \rightarrow \psi) \leftrightarrow (@\varphi \rightarrow @\psi)$

(A7) $@\Box\varphi \to \varphi$

(A8) $\Box@\varphi \leftrightarrow @\varphi$

(A9) $@\exists\mathbf{x}\varphi \leftrightarrow @\exists\mathbf{x}@\varphi$

(A10) $@\Diamond Ex$

The rules of inference are GEN, from φ infer $\forall\mathbf{x}\varphi$; MP, from $\varphi \to \psi$ and φ infer ψ; NEC, if φ is a theorem, then so are $\Box\varphi$ and $@\varphi$; and $\forall$NEC, if:

$$\varphi_0 \to \Box(\varphi_1 \to ... \to \Box(\varphi_{n_1} \to \Box\varphi_n)...)$$

then so is:

$$\varphi_0 \to \Box(\varphi_1 \to ... \to \Box(\varphi_{n_1} \to \Box\forall\mathbf{x}\varphi_n)...)$$

where x is not free in $\varphi_0, ..., \varphi_{n-1}$. The rule which results from $\forall$NEC by replacing the $\Box$'s for $@$'s is then derivable using A7, A8, and A9.[20]

Remarks. Before I discuss the features of this logic, let me first use it to define an existence predicate for concepts. Let $F \equiv G$ abbreviate $@\Box\forall x@\Box(x \in F \leftrightarrow x \in G)$ and EF abbreviate $\exists G(G \equiv F)$. Then, in the presence of A10, $F \equiv G$ will satisfy the identity axioms for $\mathcal{L}_\Diamond$ and thus EF will behave as required.[21]

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[20]*Proof*: Suppose $\varphi_0 \to @\psi$ is provable with $\mathbf{x}$ not free in φ_0. It follows that $\Diamond\varphi_0 \to \forall\mathbf{x}\Diamond@\psi$ is provable and thus that $\Diamond\varphi_0 \to \forall\mathbf{x}@\psi$ is provable by A8. So $@\Diamond\varphi_0 \to @\forall\mathbf{x}@\psi$ is provable and thus $@\Diamond\varphi_0 \to @\forall\mathbf{x}\psi$ is provable by A9. Finally, $\varphi_0 \to @\forall\mathbf{x}\psi$ is provable by A7. When ψ is of the form $(\varphi_1 \to @(\varphi_2 \to ... \to @\varphi_n)...)$, $\forall\mathbf{x}$ can be pushed inwards by repeated application of A9.

[21]*Proof:* It suffices to show that $F \equiv G \to (x \in F \leftrightarrow x \in G)$, $F \equiv G \to \Box(F \equiv G)$, $F \equiv G \to @(F \equiv G)$, and $F \equiv F$. For the first, note that since $@\Diamond Ex$ it follows from $F \equiv G$ that $@\Diamond@\Box(x \in F \leftrightarrow x \in G)$ and thus $x \in F \leftrightarrow x \in G$ by A8, A9, and A7. The second and third follow from A8 and A9, and the last is trivial.

Axioms A1-A5 constitute an axiomatisation of positive free quantificational logic. Classical logic can be obtained by adding as a further axiom: $E\mathbf{x}$. The choice of a free logic is forced by @-COLLAPSE$^\Diamond$. As I pointed out in section 1.1, it implies that there could have been a set which does not actually exist; that is, $\Diamond\exists x\neg@Ex$. But if Ex were provable, it would follow from NEC and GEN that necessarily every set actually exists; that is, $\Box\forall x@Ex$. Nonetheless, the difference between classical and positive free quantification logic only shows up in arguments using NEC on @ or $\forall$NEC. More precisely, a simple induction on the length of proofs establishes that $\varphi \in \mathcal{L}_\Diamond$ is provable from S4.2 + A1-5 + CBF + $E\mathbf{x}$ + Γ using MP, GEN, and NEC on $\Box$ just in case $E\vec{\mathbf{x}} \to \varphi$ is provable from S4.2 + A1-5 + CBF + Γ using MP, GEN, and NEC on $\Box$.

A6, A8, and A9 express the fact that the actual world is the same from the perspective of any world. A7 corresponds to the frame condition on Kripke models that the actual world access all other worlds. On the interpretation of $\Diamond$ as concerned with a well-founded process of extending the mathematical ontology, it essentially indicates that we are only considering extensions of the actual mathematical ontology. Given that assumption, A10 corresponds to the condition on Kripke models that we only consider variables assignments which take values in the domains of worlds.

The rule $\forall$NEC corresponds to the condition on Kripke models that we consider variable assignments which take values in arbitrary worlds. For example, for Fx to be valid in such models is for any x in the domain of any world to satisfy F at any other world. $\forall$NEC was not added to the potentialist theory PMST in chapter 1, because it would have lead to inconsistency. To see this, note that the following is provable in the modal logic K over positive free quantificational logic:

$$\forall x(x \in X \leftrightarrow x \notin x) \land \forall x(\Diamond(x \equiv X) \to x \equiv X) \to \forall x \Box \neg (x \equiv X)$$

where $x \equiv X$ abbreviates $\forall y(y \in x \leftrightarrow y \in X)$. If Ex were provable, it would then also be provable that:

$$\forall x(x \in X \leftrightarrow x \notin x) \land \forall x(\Diamond(x \equiv X) \to x \equiv X) \to \Box \neg (x \equiv X)$$

and an application of $\forall$NEC would yield:

$$\forall x(x \in X \leftrightarrow x \notin x) \land \forall x(\Diamond(x \equiv X) \to x \equiv X) \to \Box \forall x \neg (x \equiv X)$$

But PMST proves both that there are pluralities satisfying the antecedent (by P-COMP and lemma 1) but that no plurality satisfies the consequent (by COLLAPSE$^\Diamond$). Furthermore, I did not add it to the modal structuralist theory MSST in chapter 2 because it is provable from the B axiom; that is, from $\varphi \to \Box \Diamond \varphi$.[22]

It is particularly useful because it allows for existential instantiation within the scope of modal operators. For example, suppose we have derived $\Diamond \exists \mathbf{x} \varphi$ from some premises Γ and suppose we can further derive ψ from $\Gamma, \Diamond(E\mathbf{x} \land \varphi)$. Then, as long as we did not apply GEN to any variables free in $\Gamma, \Diamond(E\mathbf{x} \land \varphi)$, we can conclude by the deduction theorem that $\bigwedge \Gamma \land \neg \psi \to \Box(E\mathbf{x} \to \neg \varphi)$ is provable. So, assuming $\mathbf{x}$ is not free in Γ, $\bigwedge \Gamma \land \neg \psi \to \Box \forall \mathbf{x} \neg \varphi$ will be provable by $\forall$NEC and thus ψ will be derivable from Γ alone. The same move can be used for arguments from $@E\mathbf{x} \land @\varphi$ using $\forall$NEC for $@$.

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[22]See (Hughes and Cresswell, 1996, p.293-6) for details and discussion.

Basic set-theoretic axioms

The basic set-theoretic axioms are Extensionality, Foundation, STAB$_=$, STAB$_\in$, and INEXT which were discussed in section 1.1. For convenience, I will repeat the last three here.

(STAB$_=$) $$\Box(x = y) \vee \Box(x \neq y)$$

(STAB$_\in$) $$\Box(x \in y) \vee \Box(x \notin y)$$

(INEXT) $$Ex \wedge \Diamond(y \in x) \rightarrow Ey$$

Given A7, STAB$_=$ implies $@(x = y) \leftrightarrow x = y$ and STAB$_\in$ implies $@(x \in y) \leftrightarrow x \in y$. I will appeal to these facts in what follows by citing STAB$_=$ and STAB$_\in$ respectively.

@-COLLAPSE$^\Diamond$, R, and C-COMP

Finally, we have @-COLLAPSE$^\Diamond$, R, and C-COMP which were introduced in section 1.1. For convenience, I will repeat them here.

(@-COLLAPSE$^\Diamond$) $$\Diamond \exists x \forall y (y \in x \leftrightarrow @Ey \wedge \varphi)$$

(R) $$\forall \vec{\mathbf{x}} [\Diamond \exists x \varphi \rightarrow \exists x \varphi]$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{\mathbf{x}}$.

(C-COMP) $$\exists F \forall x (x \in F \leftrightarrow \varphi)$$

3.2.2   RMST and ZF2 + SCR

In this section I determine the strength of RMST. The main result is that it proves exactly the same sentences of second-order set theory as ZF2 + SCR.

As a first step towards the main result, the next lemma highlights a simple consequence of R; namely, that the actual world is a second-order elementary substructure of any other world for formulas in $\mathcal{L}^2_\in$.

Lemma 21 (RMST)**.** *Suppose* $@E\vec{\mathbf{x}}$*. Then:*

$$@\varphi \leftrightarrow \varphi$$

where $\varphi \in \mathcal{L}^2_\in$ *with free variables among* $\vec{\mathbf{x}}$*.*

Proof. First, note that R is trivially equivalent to the schema: $E\vec{\mathbf{x}} \to (\varphi \to \Box\varphi)$. Then it follows from A7 that $@E\vec{\mathbf{x}} \to (@\varphi \to \varphi)$. Since we can substitute $\neg\varphi$ for φ, we are done. $\qquad\qquad\square$

An immediate corollary of lemma 21 given A8 is that $@\varphi \leftrightarrow \Diamond\varphi$ whenever $@E\mathbf{x}$. I will appeal to this fact in what follows by citing lemma 21.

Theorem 12. *RMST proves every sentence ZF2 proves.*

Proof. Above I observed that classical logic proves a sentence in $\mathcal{L}^2_\in$ only if positive free logic does. It follows that RMST proves every sentence in $\mathcal{L}^2_\in$ provable in classical logic. To establish our result, it thus suffices by lemma 21 to show that

RMST proves that every non-logical axiom of ZF2 is actually true. There are two cases to consider.

CASE 1: Pairing, Union, Powerset, and the consequent of second-order Replacement all have the form:

$$\exists x \forall y (y \in x \leftrightarrow \varphi)$$

For Pairing, $\varphi =$ "$y = z \vee y = z'$"; for Union, $\varphi =$ "$\exists z \in w(y \in z)$"; for Powerset, $\varphi =$ "$y \subseteq z$"; and for Replacement, $\varphi =$ "$\exists z \in w(\langle z, y \rangle \in F)$". I will now outline a general strategy for proving claims of this form and then apply it to these particular φ. The idea is to show that RMST proves:

$$@E\vec{\mathbf{x}} \rightarrow \forall y(\varphi \rightarrow @Ey) \tag{3.1}$$

where φ's free variables are among $\vec{\mathbf{x}}, y$. For if it does, it will also prove:

$$@E\vec{\mathbf{x}} \rightarrow \Diamond \exists x \forall y(y \in x \leftrightarrow \varphi)$$

by A8 and @-COLLAPSE$^{\Diamond}$ and thus:

$$@E\vec{\mathbf{x}} \rightarrow @\exists x \forall y(y \in x \leftrightarrow \varphi)$$

by A8 and lemma 21.

It is easy to see that 3.1 holds for Pairing. For Union, we use INEXT. For

Powerset, assume $@Ex$, Ey, and $y \subseteq x$. By C-COMP, let $@EF$ be such that $@\forall z(z \in F \leftrightarrow z \in y)$. Now, if $z \in y$, then $@Ez$ by INEXT and $@(z \in y)$ by STAB$_\in$. Thus, $@(z \in F)$ and so $z \in F \wedge z \in x$ by lemma 21. Conversely, if $z \in F \wedge z \in x$, then $@Ez$ by INEXT and $@(z \in F)$ by lemma 21. Thus, $@(z \in y)$ and so $z \in y$ by STAB$_\in$. So:

$$\forall z(z \in y \leftrightarrow z \in F \wedge z \in x)$$

and:

$$\exists y \forall z(z \in y \leftrightarrow z \in F \wedge z \in x)$$

since Ey. Thus:

$$@\exists y \forall z(z \in y \leftrightarrow z \in F \wedge z \in x)$$

by lemma 21. So let $@Ey'$ be a witness; that is, $@\forall z(z \in y' \leftrightarrow z \in F \wedge z \in x)$. Then by lemma 21 again we have $\forall z(z \in y' \leftrightarrow z \in F \wedge z \in x)$ and thus $\forall z(z \in y' \leftrightarrow z \in y)$. Finally, by Extensionality, $y = y'$ and so $@Ey$ as required.

For Replacement, assume $@EF$, $@Ew$, F is actually functional, and that y is in the range of F on w; that is, $\exists z \in w(\langle z, y \rangle \in F)$. Then $@Ez$ by INEXT and $@(z \in w)$ by lemma 21. Since F is actually functional, there will be some $@Ey'$ such that $@(\langle z, y' \rangle \in F)$ and so Ey' and $\langle z, y' \rangle \in F$ by lemma

21. Since F is actually functional, lemma 21 implies that it is functional. It follows that $y = y'$ and so $@Ey$ as required.

CASE 2: Extensionality, Foundation, and C-COMP are already axioms of RMST. For Infinity, note that by $@$-COLLAPSE$^\diamondsuit$ there could have been an empty set and so $@\exists y(y = \emptyset)$ by lemma 21. Let $@Ey$ be a witness. By lemma 21 again, Ey and $y = \emptyset$ and so $\exists y(@Ey \wedge y = \emptyset)$. So the actually existing sets contain an empty set. Then, since Pairing and Union are actually true by CASE 1, the actually existing sets are closed under successor. To see this, suppose $@Ex$. Then $@\exists y(y = x \cup \{x\})$. Let $@Ey$ be a witness. By lemma 21, Ey and $y = x \cup \{x\}$ and so $\exists y(@Ey \wedge y = x \cup \{x\})$. Finally, since $@$-COLLAPSE$^\diamondsuit$ implies that there could have been a set of the actually existing sets, it implies that there could have been an infinite set and our conclusion follows immediately from lemma 21.

$\square$

I will now extend theorem 12 by showing that RSMT proves all instances of the principle of Structural Complete Reflection introduced in chapter 1.

Definition 15. For any $\varphi \in \mathcal{L}^2_\in$, let φ^{*z} be the result of replacing all first-order quantifiers $\exists x$ in φ with $\exists x \in z$ and all second-order quantifiers $\exists F$ with $\exists y \subseteq z$ (making sure to avoid clashes of variables).

Definition 16. The principle of *Structural Complete Reflection* (SCR) is the following schema in $\mathcal{L}^2_\in$: There is a non-empty V_κ and concept F such that for all $\vec{x} \in V_\kappa$, $\vec{y} \subseteq V_\kappa$, and $\vec{G}$, if:

$$\forall z(\langle \vec{y}, z \rangle \in F \leftrightarrow z \in \vec{G})$$

then:

$$\varphi^{*V_\kappa}(\vec{x}, \vec{y}) \leftrightarrow \varphi(\vec{x}, \vec{G})$$

where φ's free variables are among $\vec{x}, \vec{G}$.

The next lemma shows, in effect, that truth in the set of actually existing sets is equivalent to actual truth for formulas in $\mathcal{L}^2_\in$.

Lemma 22 (RMST). *Suppose $z = \{w : @Ew\}$, $@E\vec{x}$, and $@\forall w(w \in \vec{y} \leftrightarrow w \in \vec{F})$. Then:*

$$\varphi^{*z}(\vec{x}, \vec{y}) \leftrightarrow @\varphi(\vec{x}, \vec{F})$$

where $\varphi \in \mathcal{L}^2_\in$ with free variables among $\vec{x}, \vec{F}$.

Proof. By induction in the complexity of φ. The base cases are immediate given STAB$_=$ and STAB$_\in$. Conjunction and negation are trivial. For $\exists w$, the induction hypothesis implies:

$$(\exists w \varphi)^{*z}(\vec{x}, \vec{y}) \leftrightarrow \exists w(@Ew \wedge @\varphi(\vec{x}, \vec{F}))$$

Then note that by $\forall$NEC and lemma 21, $\exists w(@Ew \wedge @\varphi)$ is equivalent to $@\exists w\varphi$. The case for $\exists F$ is similar, but uses C-COMP and Separation. $\square$

It is a simple corollary of lemma 22 that the set of actually existing sets is an inaccessible rank.[23]

Corollary 2 (RMST). *If $x = \{y : @Ey\}$, then $x = V_\kappa$ for κ inaccessible.*

Proof. By theorem 12 and lemma 21, $@ZF2$ and so $ZF2^{*x}$ by lemma 22. From INEXT it follows that x is transitive. Thus, $x = V_\kappa$. □

Theorem 13. *RMST proves every sentence ZF2 + SCR proves.*

Proof. Given theorem 12, it suffices to show that RSMT proves each instance of SCR. Now, working in RMST, suppose that $V_\kappa = \{x : @Ex\}$. By C-COMP, let F be such that:

$$\forall y \subseteq V_\kappa \forall z (\langle y, z \rangle \in F \leftrightarrow \exists H (@EH \wedge @\forall w(w \in y \leftrightarrow w \in H) \wedge z \in H))$$

Then I claim that V_κ and F are witnesses to SCR. To see this, first note that any two $@EH, H'$ such that $@\forall w(w \in y \leftrightarrow w \in H)$ and $@\forall w(w \in y \leftrightarrow w \in H')$ will be co-extensive by lemma 21. Thus, for any such H:

$$\forall z (\langle y, z \rangle \in F \leftrightarrow z \in H)$$

Now, suppose $\vec{x} \in V_\kappa$, $\vec{y} \subseteq V_\kappa$, $\forall z (\langle \vec{y}, z \rangle \in F \leftrightarrow z \in \vec{G})$. Then there are $@E\vec{H}$ such that $@\forall w(w \in \vec{y} \leftrightarrow w \in \vec{H})$ by C-COMP. It follows from lemmas 22 and 21 that:

$$\varphi^{*V_\kappa}(\vec{x}, \vec{y}) \leftrightarrow \varphi(\vec{x}, \vec{H})$$

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[23]Here I use a slightly non-standard definition of inaccessibility; namely, that κ is *inaccessible* just in case $ZF2^{*V_\kappa}$. In ZFC, this is equivalent to the usual definition. But in the absence of Choice, 2^α is not always well-defined and the two definitions can come apart. See Blass et al. (2007).

Finally, by the observation above, $\vec{H}$ will be co-extensive with $\vec{G}$ and so:

$$\varphi(\vec{x}, \vec{H}) \leftrightarrow \varphi(\vec{x}, \vec{G})$$

$\square$

I will now prove a converse of theorem 13; namely, that any sentence in $\mathcal{L}_{\in}^2$ provable in RMST is provable in ZF2 + SCR.

Definition 17. Let $\,_{i,F,z}^{tr}$ be the following translation from $\mathcal{L}_\diamond$ to $\mathcal{L}_{\in}^2$.

- $\,_{i,F,z}^{tr}$ is the identity on atomic formulas and commutes with the connectives.

- $(\exists x \varphi)_{i,F,z}^{tr} = \exists x \varphi_{i,F,z}^{tr}$ if $i = 1$ and:

- $(\exists x \varphi)_{i,F,z}^{tr} = \exists x \in z \varphi_{i,F}^{tr}$ if $i = 0$.

- $(\exists G \varphi)_{i,F,z}^{tr} = \exists G \varphi_{i,F,z}^{tr}$ if $i = 1$ and:

- $(\exists G \varphi)_{i,F,z}^{tr} = \exists G(\exists y \subseteq z \forall w[\langle y, w \rangle \in F \leftrightarrow w \in G] \wedge \varphi_{i,F,z}^{tr})$ if $i = 0$.

- $(@\varphi)_{i,F,z}^{tr} = \varphi_{0,F,z}^{tr}$

- $(\Diamond \varphi)_{i,F,z}^{tr} = \varphi_{1,F,z}^{tr}$ if $i = 1$ and:

- $(\Diamond \varphi)_{i,F,z}^{tr} = \varphi_{0,F,z}^{tr} \vee \varphi_{1,F,z}^{tr}$ if $i = 0$.

Lemma 23 (ZF2). *Let V_κ and F witness SCR for φ, and suppose $\vec{x} \in V_\kappa$, $\vec{y} \subseteq V_\kappa$, and $\forall w(\langle \vec{y}, w \rangle \in F \leftrightarrow w \in \vec{G})$. Then:*

$$\varphi^{*V_\kappa}(\vec{x}, \vec{y}) \leftrightarrow \varphi_{0,F,V_\kappa}^{tr}(\vec{x}, \vec{G})$$

where $\varphi \in \mathcal{L}_{\in}^2$ with free variables among $\vec{x}, \vec{G}$.

Proof. By induction on the complexity of φ. The cases for $x = y$ and $x \in y$ are trivial. The case for $x \in G$ is just the fact that $x \in y \leftrightarrow x \in G$ by SCR. Conjunction, negation, and the case for $\exists x$ are trivial. For $\exists G \varphi$, note that for any $y \subseteq V_\kappa$ there is some G such that $\forall w(\langle y, w \rangle \in F \leftrightarrow w \in G)$ by C-COMP. $\qquad \square$

Theorem 14. *ZF2 + SCR proves every sentence in $\mathcal{L}^2_\in$ that RMST proves.*

Proof. My strategy will be to provide, for any finite set of RMST's axioms, an interpretation in ZF2 + SCR which is preserved under RMST's rules of inference and which is the identity on $\mathcal{L}^2_\in$. Our result will then be immediate.

Working in ZF2 + SCR, let V_κ and F witness SCR for $\varphi_0, ..., \varphi_n$ and let $^{tr}_i$ be the translation $^{tr}_{i,F,V_\kappa}$ from definition 17. It is straightforward but tedious to verify that ψ^{tr}_0 and ψ^{tr}_1 hold whenever ψ is an axiom of RMST other than an instance of R. I will now show that ψ^{tr}_0 and ψ^{tr}_1 hold when ψ is the instance of R for φ_i.

Since $\psi^{tr}_1 = \forall \vec{\mathbf{x}}[\varphi_i \to \varphi_i]$, it trivially holds. ψ^{tr}_0 is:

$$[\vec{x} \in V_\kappa \wedge \vec{y} \subseteq V_\kappa \wedge \forall w(\langle \vec{y}, w \rangle \in F \leftrightarrow w \in \vec{G})] \to [(\varphi_i)^{tr}_0 \vee \varphi_i \to (\varphi_i)^{tr}_0]$$

So suppose the antecedent and $\varphi_i(\vec{x}, \vec{G})$. Then, by SCR, it follows that $\varphi_i^{*V_\kappa}(\vec{x}, \vec{y})$. Thus, $(\varphi_i)^{tr}_0$ by lemma 23. Finally, the rules of inference for RMST are easily seen to be preserved by $\varphi^{tr}_0 \wedge \varphi^{tr}_1$. $\qquad \square$

3.3 Criticisms

In this section I will look at criticisms of the modal, modal realist, and model accounts outlined in section 1.

3.3.1 Distinctions without a difference

Sets and concepts

As discussed in section 1.3, sets and concepts are very similar on the model account. For example, they both satisfy Extensionality. This presses the question: how do concepts differ from sets? As Reinhardt notes, without a satisfactory answer to this question, the distinction between sets and concepts looks like a "distinction without a difference" (p. 196, 1974a).[24]

In contrast, there is a significant difference between sets and concepts on the modal account. Sets are inextensible – they could not have existed without their elements – whereas concepts can be extensible. In particular, since there could have been a set which does not actually exist by @-COLLAPSE$^\Diamond$, any actually existing concept which applies to all actual sets could have applied to something which does not actually exist. An analogous difference arises on the modal realist account. But the model account rejects the modal operators and the distinction between actual and non-actual sets and concepts on which those differences are based. So how *do* sets and concepts differ on the model account?

Reinhardt proposes that the distinction can be drawn in terms of the standard models of RMST which are taken to exist on the model account.[25] For example, sets are inextensible in such models whereas concepts can be extensible in them. The problem with this response is that the proposed difference is not particularly interesting. There are simple Kripke models in which concepts are inextensible and sets are extensible, and in which some sets are inextensible and others are extensible. In general, the existence of a model of an interesting difference does

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[24]See Maddy (1983) for discussion.
[25]See, for example, (Reinhardt, 1974a, p.198-99).

not imply that there is an interesting difference. It is therefore unclear how a proponent of the model account might spell out the difference.

Actual and non-actual sets

Although the modal realist account avoids the previous problem, it does face an analogous problem. In particular, it raises the question: how do actual sets differ from non-actual sets? It is tempting to think that this is another 'distinction without a difference'.

It is instructive to compare the distinction between actual and non-actual sets with the distinction that the metaphysical modal realist draws between the inhabitants of the metaphysically actual world and the inhabitants of metaphysically merely possible worlds. Lewis (1986), for example, draws the distinction in terms of spatiotemporal and causal connections. For him, an object actually exists just in case it is spatiotemporally or causally connected to us. Such resources are, of course, unavailable to a proponent of the modal realist account, since sets are not spatiotemporally located and do not enter into causal relations. The problem is that it is unclear what might be used by a proponent of the modal realist account instead.

3.3.2   The universality of set theory

Recall from section 1 that the central principles of the modal realist and model accounts were very similar. For example, $\mathcal{V}$-R is just:

$$\forall \vec{x} \in \mathcal{V}[\varphi^{\mathcal{V}} \leftrightarrow \varphi]$$

and $\mathcal{S}$-R is just:

$$\forall \vec{x} \in \mathcal{S}[\varphi^{\mathcal{S}} \leftrightarrow \varphi]$$

for $\varphi \in \mathcal{L}_{\in}$. In essence, $\mathcal{V}$-R says that the actual sets are an elementary substructure of the actual and non-actual sets taken together, and $\mathcal{S}$-R says that the sets are an elementary substructure of the sets and concepts taken together.

As it turns out, Reinhardt sees both principles as following from a more general claim he calls the *universality of set theory*. Applying it to get $\mathcal{S}$-R he says, for example:

> Since we regard set theory (the theory of $[V]$) as *the* theory of extensional objects such as sets, collections, etc., we assume this theory applies to [the sets and concepts taken together]. (1974, p.198-199)

The sets and concepts taken together are one example of extensional objects, according to Reinhardt. Actual and non-actual sets are another. Thus, the universality of set theory is supposed to imply both $\mathcal{S}$-R and $\mathcal{V}$-R.

The universality of set theory has been criticised in the literature, most notably by Maddy (1988). In particular, she challenges its use in obtaining $\mathcal{S}$-R (p. 754).[26] She claims that we have little reason to think that concepts *are* extensional in the relevant sense and so little reason to think that the universality of set theory applies to them. After all, given the problem raised in section 3.1, sets and concepts need to be importantly different.

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[26]She also challenges its use to obtain S4 (see section 1). She points out that even granting that the actual sets have the same theory as the possible sets, it is unclear how that fact would give rise to the j operation (p. 753-4). However, once we formulate the modal realist account with $\mathcal{V}$-R instead of S4, this worry disappears.

As long as we do not follow Reinhardt in adopting the model account, though, we can agree with Maddy. Since we need not accept $\mathcal{S}$-R, we need not try to justify it on the basis of the universality of set theory and so we need not claim that concepts are relevantly like sets. But regardless of whether it is applicable to concepts, there is a deeper problem with the claim. To see this, consider its application to get $\mathcal{V}$-R. In order for it to do so, the non-actual sets have to be extensional objects in the relevant sense. So it looks like the universality of set theory should also imply:

$$\forall \vec{x} \in \mathcal{V}[\varphi^{\mathcal{V}} \leftrightarrow \varphi^{\mathcal{V}*}]$$

where $\varphi^{\mathcal{V}*}$ is the result of restricting all quantifiers in φ to $\notin \mathcal{V}$; that is, to the non-actual sets. But this is plainly false. Every actual set is equal to some actual set, but no actual set is equal to a non-actual set. Or, consider the sets taken together with the (possibly non-actual) subsets of the actual sets. Since they seem to be extensional objects in the relevant sense, it looks like the universality of set theory should apply. But they do not satisfy Pairing.

To see the problem more clearly, consider the following formulation of the universality of set theory.

(UoS) $$\forall X \subseteq \mathrm{Ext} \forall \vec{x} \in \mathcal{V}[\varphi^{\mathcal{V}} \leftrightarrow \varphi^{X}]$$

for $\varphi \in \mathcal{L}_{\in}$ and where X is a totality and where Ext is the totality of all extensional objects. Then there are many counterexamples to UoS. For example, $X = \emptyset$, $X = \mathcal{V}^{*}$, and $X = \{x : x \subseteq \mathcal{V}\}$. So even assuming we can make good sense of

the general notion of an extensional object, UoS will have to be restricted. The problem is that it is unclear whether there is an interesting range of X for which it holds.

3.3.3   Extendability to inconsistency

In section 1.1 I noted that R does not hold for arbitrary formulas. In particular, I noted that since @-COLLAPSE$^\diamond$ implies that there could have been a set which does not actually exist, it does not hold for $\exists x \neg @Ex$. Similarly, since the principle of plural collapse (COLLAPSE$^\diamond$) from chapter 1 implies that there could have been a set not among the X's, R does not hold for $\exists x(x \notin X)$. Furthermore, since $\exists x \neg @Ex$ is equivalent to:

$$\exists x(x \neq x_0 \wedge, ..., \wedge x \neq x_\alpha \wedge, ...)$$

when $x_0, ..., x_\alpha, ...$ are all the actually existing sets, it does not hold for that infinitary formula either.

Since the modal account takes R to hold for formulas in $\mathcal{L}^2_\in$, these observations press the question: for which formulas does R hold and why? The problem is that it is unclear whether a satisfactory answer to this question can be given.

The same problem arises for reflection principles quite generally, and for similar reasons. For example, SCR does not hold for arbitrary formulas involving a predicate $\mathcal{C}$ which is intended to apply to a concept just in case that concept applies to all sets. Formally:

$$\forall F(F \in \mathcal{C} \leftrightarrow \forall x(x \in F))$$

To see this, note that there is a concept satisfying $\mathcal{C}$ by C-COMP but there is no subset of any V_α satisfying it.[27] Similarly, since:

$$\forall x(x = x_0 \vee, ..., \vee x = x_\alpha \vee, ...)$$

is true when $x_0, ..., x_\alpha, ...$ are all sets but not true in any V_β, SCR does not hold for arbitrary infinitary formulas.

On the face of it, the problem as it applies to R is no worse than the problem as it applies to SCR. In so far as we are interested in whether the potentialist can formulate principles as attractive as SCR, it therefore does not appear to be a particularly pressing problem. Nonetheless, there is a different example which has consistently been raised against R and which does not seem to have an analogue for SCR. Roughly, the idea is that R generalises to imply the existence of a non-trivial elementary embedding from the universe of sets into itself, which was shown to be inconsistent with Choice by Kunen (1971).[28] Exactly how the generalisation is supposed to go is unclear.[29] I will now propose a natural way of spelling it out and argue that it does have an analogue for SCR after all.

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[27]This is somewhat similar to the problem Linnebo (2007) raises for the predicate $x \equiv X$ used in Burgess (2004).

[28]See, for instance, (Koellner, 2009, p.117), (Reinhardt, 1980, p.200), and (Wang, 1977, p.333, fn. 14).

[29]Koellner (2009) and (Reinhardt, 1980, p.200) are clear on the intended generalisation, but they are explicitly considering principles other than R. Koellner, for example, focuses on a principle suggested at the end of Reinhardt's dissertation (1967); and Reinhardt focuses on generalisations of the model account where we assert that Ω is not only 1-extendible but also α-extendible for any α.

Higher-order set theory and R

I will spell out the generalisation as the claim that R does not hold for arbitrary formulas in the language of higher-order set theory. For simplicity, I will work in ZF2 plus Choice and show that there are no standard models of RMST in such languages. It should then be straightforward to convert this into a proof of inconsistency in a suitable higher-order extension of RMST.[30]

First, let me define the language of higher-order set theory. Let $\mathcal{L}_\in^\Omega$ extend $\mathcal{L}_\in$ with variables $x_0^\alpha, ..., x_i^\alpha, ...$ for each ordinal $\alpha \leq \Omega$. I will call α the *type* of the variable x_i^α and use α-*concept* for the entities over which they range. The x_i^0's play the same role in $\mathcal{L}_\in^\Omega$ as the x_i's play in $\mathcal{L}_\in^2$ and so the 0-concepts are just the sets. The x_i^1's play the same role in $\mathcal{L}_\in^\Omega$ as the F_i's did in $\mathcal{L}_\in^2$ and so the 1-concepts are just what I have been calling 'concepts'. For simplicity, $x^\alpha \in y^\beta$ *and* $x^\alpha = x^\beta$ are taken to be well-formed. I will say that an α-concept x^α applies to a β-concept y^β whenever $y^\beta \in x^\alpha$ and I will say that it contains y^β when $\alpha = 0$. Let $\mathcal{L}_\in^\alpha$ be the restriction of $\mathcal{L}_\in^\Omega$ to formulas with variables of type less than or equal to α. I will use *concept* for Ω-concepts.

One of the main principles governing concepts in $\mathcal{L}_\in^2$ is C-COMP, the claim that there is a concept co-extensive with any condition on the sets. Generalising this, one of the main principles governing concepts in $\mathcal{L}_\in^\Omega$ is the claim that there is an $\alpha + 1$-concept co-extensive with any condition on the α-concepts. Formally:

(COMP) $$\exists x^{\alpha+1} \forall y^\alpha (y^\alpha \in x^{\alpha+1} \leftrightarrow \varphi)$$

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[30]See Degen and Johannsen (2000) and Linnebo and Rayo (2012) for discussion.

In addition to there being an $\alpha + 1$-concept which at least applies to all and only the α-concepts satisfying φ, it is natural to also take $\alpha + 1$-concepts to only apply to α-concepts. Formally:

(REG) $$\forall x^\beta \in y^{\alpha+1} \exists z^\alpha (x^\beta = z^\alpha)$$

For limits λ, it is natural to take the λ-concepts to be an accumulation of the α-concepts for $\alpha < \lambda$. That is, (i) every α-concept for $\alpha < \lambda$ is a λ-concept and (ii) every λ-concept is an α-concept for some $\alpha < \lambda$. The first claim can formalised as:

(ACC$_1$) $$\forall x^\alpha \exists y^\lambda (x^\alpha = y^\lambda)$$

for $\alpha < \lambda$. Since we cannot quantify into the types of the language, the second claim is usually taken to be formalised by the following infinitary rule of inference:

(ACC$_2$) From $\forall x^0 \varphi(x^0), ..., \forall x^\alpha \varphi(x^\alpha), ...$ infer $\forall x^\lambda \varphi(x^\lambda)$

where $0, ..., \alpha, ...$ are all the ordinals less than λ.[31]

For simplicity, I will also add a principle of extensionality for concepts. Formally:

(EXT) $$\forall x^\lambda (x^\lambda \in y^\alpha \leftrightarrow x^\lambda \in z^\beta) \rightarrow y^\alpha = z^\beta$$

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[31]See, for example, Degen and Johannsen (2000) and Linnebo and Rayo (2012).

As I will show in the appendix to this section (theorem 16), nothing hangs on assuming EXT.

These principles strongly suggest simple intended models for $\mathcal{L}_\in^\Omega$. In particular, let $\mathcal{M} = \langle\langle X_\alpha : \alpha \leq \Omega\rangle, E\rangle$ where $X_0 = V_\kappa$ for some κ and E is a relation on $\bigcup X_\alpha$ such that $E \cap X_0 \times X_0 = {\in} \cap X_0 \times X_0$. Corresponding to COMP, we impose the constraint that for any $x \subseteq X_\alpha$ there is some $y \in X_{\alpha+1}$ such that $\forall z(\langle z, y\rangle \in E \leftrightarrow z \in x)$; corresponding to REG, we impose the constraint that if $\langle x, y\rangle \in E$ and $y \in X_{\alpha+1}$, then $x \in X_\alpha$; corresponding to ACC$_1$ and ACC$_2$ we impose the constraint that $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$; and corresponding to EXT, we impose the constraint that E be an extensional relation.[32] Intended models for $\mathcal{L}_\in^\beta$ are defined in the same way with β replacing Ω. Satisfaction in $\mathcal{M}$ is defined in the obvious way, with variables of type α ranging over X_α.

Now we are in a position to define the notion of a *standard two world* Kripke model for RMST in $\mathcal{L}_\in^\Omega$. In particular, let $\langle\mathcal{M}, \mathcal{N}\rangle$ be a pair of intended models for $\mathcal{L}_\in^\Omega$ such that $\kappa^\mathcal{M} < \kappa^\mathcal{N}$, $X_\alpha^\mathcal{M} \subseteq X_\alpha^\mathcal{N}$, and for which:

$$\forall \vec{x} \in \bigcup X_\alpha(\mathcal{M} \vDash \varphi[\vec{x}/\vec{\mathbf{y}}] \leftrightarrow \mathcal{N} \vDash \varphi[\vec{x}/\vec{\mathbf{y}}])$$

where $\varphi \in \mathcal{L}_\in^\Omega$ with free variables among $\vec{\mathbf{y}}$. With $\mathcal{M}$ as the actual world, $\mathcal{M}$ accessing $\mathcal{N}$, and $\mathcal{M}, \mathcal{N}$ accessing themselves, this guarantees that R holds for formulas in $\mathcal{L}_\in^\Omega$ and it is routine to verify that the model validates all the other axioms of RMST in $\mathcal{L}_\in^\Omega$. Two world models for RMST in $\mathcal{L}_\in^\beta$ are defined in the same way with β replacing Ω.

With these notions in place, we can then prove:

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[32]That is, if $\forall z(\langle z, x\rangle \in E \leftrightarrow \langle z, y\rangle \in E)$, then $x = y$.

Theorem 15 (ZFC2). *There is a standard two world model for RMST in $\mathcal{L}_{\in}^{\Omega}$ just in case:*

$$\exists j : V \prec V, \text{with } crit(j) = \kappa^{\mathcal{M}}$$

and there is a standard two world Kripke model for RMST in $\mathcal{L}_{\in}^{\lambda}$ just in case:

$$\exists j : V_{\kappa^{\mathcal{M}}+\lambda} \prec V_{\kappa^{\mathcal{N}}+\lambda}, \text{with } crit(j) = \kappa^{\mathcal{M}}$$

Proof. See the appendix to this section. □

But Kunen (1971) showed that there cannot be a $j : V \prec V$ given Choice. So there are no standard models of RMST in $\mathcal{L}_{\in}^{\Omega}$.

Put like this, the problem appears to have a straightforward response. A proponent of RMST can just deny that $\mathcal{L}_{\in}^{\Omega}$ exists. Someone living in the actual world of a standard two world Kripke model will of course accept that $\mathcal{L}_{\in}^{\alpha}$ exists for $\alpha < \kappa^{\mathcal{M}}$. They may even accept that $\mathcal{L}_{\in}^{\alpha}$ exists whenever it is definable in some $\mathcal{L}_{\in}^{\beta}$ which they have already adopted. But there is little reason for them to accept that $\mathcal{L}_{\in}^{\Omega}$ exists. In any case, the problem does not in fact rely on $\mathcal{L}_{\in}^{\Omega}$. As I will now show, it arises already for $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$.

By theorem 15, any standard two world Kripke model for RMST in $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ yields an elementary embedding j from $V_{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ to $V_{\kappa^{\mathcal{N}}+\kappa^{\mathcal{M}}}$. Now, consider the following true claim in $V_{\kappa^{\mathcal{N}}+\kappa^{\mathcal{M}}}$:

$$\exists\alpha\exists\beta < j(\kappa^{\mathcal{M}})\forall\gamma\exists\delta < \beta(\gamma = \alpha + \delta)$$

To see that this is true in $V_{\kappa^{\mathcal{N}}+\kappa^{\mathcal{M}}}$, note that $\alpha = \kappa^{\mathcal{N}}$ and $\beta = \kappa^{\mathcal{M}}$ are witnesses. But:

$$\exists \alpha \exists \beta < \kappa^{\mathcal{M}} \forall \gamma \exists \delta < \beta (\gamma = \alpha + \delta)$$

is clearly false in $V_{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ since $\kappa^{\mathcal{M}}$ is a cardinal. Since $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ will be definable over $V_{\kappa^{\mathcal{M}}}$, this instance of the problem cannot be avoided by denying its existence.

Let me now briefly indicate how an analogous problem arises for SCR. Just as $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ is definable over $V_{\kappa^{\mathcal{M}}}$, $\mathcal{L}_{\in}^{\Omega}$ is definable over V in ZF. In that language it will be true over V that:

$$\exists \alpha^{\Omega} \exists \beta^{\Omega} \forall \gamma^{\Omega} \exists \delta^{\Omega} < \beta^{\Omega} (\gamma^{\Omega} = \alpha^{\Omega} + \delta^{\Omega})$$

To see that this would be true, note that since its quantifiers would range over $\Omega + \Omega$, $\alpha = \Omega$ and $\beta = \Omega$ are witnesses. But it is false over any V_{κ}, since in that case its quantifiers will range over $\kappa + \Omega = \Omega$ and it will just be equivalent to:

$$\exists \alpha \exists \beta \forall \gamma \exists \delta < \beta (\gamma = \alpha + \delta)$$

Appendix

Proof of theorem 15. Let $\langle \mathcal{M}, \mathcal{N} \rangle$ be a standard two world Kripke model for RMST in $\mathcal{L}_{\in}^{\Omega}$. Since X_{λ} is a union of the X_{α}'s for $\alpha < \lambda$ and since elements of $X_{\alpha+1}$ only apply to elements of X_{α}, E will be well-founded. So, given that it is also extensional, we can apply Mostowski's collapse lemma to get an isomorphism j from a

transitive class Y to $\langle X_\Omega^{\mathcal{M}}, E^{\mathcal{M}} \rangle$. Since every subset of X_α determines a concept at $X_{\alpha+1}$, a simple induction establishes that $Y = V$. Let i be a similar isomorphism between $\langle X_\Omega^{\mathcal{N}}, E^{\mathcal{N}} \rangle$ and V.

Now, a simple induction on the complexity of φ shows that:

$$\langle X_\Omega^{\mathcal{M}}, E^{\mathcal{M}} \rangle \vDash \varphi \leftrightarrow \mathcal{M} \vDash \varphi^{*\Omega}$$

where $\varphi \in \mathcal{L}_\in$ and $\varphi^{*\Omega}$ is the result of replacing each variable x in φ with x^Ω. Similarly for $\mathcal{N}$. By R, we have:

$$\mathcal{M} \vDash \varphi^{*\Omega} \leftrightarrow \mathcal{N} \vDash \varphi^{*\Omega}$$

It follows that $i \circ j$ is an elementary embedding from V to V and it is easy to check that $\mathrm{crit}(i \circ j) = \kappa^{\mathcal{M}}$ as required.

Given $j : V \to V$, it is easy to see that the pair $\langle \mathcal{M}, \mathcal{N} \rangle$ with $X_\alpha^{\mathcal{M}} = j[V_{\mathrm{crit}(j)+\alpha}]$ and $X_\alpha^{\mathcal{N}} = V_{j(\mathrm{crit}(j))+\alpha}$ will be a two world Kripke model for RMST in $\mathcal{L}_\in^\Omega$.

An analogous argument with λ replacing Ω will establish the second conjunct of theorem 15. $\qquad\square$

I mentioned above that nothing hangs on the assumption of extensionality. Let me now prove that.

Definition 18. Let $\mathcal{M}$ be an intended model for $\mathcal{L}_\in^\alpha$. Say that $Y \subseteq \bigcup_{\beta \leq \alpha} X_\beta$ is an *extensional core* of $\mathcal{M}$ if (i) $X_0 \subseteq Y$, (ii) $E \cap Y \times Y$ is an extensional relation, and (iii) for any $x \subseteq X_\beta \cap Y$ there is some $y \in X_{\beta+1} \cap Y$ such that $\forall z(\langle z, y \rangle \in E \leftrightarrow z \in x)$.

Lemma 24. *Every intended model of $\mathcal{L}_\in^\alpha$ has an extensional core.*

Proof. Straightforward transfinite recursion. □

Lemma 25. *Suppose Y is an extensional core of an intended model $\mathcal{M}$ for $\mathcal{L}_\in^\alpha$. Then there is an isomorphism j from $\langle Y, E \cap Y \times Y \rangle$ to $V_{\kappa^\mathcal{M}+\alpha}$ which is the identity on $V_{\kappa^\mathcal{M}}$.*

Proof. Since $E \cap Y \times Y$ is well-founded and extensional, Mostowski's collapse lemma implies that there is an isomorphism j from $\langle Y, E \cap Y \times Y \rangle$ to some transitive set x. By induction we can show that $j[Y \cap X_\alpha] = V_{\kappa^\mathcal{M}+\alpha}$. The base case is trivial since $Y \cap X_0 = X_0 = V_{\kappa^\mathcal{M}}$ and since $V_\kappa^\mathcal{M}$ is transitive in Y, j will be the identity on it. The successor case follows by condition (ii), and the limit case is trivial. □

Lemma 26. *Suppose $\mathcal{M}$ is an intended model for $\mathcal{L}_\in^{\lambda+1}$, $Y \in X_{\lambda+1}^\mathcal{M}$ is an extensional core of $\mathcal{M}$ restricted to λ, j is an isomorphism from $V_{\kappa^\mathcal{M}+\lambda}$ to Y, and $\vec{x} \in V_{\kappa^\mathcal{M}+\lambda}$. Then:*

$$V_{\kappa^\mathcal{M}+\lambda} \vDash \varphi(\vec{x}) \leftrightarrow \mathcal{M} \vDash (\varphi^\lambda)^Y(j(\vec{x}))$$

where $\varphi \in \mathcal{L}_\in$, φ^λ is the result of replacing each variable x in φ with x^λ, and φ^Y is the result of binding all quantifiers in φ to Y.

Proof. A simple induction on the complexity of φ. □

Theorem 16 (ZFC2). *If there is a two world Kripke model for RMST in $\mathcal{L}_\in^{\lambda+1}$, then:*

$$\exists j : V_{\kappa^\mathcal{M}+\lambda} \prec V_{\kappa^\mathcal{N}+\lambda}, \text{ with } crit(j) = \kappa^\mathcal{M}$$

Proof. Suppose $\langle \mathcal{M}, \mathcal{N} \rangle$ is a two world Kripke model for RMST in $\mathcal{L}_{\in}^{\lambda+1}$. By lemmas 24 and 25, let j be an isomorphism from $V_{\kappa^{\mathcal{M}}+\lambda}$ to an extensional core $Y \in X_{\lambda+1}^{\mathcal{M}}$. By lemma 26:

$$V_{\kappa^{\mathcal{M}}+\lambda} \vDash \varphi(\vec{x}) \leftrightarrow \mathcal{M} \vDash (\varphi^{\lambda})^{Y}(j(\vec{x}))$$

and by R:

$$\mathcal{M} \vDash (\varphi^{\lambda})^{Y}(j(\vec{x})) \leftrightarrow \mathcal{N} \vDash (\varphi^{\lambda})^{Y}(j(\vec{x}))$$

Finally, note that the fact Y is an extensional core can be expressed using sentences in $\mathcal{L}_{\in}^{\lambda}$. For example, condition (iii) can expressed by $\forall x^{\beta+1} \exists y^{\beta+1} \in Y \forall z^{\beta}(z^{\beta} \in y^{\beta} \leftrightarrow z^{\beta} \in x^{\beta+1})$ for all $\beta < \lambda$. It follows from R that Y is an extensional core of $\mathcal{N}$ restricted to λ. So by lemma 24 let i be an isomorphism from Y in $\mathcal{N}$ to $V_{\kappa^{\mathcal{N}}+\lambda}$. Then, by lemma 25:

$$\mathcal{N} \vDash (\varphi^{\lambda})^{Y}(j(\vec{x})) \leftrightarrow V_{\kappa^{\mathcal{N}}+\lambda} \vDash \varphi(i(j(\vec{x})))$$

Thus $i \circ j$ is an elementary embedding from $V_{\kappa^{\mathcal{M}}+\lambda}$ to $V_{\kappa^{\mathcal{N}}+\lambda}$ and it is easy to check that $\text{crit}(i \circ j) = \kappa^{\mathcal{M}}$. $\qquad \square$

Just as $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}}$ is definable over $V_{\kappa^{\mathcal{M}}}$, $\mathcal{L}_{\in}^{\kappa^{\mathcal{M}}+\kappa^{\mathcal{M}}+1}$ is definable over $V_{\kappa^{\mathcal{M}}}$. So theorem 16 shows that the problem persists in the absence of extensionality.

3.3.4   When does a condition determine a set/concept?

As I pointed out in chapter 1, there are a number of general and pressing questions which arise in the language of first-order set theory. For example, one upshot of the set-theoretic paradoxes is that the question:

(1) When does a condition φ determine a set?

[*In other words:* Given any condition φ, when is there a set x such that

$\forall y(y \in x \leftrightarrow \varphi)$?]

requires a more sophisticated answer than "never" and "always". Once we add the modal operator $\Diamond$ to that language, such questions bifurcate. In addition to (1), for example, we have:

(1*) When could there have been a set of all possible φ's?

[*In other words:* Given any condition φ, when could there have been a set x such that $\Box\forall y(y \in x \leftrightarrow \varphi)$?]

Analogous questions arise in the language of second-order set theory and its extension with $\Diamond$ and @. For example, we have:

(2) When is there a concept which applies to all and only the φ's?

[*In other words:* Given any condition φ, when is there a concept F such that $\forall x(x \in F \leftrightarrow \varphi)$?]

(2*) When could there have been a concept which necessarily applies to all and only the φ's?

[*In other words:* Given any condition φ, when could there have been a concept F such that $\Box\forall x(x \in F \leftrightarrow \varphi)$?]

(2**) When is there a concept which necessarily applies to all and only the φ's?

[*In other words:* Given any condition φ, when is there a concept F such that $\Box \forall x (x \in F \leftrightarrow \varphi)$?]

Since RMST includes C-COMP, it will answer question (2) "always". In the rest of the chapter, I want to look at how RMST fairs with respect to the other questions.

When does a condition determine a set?

In chapter 1 I noted that the potentialist theory PMST can answer question (1*) using plural resources. In particular, it implies that there could have been set of all possible φ's just in case there could have been a plurality of all possible φ's (theorem 2). As I noted, this begs the question: when could there have been a plurality of all possible φ's. A natural answer can be given on the basis of a general conception of pluralities as nothing over and above the things they comprise. In particular, it will follow from such a conception that there could have been a plurality of all possible φ's if the possible φ's could have all co-existed. But even expressing this requires resources which go beyond PMST. In effect, it requires the ability to cross-reference worlds; to say that there is a world w such that anything which is φ in any world accessible from w already exists at w. For example, the backtracking operators in Hodes (1984) would suffice (see the next section for discussion).

I also noted that PMST does not provide an answer to (1). Although it implies that there could have been all the finite ordinals, it is consistent with there only being the empty set or only 7. But once we have the ability to cross-reference

worlds, a neat answer to (1) becomes available. We can say that there is a set of the φ's just in case all the φ's had previously co-existed. Formally: $w \vDash \exists x \forall y (y \in x \leftrightarrow \varphi)$ just in case there is some $w' < w$ such that every y which exists and is φ at w already exists at w'.[33]

The problem for RMST is that this answer to (1) is not available. To see this, note that since no world strictly accesses the actual world, the answer would imply that actually there are no sets. But as theorem 12 shows, there are many sets which actually exist according to RMST. Indeed, the actually existing sets satisfy ZF2. Given this, it is unclear how a proponent of RMST might give a satisfactory answer to (1).[34]

When could there have been a concept which necessarily applies to all and only the φ's?

Recall from chapter 1 that the modalisation of φ is the result of prefixing all of its universal quantifiers with $\square$ and all of its existential quantifiers with $\lozenge$. One of the central results concerning PMST was that it proves the modalisations of many of the axioms of Zermelo set theory (theorem 1). It turns out that RMST proves something much stronger. In particular, it proves that every formula in $\mathcal{L}_\in^2$ is equivalent to its modalisation and it thus proves the modalisations of the

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[33]As I pointed out, this in turn can be seen to follow from COLLAPSE$^\lozenge$ together with a principle of priority – which says if $x \in \text{dom}(w)$, then there is some other world w' which accesses such that $x \subseteq \text{dom}(w')$ – and a principle of maximality – which says that if $x \subseteq \text{dom}(w)$ and w accesses w', then $x \in \text{dom}(w')$.

[34]In laying down axioms for the actuality operator, I assumed that the actual world accesses every world and is thus not strictly accessed by any other world. This was expresses by A8; i.e. @$\square\varphi \to \varphi$. But we might want to deny that assumption and just modify the axioms accordingly. However, this would not help with the closely related question: which sets actually exist and why? RMST's commitment to a plethora of actually existing sets makes it hard to see how a proponent could answer this question.

axioms of ZF2 + SCR by theorem 12. More precisely, we have:

Theorem 17 (RMST). *Suppose $E\vec{\mathbf{x}}$. Then:*

$$\varphi \leftrightarrow \varphi^\Diamond$$

where $\varphi \in \mathcal{L}_\in^2$ with free variables among $\vec{\mathbf{x}}$.

Proof. By induction on the complexity of φ. The base, conjunction, and negation cases are trivial. So suppose $E\vec{\mathbf{x}}$ and $\exists\mathbf{y}\varphi$. Then $\exists\mathbf{y}\varphi^\Diamond$ by the induction hypothesis and thus $\Diamond\exists\mathbf{y}\varphi^\Diamond = (\exists\mathbf{y}\varphi)^\Diamond$. Conversely, suppose $E\vec{\mathbf{x}}$ and $\Diamond\exists\mathbf{y}\varphi^\Diamond$. Then $\Box E\vec{\mathbf{x}}$ by CBF and thus $\Diamond(E\vec{\mathbf{x}} \wedge \exists\mathbf{y}\varphi^\Diamond)$. It follows that $\Diamond\exists\mathbf{y}\varphi$ by the induction hypothesis and then $\exists\mathbf{y}\varphi$ by R. □

Given C-COMP, it is an immediate corollary that:

$$(\text{C-COMP}^\Diamond) \qquad\qquad \Box\forall\vec{\mathbf{x}}\Diamond\exists F\Box\forall x(x \in F \leftrightarrow \varphi)$$

where φ is either in $\mathcal{L}_\in^2$ or the modalisation of a formula in $\mathcal{L}_\in^2$. In other words, RMST places a significant constraint on any answer to the question "when could there have been a concept which necessarily applies to all and only the φ's?" It tells us that there could have been such a concept for any φ in $\mathcal{L}_\in^2$ and any modalisation of any φ in $\mathcal{L}_\in^2$.

I will now argue that although a proponent of RMST is committed to all of those instances of C-COMP$^\Diamond$, there are formulas in natural extensions of $\mathcal{L}_\Diamond$ for which C-COMP$^\Diamond$ fails. To do this, I will make limited use of the backtacking operators of Hodes (1984) which, as mentioned above, may be needed anyway in order

to give a satisfactory answer to question (1^*).[35] For my purposes, the following remarks should suffice. A backtracking operator is a downward arrow $\downarrow$ which attaches to formulas as $\Diamond$ does. It is intended to 'refer back' from within the scope of modal operators to what had been the case. For example, the formula $\Box \forall x(\varphi \rightarrow \downarrow Ex)$ will be true just in case anything which could have been φ already exists. Similarly, we can use $\downarrow$ to formulate a generalisation of @-COLLAPSE$^\Diamond$:

$$(\text{COLLAPSE}^{\Diamond\downarrow}) \qquad \forall \vec{\mathbf{x}} \Diamond [\downarrow E\vec{\mathbf{x}} \wedge \exists x \forall y(y \in x \leftrightarrow \downarrow Ey \wedge \varphi)]$$

COLLAPSE$^{\Diamond\downarrow}$ says that given any sets and concepts it could have been the case that they had existed and there is a set of all the φ's which had existed. Since the potentialist is committed to this generalisation of @-COLLAPSE$^\Diamond$, I will assume it in what follows. I will also assume the following axioms and rules governing $\downarrow$, which are clearly true and truth preserving on the intended interpretation.

(A6*) $\downarrow (\varphi \rightarrow \psi) \leftrightarrow (\downarrow \varphi \rightarrow \downarrow \psi)$

(A7*) $\downarrow \Box \varphi \rightarrow \varphi$

The rules are $\downarrow$NEC, if φ is a theorem, then so is $\downarrow \varphi$; and $\downarrow \forall$NEC, if $\varphi \rightarrow \downarrow \psi$ is a theorem, then so is $\varphi \rightarrow \downarrow \forall x \psi$ when x does not occur free in φ.

My strategy will be to use $\downarrow$ to define a relation R which obeys the following existence and uniqueness claims.

(E) $\forall F \Diamond \exists x R(F, x)$

(U) $\forall F, G(\exists x[R(F, x) \wedge R(G, x)] \rightarrow \forall x(x \in F \leftrightarrow x \in G))$

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[35]See Payne (2015) for discussion and a nice proof theory for $\downarrow$.

In essence, E and U say that R is an injection from the concepts into the sets.

By a Russell style argument, E and U can be shown to be inconsistent with the following instance of C-COMP$^\Diamond$:

$$\Diamond \exists F \Box \forall x (x \in F \leftrightarrow \exists G(R(G,x) \wedge x \notin G))$$

To see this, suppose that EF is a witness; that is:

$$\Box \forall x (x \in F \leftrightarrow \exists G(R(G,x) \wedge x \notin G))$$

By E and CBF:

$$\Diamond(EF \wedge \exists x R(F,x) \wedge \forall x (x \in F \leftrightarrow \exists G(R(G,x) \wedge x \notin G)))$$

Then $\Diamond(x \in F \leftrightarrow x \notin F)$. Contradiction.

I will now define my relation R. Roughly, it will map a concept to the set it had been co-extensive with. Formally:

$$R(F,x) =_{df} \downarrow EF \wedge \downarrow \forall y (y \in F \leftrightarrow y \in x)$$

To see that E holds for R, let EF. By COLLAPSE$^{\Diamond\downarrow}$, it follows that:

$$\Diamond[\downarrow EF \wedge \exists x \forall y (y \in x \leftrightarrow \downarrow Ey \wedge y \in F)]$$

Let $\downarrow EF$, Ex, and $\forall y(y \in x \leftrightarrow \downarrow Ey \wedge y \in F)$. Then we want to show that $\downarrow \forall y(y \in F \leftrightarrow y \in x)$. So suppose that $\downarrow Ey$. Then Ey by CBF and A7*. So

$y \in F \leftrightarrow y \in x$. By R and A7*, $y \in F \leftrightarrow\downarrow y \in F$ and by STAB$_\in$ and A7*,
$y \in x \leftrightarrow\downarrow y \in x$. Thus, $\downarrow \forall y(y \in F \leftrightarrow y \in x)$ as required.

To see that U holds for R, let EF, G, x and suppose that $\downarrow EF, G, \downarrow \forall y(y \in F \leftrightarrow y \in x)$, and $\downarrow \forall y(y \in G \leftrightarrow y \in x)$. Then, $\downarrow \forall y(y \in F \leftrightarrow y \in G)$ and so $\forall y(y \in F \leftrightarrow y \in G)$ by R and A7*.

It is unclear how a proponent of RMST might answer the question of this section in a way which explains why C-COMP$^\diamond$ holds for all formulas in $\mathcal{L}_\in^2$ and modalisations thereof but not for $\exists G(R(G, x) \wedge x \notin G)$.

When is there a concept which necessarily applies to all and only the φ's?

As with the previous question, RMST puts two severe constraints on any answer to the question: when is there a concept which necessarily applies to all and only the φ's? The first constraint is given by C-COMP. It tells us that for any condition whatsoever, there is always a concept which applies to all and only the things satisfying that condition. In particular, for any possible set of actually existing sets, there is actually a concept which applies to all and only its elements. Formally:

$$\Box \forall x @ \exists F \forall y(y \in F \leftrightarrow y \in x)$$

Such instances of C-COMP were integral to the proofs of Powerset in theorem 12 and SCR in theorem 13, and appear to be necessary for RMST's strength.[36]

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[36]For example, let κ be an inaccessible cardinal and let $\langle V_\alpha, X \rangle$ with $\alpha < \kappa$ and $|X| \leq |V_\alpha|$ be

The second constraint is given by R and @-COLLAPSE$^\diamond$. By @-COLLAPSE$^\diamond$, there could have been a set of all actually existing sets. But there is no actually existing concept which necessarily applies to all and only the actually existing sets. For if there were, it would necessarily apply to every set by R. But @-COLLAPSE$^\diamond$ implies that there could have been a set which does not actually exist. So it would both apply and not apply to such a set. In general, there could have been many sets for which there is no actually existing concept which necessarily applies to all and only their members.

There has to be enough actually existing concepts for there to be one for every set of actually existing sets, but there cannot be so many actually existing concepts that there is one necessarily co-extensive with each such set. The problem is that it is unclear how a proponent of RMST might answer the question of this section in a way which explains why there are the former concepts but not the latter.[37]

Conclusion

Some of the criticisms I have considered in this section are easily avoided. A proponent of the modal, modal realist, or model accounts can simply forgo commitment to the universality of set theory and thus avoid its problems. Similarly, a proponent of the modal account can avoid the problems arising from distinctions without a difference. Furthermore, the problem of extendability to inconsistency

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a second-order elementary substructure of $\langle V_\kappa, V_{\kappa+1}\rangle$. A simple Lowenheim-Skolem construction will yield such a pair. Then the associated two world Kripke model will validate all of the axioms of RMST minus C-COMP. But notice that:

(C-COMP*) $$\forall \vec{\mathbf{x}} \exists F \forall y (y \in F \leftrightarrow \varphi)$$

will be validated, where $\varphi \in \mathcal{L}_\diamond$ with free variables among $\vec{\mathbf{x}}$.

[37](Koellner, 2009, p.217) raises a similar problem, which he calls the *problem of tracking*.

does not appear to be uniquely problematic for the modal account, but rather targets all reflection principles equally. The problems of comprehension, however, do not have analogues for the actualist. For them, as I argued in chapter 1, each of the concept comprehension questions are equivalent and all have a particularly neat answer when concepts are taken to be pluralities; namely, that every condition determines a plurality because pluralities are nothing over and above the things which they comprise.

Although potentialism may do better than actualism on questions like (1^*) and "when could some things have determined a set?",[38] the problems discussed in this section outweigh those benefits. For this reason, the potentialist should not adopt RMST.

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[38]See chapter 1 for discussion.

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