Affine Coxeter groups, involution classes and commuting involution graphs

AFFINE COXETER GROUPS, INVOLUTION CLASSES AND COMMUTING INVOLUTION GRAPHS

A Thesis Submitted to Birkbeck University of London For the Degree of Doctor of Philosophy in Mathematics

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Declaration

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Abstract

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Abstract of thesis submitted by **Amal S. Clarke** and entitled **Affine Coxeter Groups, Involution Conjugacy Classes and Commuting Involution Graphs.** For a group G and X a subset of G the commuting graph of G on X, denoted $\mathcal{C}(G, X)$, is the graph whose vertex set is X where there is an edge joining $x, y \in X$ whenever x commutes with y and $x \neq y$. If the elements of X are involutions, then $\mathcal{C}(G, X)$ is called a commuting involution graph. In this thesis, we investigate conjugacy classes of involutions, studying the connectedness of the commuting involution graph and determining the size of the diameter of the connected $\mathcal{C}(G, X)$, where X is a conjugacy class of involutions of G and G is an affine Coxeter group. We show that if G is of type $\tilde{C}_n$, $\tilde{B}_n$ or $\tilde{D}_n$ and $\mathcal{C}(G, X)$ is connected, then Diam $\mathcal{C}(G, X)$ is at most $n+2$. If G is of type $\tilde{G}_2$, then $\mathcal{C}(G, X)$ is disconnected. If G is of type $\tilde{F}_4$, then $\mathcal{C}(G, X)$ is connected when X is a conjugacy class of $(r_2r_3)^2$ or r_3r_5. Otherwise, it is disconnected. Finally, we examine the connectedness of $\mathcal{C}(W, X)$ where W is an arbitrary Coxeter group, R is its set of simple reflections, and $X = R^W$. In this case we call $\mathcal{C}(W, X)$ the commuting reflection graph of W.

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Throughout my life, my academic ambition has always been more of a struggle with tradition, patriarchal society, and wars back home, than a serene experience at a desk. This thesis is the culmination of this lifetime struggle. Although I have finally managed to fulfil my father's wish, but time has unfortunately beaten me.

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I dedicate this thesis to my father, who made me believe in something bigger than myself.

Contents

Chapter 1

Introduction and Preliminary Results

One of the tools to study the structure of groups has been via their non-commuting and commuting graphs. Scholars have investigated the action of groups on graphs in order to classify them and explore their properties. Consider X a subset of a group G, we define the commuting graph, denoted $\mathcal{C}(G, X)$ to be the graph whose vertex set is X, in which distinct $x, y \in X$ are joined by an edge whenever $xy = yx$.
Brauer and Fowler [12] were the first who examined commuting graphs in order to prove that only finitely many simple groups can contain a given involution centraliser. Commuting involution graphs are used to prove first that if G is a group of even order g greater than 2, then there exists a proper subgroup of order $v > \sqrt[3]{g}$. The commuting graph used in [12] had $X = G/\{1\}$; such graphs have played an important role in results linking to the Margulis and Platanov conjecture: if D is a finite dimensional division algebra, then any finite quotient of the multiplicative group D^x is solvable (see [24]).
On the other hand, some scholars studied the non-commuting graphs where x, y are joined when $xy \neq yx$ to investigate groups. Abdollahi, Akbari and Maimani [1] obtained this important conjecture: if G is a finite simple group and M is a finite group with trivial centre, and the non-commuting graphs of G and M are isomorphic as graphs, then G and M are isomorphic as groups. This conjecture has been proven by Chen [14]. Darafsheh [16] used this conjecture to show that a finite group and the alternating group A_{p+3} of degree $p + 3$ (p is a prime number) are isomorphic if the non-commuting graph of the finite group and non-commuting graph of A_{p+3} are isomorphic. Moreover, Neumann [22] analysed non-commuting graphs, in response to a question posed by Erdös and proved this result: in a group with every subset of pair-

wise non-commuting elements finite, there is a bound on the size of these sets.
In order to study and to characterise finite groups, different types of commuting graphs have been investigated, such as commuting involution graphs whose vertices are involutions. For instance, Bernd Fischer [17] was the first who investigated commuting involution graphs during his study of the 3-transposition groups. The vertices of Fischer's graph had the property that the product of any two involutions had order at most 3. Fischer classified all the finite groups generated by 3-transpositions. A 3-transposition group is a group generated by a conjugacy class D of involutions such that for all $a, b \in D$ the order of ab is either 1,2 or 3. As an example of 3-transposition group where D is a conjugacy class of 3-transpositions of a finite group G, for $d \in D$ define D_d, A_d, to be the set of elements $b, c \in D$ such that db, dc has order $2, 3$ respectively. Let $d \in D$, $H = \langle D_d \rangle$, $c \in A_d$, and $L = \langle D_d \cap D_c \rangle$. Then G is of type $M(22)$ if $Z(G) = 1, H/Z(H) \cong U_6(2)$, and $L/Z(L) \cong PO_6^{+,\pi}(3)$. G is of type $M(23)$ if $Z(G) = 1, H/Z(H)$ is of type $M(22)$, and $L/Z(L) \cong PO_7^{-\pi,\pi}(3)$, Finally G is of type $M(24)$ if $Z(G) = 1, H/Z(H)$ is of type $M(23)$, and $L/Z(L) \cong S_3/P\Gamma_8^+(3)$.

S. Hart, D. Bundy, C. Bates and P. Rowley [6] studied in detail the commuting involution graphs for G a symmetric group of degree n where X is a conjugacy class of involutions, analysed such graphs, their connectedness and came up with results about the size of their diameters. It was shown that $\mathcal{C}(G, X)$ is connected unless $n = 2m + 1$ or $n = 4, m = 1$ (m being the number of 2-cycles in the involution), and that the diameter of $\mathcal{C}(G, X)$ is at most 3 except for three specifically given graphs with diameter 4 (when $n \in \{6, 8, 10\}$). For more details see [6]. Furthermore, for G a finite Coxeter group, they proved a powerful theorem about the sizes of diameters of their commuting involution graphs being at most 5 [7]. This was followed by [8] where G is a special linear group and by [9] where G is a sporadic simple group.
S. Hart [23] analysed the affine Coxeter group of type $\tilde{A}_{n-1}$ and she established two main theorems. In this group involutions can be considered as so-called labelled permutations, where each involution is a product of m labelled transpositions (with $m \leq 2n$). It turns out that the commuting involution graphs are disconnected if and only if either $n = 2m + 1$, or $m = 1$ with $n \in \{2, 4\}$, and if they are connected, then they have diameter at most 6.
Further commuting graphs of symmetric groups were analysed in [10]. Results in [2], [3] stated the diameters of connected graphs for a ring. The commuting involution graphs for the affine Coxeter groups of type $\tilde{C}_n$, $\tilde{B}_n$ and $\tilde{D}_n$ have been studied in this thesis, by taking into consideration the previous results proved in [6], [7] and [23].
In Chapter 2, our group G is an affine Coxeter group of type $\tilde{C}_n$. We determine the conjugacy classes of involutions in order to be able to describe the commuting involution graphs in these classes. This chapter offers key results which are the basis for the remaining chapters. We prove three main theorems: one shows the conditions necessary

for commuting involution graphs to be disconnected and the other two state the Diam $\mathcal{C}(G, X)$ when the commuting involution graphs are connected. Our main result is that when the commuting involution graph is connected, then Diam $\mathcal{C}(G, X)$ is at most $n+2$.

In Chapter 3, G is an affine Coxeter group of type $\tilde{B}_n$ or $\tilde{D}_n$. We know that $W(\tilde{D}_n)$ (The affine weyl group of type $\tilde{D}_n$) is a subgroup of $W(\tilde{B}_n)$ which is a subgroup of $W(\tilde{C}_n)$. Therefore, in this chapter we see that conjugacy classes in G of type $\tilde{C}_n$ split into two or in some cases four conjugacy classes in the affine Coxeter group G_n of type $\tilde{B}_n$ or $\tilde{D}_n$. We state many examples to illustrate our results. Theorems in Chapter 3 derive from theorems in Chapter 2 with adjustments for the split cases; again we have that Diam $\mathcal{C}(G, X)$ is at most $n + 2$ when $\mathcal{C}(G, X)$ is connected.

Chapter 4 explores the conjugacy classes in the affine Coxeter group of type $\tilde{G}_2$. We verify that $\mathcal{C}(G, X)$ is a disconnected graph of involutions. We show that in the exceptional affine Weyl group of type $\tilde{F}_4$ the graph $\mathcal{C}(G, X)$ is connected only in two cases when X is a conjugacy class of $(r_2r_3)^2$ or r_3r_5. Otherwise the graph is disconnected.

In Chapter 5, we have $X = R^W$ is the set of reflections of a Coxeter group W and $\mathcal{C}(W, X)$ is a commuting graph of reflections. We study the connectedness of $\mathcal{C}(W, X)$. We prove that if W is reducible, then $\mathcal{C}(W, X)$ is connected with diameter at most 2. If W is irreducible and $\Gamma(W)$ has diameter at least 3, then $\mathcal{C}(W, X)$ is connected. We show that $\mathcal{C}(W, X)$ is disconnected if W is a finite Coxeter group of type A_2, A_3, B_2, D_4, H_3, dihedral, or if W is an affine Weyl group of type $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{B}_2$ or $\tilde{D}_4$ or G_2. For all other finite Coxeter groups and affine Weyl groups the commuting reflection graph is connected. In the remainder of this chapter we gather together the terminology, definitions and basic results that we will use throughout this thesis.

1.1 Basic Definitions and Context

In the rest of chapter we recall what is meant by **finite and affine Coxeter groups**, their **root systems**, their **coroots** and their sets of **generators**. We also define **finite** and **afine Weyl** groups. Finally, we present **commuting involution graphs** which are thesis's main area of study. For more on Coxeter groups and root systems see [21] or [11].

1.1.1 Reflections and Affine Reflections

Let α be any non-zero element in a real Euclidean space V. The reflection s_α is reflection in H_α, the hyperplane orthogonal to α, $H_\alpha := \{\mathbf{v} \in V : \langle \mathbf{v}, \alpha \rangle = 0\}$. Hence, s_α maps non-zero α to its negative and fixes any vector in H_α. Then s_α is defined as follows. For all $\mathbf{v} \in V$, define

$$s_\alpha(\mathbf{v}) = \mathbf{v} - \tfrac{2\langle \mathbf{v}, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

We can see that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha^2 = 1$, so s_α has order 2 in the group $O(V)$ of all orthogonal transformations of V. Orthogonal reflections fix the origin in V, while affine reflections are relative to hyperplanes which do not necessarily pass through origin. We define an affine hyperplane $H_{\alpha,k} := \{\mathbf{v} \in V : \langle \mathbf{v}, \alpha \rangle = k\}$ and an affine reflection is defined as follows, for all $\mathbf{v} \in V$.

$$s_{\alpha,k}(\mathbf{v}) = \mathbf{v} - \tfrac{2(\langle \mathbf{v}, \alpha \rangle - k)}{\langle \alpha, \alpha \rangle}\alpha$$

We can write $s_{\alpha,k}(\mathbf{v})$ as $s_\alpha(\mathbf{v}) + 2k\frac{\alpha}{\langle \alpha, \alpha \rangle}$.

1.1.2 Roots and Coroots

Let V be a Euclidean vector space and let Φ be a finite subset in V where $\Phi \neq \emptyset$. Then Φ is a root system if the following conditions are satisfied:

(i) Φ is a finite set of V and $0 \notin \Phi$.

(ii) For all $\alpha \in \Phi$, we have $\langle \alpha \rangle \bigcap \Phi = \{\alpha, -\alpha\}$.

(iii) For all $\alpha \in \Phi$, s_α leaves Φ invariant.

Then we define W to be the finite reflection group associated to Φ. That is, W is generated by $\{s_\alpha, \alpha \in \Phi\}$.

Lemma 1.1.1. *There is at least one root system for a finite reflection group of the Euclidean vector space V.*

Proof. Let $\mathbf{T}$ be the set of $\alpha \in V$ for which $s_\alpha \in W$. Then we define

$$\Phi = \mathbf{T} \bigcap \{\beta \in V : ||\beta|| = 1\}$$

which is the intersection of $\mathbf{T}$ with the unit sphere.

(i) $0 \notin \Phi$, clearly $||0|| \neq 1$.

(ii) Certainly, we have $||\alpha|| = || - \alpha|| = 1$, then $\alpha, -\alpha \in \Phi$ and $\langle \alpha \rangle \cap \Phi = \{\alpha, -\alpha\}$.

(iii) We know that W is a subgroup of the orthogonal group $O(V)$, then the elements of W are orthogonal which is equivalent to linear Euclidean. Hence, they preserve Euclidean norm.
Now, for all $\alpha \in \Phi$ and $w \in W$ we can see that $w(\alpha) \in \Phi$. We know that

$$ws_\alpha w^{-1}(\mathbf{v}) = w(s_\alpha(w^{-1}\mathbf{v})) = w(w^{-1}(\mathbf{v}) - 2\tfrac{\langle w^{-1}(\mathbf{v}),\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha) =$$
$$\mathbf{v} - 2\tfrac{\langle \mathbf{v}, w(\alpha)\rangle}{\langle w(\alpha), w(\alpha)\rangle} w(\alpha) = s_{w(\alpha)}(\mathbf{v}).$$

for all $\mathbf{v} \in V$. Hence, we have if $ws_\alpha w^{-1} \in W$, then $s_{w(\alpha)} \in W$, which implies $w(\alpha) \in \Phi$. We can see that all the conditions hold. Thus Φ is a root system and W is the finite reflection group associated to Φ.

$\square$

For instance, consider $W = S_n$, acting on $\mathbb{R}^n$ by permuting the subscripts of standard orthonormal basis $e_1, e_2, ..., e_n$. Hence, S_n is a reflection group which is generated by the transpositions $(i\ j)$ where $(i\ j)$ represents the reflection $s_{e_i - e_j}$. Then S_n has a root system

$$\Phi = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$$

and we have $w(\Phi) = \Phi$ for all $w \in S_n$ where $w(e_i - e_j) = e_{w(i)} - e_{w(j)}$. Hence, condition (iii) holds. In addition, we can see that $\langle e_i - e_j\rangle \cap \Phi = \{e_i - e_j, e_j - e_i\}$. Therefore, Φ is a root system.
As another example, consider the dihedral group D_8 acting on $\mathbb{R}^2$. We can take $\Phi = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$. We can see that Φ might have as many elements as W. Therefore, using Φ as a tool to classify the reflection groups is not a good idea without some more work. This leads us to divide Φ into two subsets: one, Φ^+, is the positive root system and the other, Φ^- (definition is given in next page), is the negative root system. Then we have $\Phi = \Phi^+ \bigcup \Phi^-$.
In order to classify positive roots and negative roots we need to recall the definition of total ordering.

Definition 1.1.2. A total ordering denoted by $<$ is a binary relation on V. This relation satisfies the following axioms. For v_1, v_2 and $v_3 \in V$:

(i) Transitivity, if $v_1 < v_2$ and $v_2 < v_3$, then $v_1 < v_3$.

(ii) Trichotomy, we have one of the following holds $v_1 < v_2$ or $v_1 > v_2$ or $v_1 = v_2$.

(iii) Additivity, if $v_1 < v_2$, then $v_1 + v_3 < v_2 + v_3$.

(iv) Multiplication by a real positive number, for $a \in \mathbb{R}^+$ if $v_1 < v_2$, then $av_1 < av_2$.

Lemma 1.1.3. Every finite dimensional real vector space V has a total ordering.

Proof. Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be a basis of V, $\alpha = \sum_{i=1}^n \lambda_i \mathbf{v}_i$, $\beta = \sum_{i=1}^n \mu_i \mathbf{v}_i \in V$ and $\gamma = \sum_{i=1}^n \delta_i \mathbf{v}_i \in V$. Then we shall define the relation $\alpha < \beta$ if for the first i with $\mu_i \neq \lambda_i$ we have $\mu_i - \lambda_i > 0$.

(i) If $\alpha < \beta$ and $\beta < \gamma$, then there is $j \leq n$ which is the first j such that $\mu_j - \lambda_j > 0$ and $k \leq n$ which is the first k such that $\delta_k - \mu_k > 0$.
Let $l = \min\ \{j, k\}$. Then, if $l = j$, we have $\mu_l - \lambda_l > 0$ and $\delta_l - \mu_l \geq 0$ that implies $\delta_l - \lambda_l > 0$ and we shall have same argument if $l = k$; and so transitivity holds.

(ii) If $\alpha \neq \beta$, then there is a smallest i such that $\mu_i - \lambda_i \neq 0$. Then either $\mu_i - \lambda_i > 0$ or $\mu_i - \lambda_i < 0$. Hence, trichotomy holds.

(iii) If we have the first i such that $\mu_i - \lambda_i > 0$; that implies $\mu_i - \lambda_i + \delta_i - \delta_i > 0$, then $\mu_i + \delta_i - (\delta_i + \lambda_i) > 0$ and $\alpha + \gamma < \gamma + \beta$, therefore additivity holds.

(iv) if we have $\alpha < \beta$ and $a \in \mathbb{R}^+$, then $\mu_i - \lambda_i > 0$ implies $a(\mu_i - \lambda_i) > 0$. Whence $a\mu_i - a\lambda_i > 0$ and $a\alpha < a\beta$. Hence, any finite dimensional vector space has a total ordering.

□

Now, Φ^+ is the subset of Φ consisting of all those roots which are positive (≥ 0) relative to some total ordering of V. Let Δ be a linearly independent subset of Φ^+. Then Δ is *a simple system*, if for all $\beta \in \Phi$, β is a linear combination of the elements of Δ where $\beta = \sum_{\alpha \in \Delta} \lambda_\alpha \alpha$ with either all λ_α nonpositive or all λ_α non-negative. Now recall Theorem 1.5 in [21].

Theorem 1.1.4. *For a fixed simple system Δ, W is generated by s_α with $\alpha \in \Delta$.*

Then we have

$$\begin{aligned} \Phi^+ &:= \{\textstyle\sum_{\alpha_i} \delta_i \alpha_i \in \Phi : \delta_i \geq 0 \text{ for all } \alpha_i \in \Delta\}. \\ \Phi^- &:= \{\textstyle\sum_{\alpha_i} \delta_i \alpha_i \in \Phi : \delta_i \leq 0 \text{ for all } \alpha_i \in \Delta\}. \end{aligned}$$

It can be shown that every positive system contains a unique simple system. The set of fundamental reflections is given by $R = \{r_i = s_{\alpha_i} : \alpha_i \in \Delta\}$ and any reflection of W is conjugate to a fundamental reflection. Then, we have $W = \{w^{-1} r_i w : r_i \in R, w \in W\}$. Finally, the set of coroots $\Phi^\vee$ is given by $\Phi^\vee = \{\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}, \alpha \in \Phi\}$.

Definition 1.1.5. Suppose Φ is a root system. We say that Φ is crystallographic if for all α, $\beta \in \Delta$, we have $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.
The significance of this is that $s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$, meaning every root is an *integer* linear combination of Δ.
The coroot lattice $L(\Phi^\vee)$ is the $\mathbb{Z}$-span of $\Phi^\vee$ in V, the lattice generated by $\Phi^\vee$. So $L(\Phi^\vee) = \{\lambda \alpha^\vee, \lambda \in \mathbb{Z}, \alpha^\vee \in \Phi^\vee\}$.

1.2 Coxeter Groups and their Root Systems

At the beginning of this section, it is important to state that a finite reflection group is generated by orthogonal reflections and an affine reflection group is generated by affine reflections in Euclidean space. It can be shown that every finite reflection group is of the form $W = \langle R \rangle$, where R is a set of fundamental reflections, subject only to relations of the form $(rs)^m = 1$ for $r, s \in R$.

1.2.1 Finite Weyl Groups

Finite Weyl groups form a special class of finite reflection groups whose associated root system is crystallographic. It can be shown that this is equivalent to the product of any two generators having order $2, 3, 4$ or 6 Proposition 2.8 in [21] . Furthermore, finite reflection groups are an example of a broader class of groups known as Coxeter groups.

1.2.2 Coxeter Groups

We define a Coxeter system to be a pair (W, R) where W is a group (called a Coxeter group) and $R = \{r_1, ..., r_n\}$ is a set of generators known as the fundamental, or simple, reflections in W, subject only to relations of the form $(r_i r_j)^{m_{r_i r_j}} = 1$, where $m_{r_i r_j} = m_{r_j r_i} \geq 2$ for $1 \leq i < j \leq n$ and $m_{r_i r_i} = 1$ for all $i \in \{1, ..., n\}$. In case no relation occurs for a pair (r_i, r_j), we make a convention that $m_{r_i r_j} = \infty$.

Now, any $w \in W$ can be written in the form $w = r_1 r_2 \cdots r_k$ for some $r_i \in R$.
The length of w, denoted $l(w)$, is given by

$$l(w) = \min\,\{k \in \mathbb{N} : w = r_1 r_2 ... r_k \text{ for some } r_i \in R\}.$$

Any expression $r_1 r_2 ... r_k$ for w where $k = l(w)$ is called a reduced expression.
By convention $l(1) = 0$. If $w = s_\alpha$ for $\alpha \in \Delta$, then we have $l(w) = 1$. Clearly, $l(w) = l(w^{-1})$ because $w^{-1} = r_k..r_2 r_1$.
For any subset I of R, the subgroup W_I generated by I is called a standard parabolic subgroup. Conjugates of standard parabolic subgroups are called parabolic subgroups. Any parabolic subgroup of a Coxeter group is itself a Coxeter group (Theorem 5.5 in Chapter 5 of [21]).

Coxeter Groups and their Root and Coroot Systems In this table we state Φ^+ the positive root system, $\Phi^{+\vee}$ the positive coroot system, R the set of generators and Δ the simple system for the finite Coxeter groups of types A_{n-1}, B_n, C_n and D_n. The 'type' notation is defined later.

Coxeter Groups	
Type	Δ
A_{n-1}, $n \geq 2$	$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n\}$
$B_n, n \geq 2$	$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_1\}$
$C_n, n \geq 2$	$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, 2e_1\}$
$D_n, n \geq 4$	$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$
	R
A_{n-1}	$\{s_{e_1-e_2}, \ldots, s_{e_{n-1}-e_n}\}$
$B_n, n \geq 2$	$\{s_{e_1-e_2}, \ldots, s_{e_{n-1}-e_n}, s_{e_1}\}$
$C_n, n \geq 2$	$\{s_{e_1-e_2}, \ldots, s_{e_{n-1}-e_n}, s_{2e_1}\}$
$D_n, n \geq 4$	$\{s_{e_1-e_2}, \ldots, s_{e_{n-1}-e_n}, s_{e_{n-1}+e_n}\}$
	Φ^+
A_{n-1}	$\{e_i - e_j; 1 \leq i < j \leq n\}$
B_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\} \bigcup \{e_i, 1 \leq i \leq n\}$
C_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\} \bigcup \{2e_i, 1 \leq i \leq n\}$
D_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\}$
	$\Phi^{+\vee}$
A_{n-1}	$\{e_i - e_j; 1 \leq i < j \leq n\}$
B_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\} \bigcup \{2e_i, 1 \leq i \leq n\}$
C_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\} \bigcup \{e_i, 1 \leq i \leq n\}$
D_n	$\{e_i \pm e_j; 1 \leq i < j \leq n\}$

Here $\{e_1, e_2, ..., e_n\}$ is the standard orthonormal basis for $\mathbb{R}^n$. Note that since $s_{e_1} = s_{2e_1}$, the groups of type B_n and C_n are isomorphic, so we just refer to $W(B_n)$ when looking at finite Coxeter groups.

Let W be of type A_{n-1}, B_n, C_n or D_n. We may view the elements of W as signed permutations; they act on $\mathbb{R}^n$ by permuting the subscripts of basis vectors and changing their signs. For instance, consider the finite Coxeter group $W(A_{n-1})$ of type $A_{n-1}, n \geq 2$. We have $W(A_{n-1}) \cong S_n$ which acts on $\mathbb{R}^n$ by permuting the standard basis vectors $e_1, e_2, ..., e_n$. For example, applying the formula for the reflection function given in Section 1.1.1 we have

$$s_{e_i-e_j}(e_i) = e_i - 2\frac{\langle e_i-e_j, e_i\rangle}{\langle e_i-e_j, e_i-e_j\rangle}(e_i - e_j) = e_j,$$
$$s_{e_i-e_j}(e_j) = e_j - 2\frac{\langle e_i-e_j, e_j\rangle}{\langle e_i-e_j, e_i-e_j\rangle}(e_i - e_j) = e_i.$$

Then, instead of the notation $w = s_{e_i-e_j}$ we use $w = (\overset{++}{i\ j})$, meaning $w(i) = +j$ and $w(j) = +i$.

Moreover,

$$s_{e_i+e_j}(e_i) = e_i - 2\frac{\langle e_i+e_j, e_i\rangle}{\langle e_i+e_j, e_i+e_j\rangle}(e_i + e_j) = -e_j,$$

$$s_{e_i+e_j}(e_j) = e_j - 2\frac{\langle e_i+e_j, e_j\rangle}{\langle e_i+e_j, e_i+e_j\rangle}(e_i + e_j) = -e_i.$$

In $W(B_n)$ and $W(D_n)$ the notation $w = s_{e_i+e_j}$ is replaced by $(\overset{-}{i}\ \overset{-}{j})$, meaning $w(i) = -j$ and $w(j) = -i$. Now, we have

$$s_{e_i}(e_i) = e_i - 2\frac{\langle e_i, e_i\rangle}{\langle e_i, e_i\rangle}(e_i) = -e_i,$$
$$s_{e_i}(e_j) = e_j - 2\frac{\langle e_j, e_i\rangle}{\langle e_i, e_i\rangle}(e_i) = e_j.$$

Hence, to obtain a signed permutation we write a permutation in $\mathrm{Sym}(n)$ (including 1-cycles), add a plus sign or a minus sign above each i, and say i is positive or negative accordingly. We adopt the convention of reading the sign first; for example, if $w = (\overset{-}{1}\ \overset{+}{2}\ \overset{-}{3}) \in W$, then $1^w = -2$, $2^w = 3$ and $3^w = -1$. This means, for example, that $w(\lambda e_1 + \mu e_2 + \rho e_3) = -\rho e_1 - \lambda e_2 + \mu e_3$.
Expressing σ as a product of disjoint cycles, we say that a cycle $(i_1 \cdots i_r)$ of σ is *positive* if there is an even number of minus signs above its elements, and *negative* if the number of minus signs is odd. For example, $(\overset{+}{1}\ \overset{+}{3}\ \overset{-}{2})$ is a negative cycle, whereas $(\overset{-}{5}\ \overset{-}{6})$ is positive. It is straightforward to check that an involution of W only has 1-cycles (positive or negative) and positive 2-cycles.

1.2.3 Affine Weyl Groups

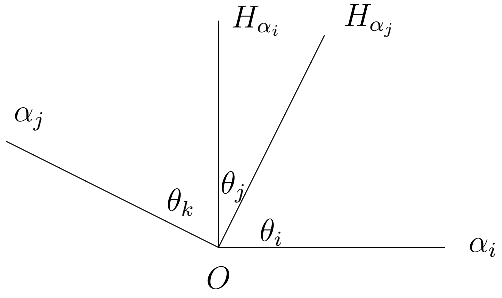
An affine Weyl group denoted by $\tilde{W}$ is defined to be the semidirect product of a finite Weyl group W with the translation group Z of the coroot lattice $L(\Phi^\vee)$ of W, where if $w \in \tilde{W}$, then $w = (\sigma, \mathbf{v})$ with $\sigma \in W$ and $\mathbf{v}$ is an element of the translation group Z. We usually write translations as row vectors in $\mathbb{R}^n$ with respect to the standard basis. It can be shown that affine Weyl groups are Coxeter groups. The reflections are $\{s_{\alpha,k} : \alpha \in \Phi, k \in \mathbb{Z}\}$.
Now, let W be a finite Weyl group with root system Φ, with $\Phi^\vee = \{\alpha^\vee = 2\frac{\alpha}{\langle\alpha,\alpha\rangle}, \alpha \in \Phi\}$ the set of coroots. If R is a set of simple reflections for W and $\tilde{\alpha}$ is the highest root (i.e. root with highest coefficient sum), then it can be shown that $R \cup \{s_{\tilde{\alpha},1}\}$ is the set of simple reflections for $\tilde{W}$ Proposition 4.2 [21].

1.2.4 Coxeter Graphs

Let (W, R) be a finite Coxeter system and V a vector space over $\mathbb{R}$ with basis $\{\alpha_r, r \in R\}$ where R is the set of fundamental reflections of the Coxeter group W. Hence, $\forall r_i, r_j \in R$ we have $(r_i r_j)^m = 1$, for some $m_{r_i r_j}$ where $m_{r_i r_j}$ can be determined from the angle θ between α_{r_i} and α_{r_j}. We know that the product of reflections in hyperplanes at an angle θ is a rotation through 2θ which will have order $2\pi/2\theta$ (This is true in this case where $\theta = \frac{\pi}{m}$ but not in general).
By Corollary 1.3 of [21], if Δ is a simple system in Φ, then $\langle\alpha, \beta\rangle \leq 0$ for all $\alpha \neq \beta$ in Δ. We have $\langle\alpha_{r_i}, \alpha_{r_j}\rangle = ||\alpha_{r_i}||.||\alpha_{r_j}|| \cos\theta$, meaning $||\alpha_{r_i}||.||\alpha_{r_j}|| \cos\theta \leq 0$ which implies

that $\frac{\pi}{2} \leq \theta < \pi$. Since α_{r_i} and α_{r_j} have an angle θ between them, by the calculation below we see that $\pi - \theta$ is the angle between H_{α_i} and H_{α_j}. Then, $m_{r_i r_j} = \frac{\pi}{\pi-\theta}$. So, we have $\theta = \pi - \frac{\pi}{m_{r_i r_j}}$. Therefore, $\cos\theta = -\cos\frac{\pi}{m_{r_i r_j}}$.

To verify this result, we draw a diagram to show the geometric configuration of the plane containing vectors α_i, α_j, H_{α_i} and H_{α_j}which are their hyperplanes.



We have that α_i and H_{α_i} are orthogonal, so

$$\theta_i + \theta_j = \frac{\pi}{2}, \tag{1.1}$$

$$\theta_j + \theta_k = \frac{\pi}{2}, \tag{1.2}$$

because α_j and H_{α_j} are also orthogonal. Let θ be the angle between the two lines α_i and α_j. Then $\theta = \theta_i + \theta_j + \theta_k$. Adding the two equations(1.1) and (1.2) we have $2\theta_j + \theta_i + \theta_k = \pi$. Hence, $\theta_j = \pi - \theta$, where θ_j is the angle between H_{α_i} and H_{α_j}which is $\pi - \theta$. Therefore, we can see that the order of $s_{\alpha_i} s_{\alpha_j}$, $m = \frac{\pi}{\pi-\theta}$. Thus $\theta = \pi - \frac{\pi}{m}$.

There is a useful graphical representation of the Coxeter group W: a Coxeter graph $\Gamma = \Gamma(W)$. The vertex set of Γ is R, where vertices r, s are joined by an edge labelled m_{rs} whenever $m_{rs} > 2$. By convention the label is omitted when $m_{rs} = 3$. We say that the Coxeter system (W, R) is irreducible if the Coxeter graph $\Gamma(W)$ is connected.
For the Coxeter group S_n of type A_{n-1}, we have $R = \{s_{e_1 - e_2}, ..., s_{e_{n-1}-e_n}\}$. Therefore, by applying the following relation

$$\cos\theta = -\cos\tfrac{\pi}{m_{r_i r_{i+1}}} = -\tfrac{\langle \alpha_{r_i}, \alpha_{r_{i+1}} \rangle}{||\alpha_{r_i}||.||\alpha_{r_{i+1}}||},$$
$$\cos\tfrac{\pi}{m_{r_i r_{i+1}}} = -\tfrac{\langle e_{i-1}-e_i, e_i - e_{i+1} \rangle}{||e_{i-1}-e_i||e_i - e_{i+1}||} = -\tfrac{-1}{2} = \tfrac{1}{2},$$

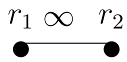
we know that $\cos\frac{\pi}{3} = \frac{1}{2}$. Hence, $m_{r_i r_{i+1}} = 3$, so there is an edge between adjacent vertices and the label 3 is omitted by convention. Now,

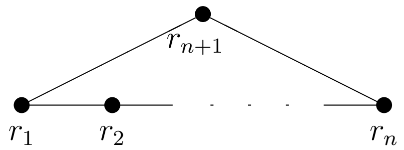
$$\cos\tfrac{\pi}{m_{r_i r_j}} = -\tfrac{\langle \alpha_{r_i}, \alpha_{r_j} \rangle}{||\alpha_{r_i}||.||\alpha_{r_j}||},$$
$$\cos\tfrac{\pi}{m_{r_i r_j}} = -\tfrac{\langle e_i - e_{i+1}, e_j - e_{j+1} \rangle}{||e_i - e_{i+1}||e_j - e_{j+1}||} = -\tfrac{0}{2} = 0,$$

and we know that $\cos \frac{\pi}{2} = 0$ where $i \neq j$. Then, $m_{r_i r_j} = 2$ and there is no edge between nonadjacent vertices. Hence, in the Coxeter graph $\Gamma(A_{n-1})$ all the edges have label 3, then the labels are omitted by convention. So, the Coxeter graph of S_n is of type $A_{n-1}, n \geq 2$ is as follows:

$A_{n-1}(n \geq 2)$ 

It can be shown that the Coxeter graphs of $\tilde{A}_1, \tilde{A}_{n-1}$ are as follows:





We may set $r_1 = ((1\ 2), 0)$, $r_i = ((i\ i+1), 0))$, for $2 \leq i \leq n-1$ and the highest root is $(e_1 - e_2) + (e_2 - e_3) + \cdots + (e_{n-1} - e_n) = (e_1 - e_n)$. Then, $r_n = ((1\ n), e_1 - e_n)$ Now, for the Coxeter graph $\Gamma(B_n)$ which has a vertex set $R = \{s_{e_1 - e_2}, ..., s_{e_{n-1} - e_n}, s_{e_1}\}$, we have

$$\cos \frac{\pi}{m_{r_i r_{i+1}}} = -\frac{\langle \alpha_{r_i}, \alpha_{r_{i+1}} \rangle}{||\alpha_{r_i}||.||\alpha_{r_{i+1}}||},$$
$$\cos \frac{\pi}{m_{r_i r_{i+1}}} = -\frac{\langle e_{i-1} - e_i, e_i - e_{i+1} \rangle}{||e_{i-1} - e_i||e_i - e_{i+1}||} = -\frac{-1}{2} = \frac{1}{2},$$

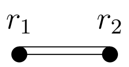
and we know that $cos\frac{\pi}{3} = \frac{1}{2}$. Then, $m_{r_i r_{i+1}} = 3$. Therefore, the label 3 is omitted by convention. Now, for

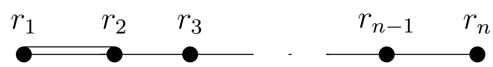
$$\cos \frac{\pi}{m_{r_i r_{i+2}}} = -\frac{\langle \alpha_{r_i}, \alpha_{r_{i+2}} \rangle}{||\alpha_{r_i}||.||\alpha_{r_{i+2}}||},$$
$$\cos \frac{\pi}{m_{r_i r_{i+2}}} = -\frac{\langle e_{i-1} - e_i, e_{i+1} - e_{i+2} \rangle}{||e_{i-1} - e_i||e_{i+1} - e_{i+2}||} = -\frac{0}{2} = 0,$$

and we know that $\cos \frac{\pi}{2} = 0$. Then, $m_{r_i r_{i+2}} = 2$. We can see that $m_{r_{n-1} r_n} = 4$ where

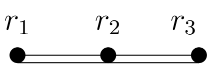
$$\cos \frac{\pi}{m_{r_{n-1} r_n}} = -\frac{\langle e_{n-1} - e_n, e_n \rangle}{||e_{n-1} - e_n||e_n||} = -\frac{-1}{\sqrt{2}} = \cos \frac{\pi}{4}.$$

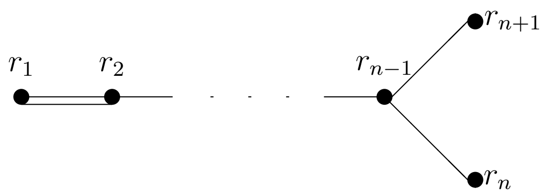
Hence, we conclude that in the Coxeter graph $\Gamma(B_n)$ the labels $m_{r_i r_{i+1}} = 3$ are omitted for $2 \leq i \leq n$ and the label $m_{r_1 r_2} = 4$ which is represented by two edges. Hence, the Coxeter graph of B_2 and B_n, $n \geq 3$ are as follows. Note that C_n and B_n have the same graphs.

B_2 

$B_n(n \geq 3)$ 

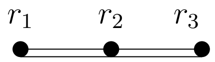
The Coxeter graphs of $\tilde{B}_2$ and $\tilde{B}_n$, $n \geq 3$ are as follows.

$\tilde{B}_2$ 

$\tilde{B}_n(n \geq 3)$ 

The Coxeter graphs of $\tilde{C}_2$ and $\tilde{C}_n$, $n \geq 3$ are as follows.

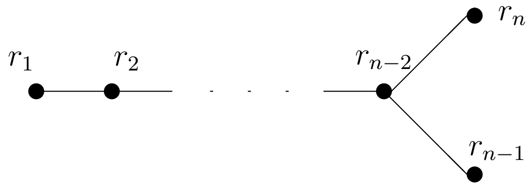
$\tilde{C}_2$



$\tilde{C}_n(n \geq 3)$

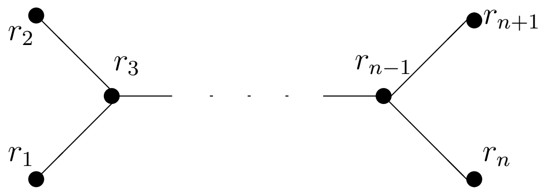
The Coxeter graph D_n, $n \geq 4$ is as follows.

$D_n(n \geq 4)$



The Coxeter graph $\tilde{D}_n$, $n \geq 4$ is as follows.

$\tilde{D}_n(n \geq 4)$



1.3 Involutions in Coxeter Groups

Our first job is to establish what involutions in affine Weyl groups look like. Let W be a finite Weyl group with root system Φ and $\Phi^\vee$ the set of coroots. The affine Weyl group $\tilde{W}$ is the semidirect product of W with translation group Z of the coroot lattice $L(\Phi^\vee)$. For any $w \in \tilde{W}$, w is written of the form $(\sigma, \mathbf{v})$ where $\sigma \in W$ and $\mathbf{v} = (v_1, v_2, ..., v_n) \in Z$. See, for example, [21, Chapter 4] for more detail. For $\sigma, \tau \in W$ and $\mathbf{u}, \mathbf{v} \in Z$ where $\mathbf{v}^\tau$ is a shorthand for $\tau(\mathbf{v})$ we have

$$(\sigma, \mathbf{v})(\tau, \mathbf{u}) = (\sigma\tau, \mathbf{v}^\tau + \mathbf{u}).$$

We have $(\sigma, \mathbf{v})^{-1} = (\sigma^{-1}, -\mathbf{v}^{\sigma^{-1}})$. In $\tilde{W}$, the element $(\sigma, \mathbf{v})$ is conjugate to $(\tau, \mathbf{u})$ via some $(g, \mathbf{w})$ if and only if:

$$\begin{aligned}(\tau, \mathbf{u}) &= (\sigma, \mathbf{v})^{(g,\mathbf{w})} \\ &= (g^{-1}\sigma g, \mathbf{v}^g + \mathbf{w} - \mathbf{w}^{g^{-1}\sigma g}). \end{aligned} \tag{1.3}$$

For the rest of this thesis, let $W(\tilde{C}_n)$ be the affine Weyl group of type $\tilde{C}_n$. As the set of coroots contains $\{e_1, ..., e_n\}$, the coroot lattice here is just $\{\mathbf{v} = (\lambda_1, ..., \lambda_n) : \lambda_i \in \mathbb{Z}\}$. Let $W(\tilde{B}_n)$ be the subgroup of $W(\tilde{C}_n)$ with elements $(\sigma, \mathbf{v})$ such that the coordinate sum of $\mathbf{v}$ is even. Let $W(\tilde{D}_n)$ be the subgroup of $W(\tilde{B}_n)$ whose elements have an even number of minus signs in their signed permutation part. Finally we can take $W(\tilde{A}_{n-1})$ to be the subgroup of elements $(\sigma, \mathbf{v})$ of $W(\tilde{D}_n)$ such that the coordinate sum of $\mathbf{v}$ is zero and σ has no minus signs.

By the definition of group multiplication in $\tilde{W}$, we see that the element $(\sigma, \mathbf{v})$ of $\tilde{W}$ is an involution precisely when $(\sigma^2, \mathbf{v}^\sigma + \mathbf{v}) = (1, \mathbf{0})$. This allows us to characterise the involutions in $\tilde{W}$.

Since $W(\tilde{A}_n) \subset W(\tilde{D}_n) \subset W(\tilde{B}_n) \subset W(\tilde{C}_n)$, we assume for the moment that $\tilde{W} = W(\tilde{C}_n)$.

Lemma 1.3.1. *A non-identity element $(\sigma, \mathbf{v})$ of $W(\tilde{C}_n)$ is an involution if and only if σ, when expressed as a product of disjoint signed cycles, has the form*

$$\sigma = (\overset{+}{a_1}\,\overset{+}{b_1}) \cdots (\overset{+}{a_t}\,\overset{+}{b_t})(\overset{-}{a_{t+1}}\,\overset{-}{b_{t+1}}) \cdots (\overset{-}{a_m}\,\overset{-}{b_m})\ (\overset{-}{c_{2m+1}}) \cdots (\overset{-}{c_{n-l}})\ (\overset{+}{d_{n-l+1}}) \cdots (\overset{+}{d_n})$$

for some a_i, b_i, c_i, d_i, t, m and l; and, writing $\mathbf{v} = (v_1, \ldots, v_n)$, we have $v_{b_i} = -v_{a_i}$ when σ contains $(\overset{+}{a_i}\,\overset{+}{b_i})$, $v_{b_i} = v_{a_i}$ when σ contains $(\overset{-}{a_i}\,\overset{-}{b_i})$, and $v_{d_i} = 0$ for $n - l < i \leq n$.

Proof. If $a = (\sigma, \mathbf{v})$ is an involution, then $a^2 = (\sigma^2, \mathbf{v}^\sigma + \mathbf{v}) = (1, \mathbf{0})$. Thus σ is an involution of W, and so its cycles are all either 1-cycles or positive 2-cycles. Thus σ has the form given in the statement of the lemma. Write $\mathbf{v}^\sigma + \mathbf{v} = (u_1, \ldots, u_n)$. We must have $u_r = 0$ for all r. For $1 \leq i \leq t$, we have $u_{a_i} = u_{b_i} = v_{b_i} + v_{a_i}$. Thus $v_{b_i} = -v_{a_i}$. For $t < i \leq m$, we have $u_{a_i} = -v_{b_i} + v_{a_i}$ and $u_{b_i} = -v_{a_i} + v_{b_i}$. So $v_{b_i} = v_{a_i}$. When $m < i < n - l$ we have $u_{c_i} = -v_{c_i} + v_{c_i} = 0$, so there is no restriction on v_{c_i}. For $n - l < i < n$ we have $u_{d_i} = 2v_{d_i}$, forcing $v_{d_i} = 0$. These are the necessary conditions on σ and $\mathbf{v}$; it is clear that they are also sufficient. ☐

1.3.1 Notation

Let $G = G_n = W(\tilde{C}_n)$. There is a more compact way to write elements of G: for an element $(w, \mathbf{v})$ of G, $\mathbf{v} = (v_1, \ldots, v_n)$, above each signed number i in the expression of w as a product of disjoint signed cycles, we will write v_i. For example, we would write $((\overset{-}{1}\,\overset{+}{2}\,\overset{-}{3}), 4e_1 + 5e_2 + 6e_3)$ as $(\overset{4}{\overset{-}{1}}\,\overset{5}{\overset{+}{2}}\,\overset{6}{\overset{-}{3}})$. This allows us to employ a shorthand for writing involutions of G. Suppose $(\sigma, \mathbf{v})$ is an involution with the form given in Lemma 1.3.1. For transpositions $(\overset{\pm}{a_i}\,\overset{\pm}{b_i})$ of σ, where the number above a_i determines the number above b_i as described in Lemma 1.3.1, we write $(\overset{\lambda}{\overset{+}{a_i}}\,b_i)$ for $(\overset{\lambda}{\overset{+}{a_i}}\,\overset{-\lambda}{\overset{+}{b_i}})$ and $(\overset{\lambda}{\overset{-}{a_i}}\,b_i)$ for $(\overset{\lambda}{\overset{-}{a_i}}\,\overset{\lambda}{\overset{-}{b_i}})$. We will call this the *labelled cycle form* of $(\sigma, \mathbf{v})$. Where it is helpful, we adopt the convention that cycles $(\overset{0}{\overset{+}{d_i}})$ are omitted, as these fix both d_i and v_{d_i}.

In what follows we will be making a lot of use of the labelled cycle form, and so we give a couple of examples of how to multiply elements in this form. Let $w = (\overset{v_1}{\overset{-}{1}}\,\overset{v_2}{\overset{+}{2}}\,\overset{v_3}{\overset{-}{3}})$ and $w' = (\overset{u_1}{\overset{+}{1}})(\overset{u_2}{\overset{-}{2}}\,\overset{u_3}{\overset{-}{3}})$. These represent the elements $((\overset{-}{1}\,\overset{+}{2}\,\overset{-}{3}), v_1e_1 + v_2e_2 + v_3e_3)$ and $((\overset{+}{1})(\overset{-}{2}\,\overset{-}{3}), u_1e_1 + u_2e_2 + u_3e_3)$ respectively, and we would usually write the vector components in coordinate form with respect to the standard basis, so we write $((\overset{-}{1}\,\overset{+}{2}\,\overset{-}{3}),(v_1,v_2,v_3))$ and $((\overset{+}{1})(\overset{-}{2}\,\overset{-}{3}),(u_1,u_2,u_3))$. Recall that multiplication in G is given by $(\alpha,\mathbf{v})(\tau,\mathbf{u})=(\alpha\tau,\mathbf{v}^{\tau}+\mathbf{u})$. So

$$\begin{aligned}
ww' &= ((\overset{-}{1}\,\overset{+}{2}\,\overset{-}{3}),(v_1,v_2,v_3))((\overset{+}{1})(\overset{-}{2}\,\overset{-}{3}),(u_1,u_2,u_3))\\
&= ((\overset{-}{1}\,\overset{+}{2}\,\overset{-}{3})(\overset{+}{1})(\overset{-}{2}\,\overset{-}{3}),(v_1,v_2,v_3)^{(\overset{+}{1})(\overset{-}{2}\,\overset{-}{3})}+(u_1,u_2,u_3))\\
&= ((\overset{+}{1}\,\overset{-}{3})(\overset{-}{2}),(u_1+v_1,u_2-v_3,u_3-v_2))\\
&= (\overset{u_1+v_1}{\overset{+}{1}}\quad\overset{u_3-v_2}{\overset{-}{3}})(\overset{u_2-v_3}{\overset{-}{2}}).
\end{aligned}$$

In simple cases, calculations can be carried out entirely within the labelled cycle form as follows. To multiply, for example, $(\overset{4}{\overset{+}{1}}\overset{5}{\overset{+}{2}})$ and $(\overset{6}{\overset{-}{2}}\overset{7}{\overset{-}{3}})$ in G_3, we first calculate that $(\overset{+}{1}\overset{+}{2})(\overset{-}{2}\overset{-}{3})=(\overset{-}{1}\overset{-}{3}\overset{+}{2})$. Then for each i in $\{1,2,3\}$ we calculate the image of i under $(\overset{-}{2}\overset{-}{3})$ and move the number appearing above i in $(\overset{4}{\overset{+}{1}}\overset{5}{\overset{+}{2}})$ accordingly; to this we add the number appearing above $i^{(\overset{-}{2}\overset{-}{3})}$ in $(\overset{6}{\overset{-}{2}}\overset{7}{\overset{-}{3}})$. For example, the number appearing above 2 in $(\overset{4}{\overset{+}{1}}\overset{5}{\overset{+}{2}})$ is 5; now 2 is mapped to -3 by $(\overset{-}{2}\overset{-}{3})$, so the number above 3 in the labelled cycle form of our product will be $-5+7=2$ (because 7 is the number appearing above 3 in $(\overset{6}{\overset{-}{2}}\overset{7}{\overset{-}{3}})$). Hence $(\overset{4}{\overset{+}{1}}\overset{5}{\overset{+}{2}})(\overset{6}{\overset{-}{2}}\overset{7}{\overset{-}{3}})=(\overset{4}{\overset{-}{1}}\overset{2}{\overset{-}{3}}\overset{6}{\overset{+}{2}})$. The same result is obtained if we convert to the standard form:

$$\begin{aligned}
(\overset{4}{\overset{+}{1}}\overset{5}{\overset{+}{2}})(\overset{6}{\overset{-}{2}}\overset{7}{\overset{-}{3}}) &= ((\overset{+}{1}\overset{+}{2}),(4,5,0))((\overset{-}{2}\overset{-}{3}),(0,6,7))\\
&= ((\overset{+}{1}\overset{+}{2})(\overset{-}{2}\overset{-}{3}),(4,5,0)^{(\overset{-}{2}\overset{-}{3})}+(0,6,7))\\
&= ((\overset{-}{1}\overset{-}{3}\overset{+}{2}),(4,0,-5)+(0,6,7))\\
&= ((\overset{-}{1}\overset{-}{3}\overset{+}{2}),(4,6,2))\\
&= (\overset{4}{\overset{-}{1}}\overset{2}{\overset{-}{3}}\overset{6}{\overset{+}{2}}).
\end{aligned}$$

In the case of involutions, we just have to remember the rule that $(\overset{\lambda}{\overset{+}{a_i\;b_i}})$ means $(\overset{\lambda}{\overset{+}{a_i}}\;\overset{-\lambda}{\overset{+}{b_i}})$ and $(\overset{\lambda}{\overset{-}{a_i\;b_i}})$ means $(\overset{\lambda}{\overset{-}{a_i}}\;\overset{\lambda}{\overset{-}{b_i}})$. Thus, for instance, $(\overset{5}{\overset{+}{1\;2}})(\overset{6}{\overset{-}{3\;4}})\cdot(\overset{7}{\overset{+}{1\;3}})(\overset{0}{\overset{-}{2\;4}})=(\overset{13}{\overset{-}{1}}\;\overset{5}{\overset{-}{4}})(\overset{-6}{\overset{+}{2}}\;\overset{-2}{\overset{+}{3}})$. This product is not an involution, which means that these two involutions do not commute. Some general rules about when involutions commute will be derived using calculations like this in Lemmas 1.5.4 – 1.5.6. In due course, we will give a characterisation of the involution conjugacy classes in terms of *labelled cycle types*.

Definition 1.3.2. Let a be an involution in G_n. The *labelled cycle type* of a is the tuple

(m, k_e, k_o, l), where m is the number of transpositions, k_e is the number of negative 1-cycles with an even number above them, k_o is the number of negative 1-cycles with an odd number above them, and l is the number of positive 1-cycles (fixed points), in the labelled cycle form of a.

For example, the labelled cycle type of $\overset{\overset{0}{+}}{(12)}\ \overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}\overset{\overset{3}{-}}{(5)}\overset{\overset{4}{-}}{(6)}\overset{\overset{3}{-}}{(7)}\overset{\overset{1}{-}}{(8)}\overset{\overset{0}{+}}{(9)}\overset{\overset{0}{+}}{(10)}$ is $(1, 1, 5, 2)$.
It will be useful to record here for reference the characterisation of conjugacy classes in the groups $W(A_{n-1}) \cong S_n, W(B_n)$ and $W(D_n)$.
Elements of $W(B_n)$ and $W(D_n)$ are signed permutations. The 'signed cycle type' of an element is the cycle type with a (+) or a (-) over each cycle, according to whether it is positive or negative (cycles of length 1 must be included).

Theorem 1.3.3. [7]and [23]

(i) Elements of $W(A_{n-1})$ are conjugate if and only if they have the same cycle type.

(ii) Elements of $W(B_n)$ are conjugate if and only if they have the same signed cycle type.

(iii) Conjugacy classes in $W(D_n)$ are parametrised by signed cycle type, with one class for each signed cycle type except in the case where the signed cycle type contains only even length, positive cycles, where there are two classes for each signed cycle type. In this case, elements are conjugate precisely when the number of minus signs (in their expressions as products of signed cycles) is congruent modulo 4.

For example, in $W(D_4)$, $(\overset{+}{1}\ \overset{+}{2})(\overset{+}{3}\ \overset{+}{4})$ is conjugate to $(\overset{-}{1}\ \overset{-}{2})(\overset{-}{3}\ \overset{-}{4})$, but not to $(\overset{-}{1}\ \overset{-}{2})(\overset{+}{3}\ \overset{+}{4})$.
Finally in this section we describe the involution conjugacy classes in $W(\tilde{A}_{n-1})$. See [23] for details.
Since elements of $W(\tilde{A}_{n-1})$ have no minus signs, involutions here contain transpositions $(\overset{+}{a}\ \overset{+}{b})$ and fixed points (positive 1-cycles) only, and the positive 1-cycles must have zeroes above them. We may therefore omit the signs and 1-cycles and write involutions as

$$(a_1 \overset{\lambda_1}{\ } b_1)(a_2 \overset{\lambda_2}{\ } b_2) \cdots (a_m \overset{\lambda_m}{\ } b_m)$$

For example, $(1 \overset{5}{\ } 2)(3 \overset{6}{\ } 4)$= $((1\ 2)(3\ 4), (5, -5, 6, -6))$.

Theorem 1.3.4. *Involutions in $W(\tilde{A}_{n-1})$ are conjugate if and only if they have the same number of transpositions, except in the case where there are no fixed points (meaning n is even and there are $\frac{n}{2}$ transpositions). In this case there are two conjugacy classes, and elements*

$$\prod_{i=1}^{m} (a_i \overset{\lambda_i}{\ } b_i) \text{ and } \prod_{i=1}^{m} (c_i \overset{\mu_i}{\ } d_i)$$

are conjugate if and only if $\sum_{i=1}^{m} \lambda_i \equiv \sum_{i=1}^{m} \mu_i \mod 2$.

For example, in $W(\tilde{A}_3)$, $(1 \overset{0}{\ } 2)(3 \overset{0}{\ } 4)$ and $(1 \overset{1}{\ } 2)(3 \overset{1}{\ } 4)$ are conjugate to each other, but not to $(1 \overset{1}{\ } 2)(3 \overset{0}{\ } 4)$.

1.4 Commuting Involution Graphs

Let X be a subset of a group G. The elements of X constitute the vertices of the commuting graph $\mathcal{C}(G, X)$ where there is an edge joining $x, y \in X$ whenever x commutes with y. In this thesis we study the commuting involution graph $\mathcal{C}(G, X)$ where X is a conjugacy class of involutions of G and G is an affine Coxeter group. For x an element of X and $j \in \mathbb{N}$ we define $\Delta_j(x)$ to be the set of vertices of $\mathcal{C}(G, X)$ which are distance j from x, that is,

$$\Delta_j(x) = \{g \in X : d(x, g) = j\}$$

We write Diam $\mathcal{C}(G, X)$ for the diameter of $\mathcal{C}(G, X)$ when $\mathcal{C}(G, X)$ is a connected graph, in other words the maximum distance $d(x, y)$ between any $x, y \in X$ in the graph. Since conjugation by any group element induces a graph automorphism, we can determine the diameter by fixing any $a \in X$, and then Diam $\mathcal{C}(G, X) = \max\{d(x, a) : x \in X\}$. Hence, our main study is to find the Diam $\mathcal{C}(G, X)$ for a connected graph.
Let X be a conjugacy class of involutions in a Coxeter group W. In order to study the commuting involution graph $\mathcal{C}(W, X)$ we use Theorem 1.4.2 to determine the conjugacy classes of involutions.

Definition 1.4.1. Let W be an arbitrary Coxeter group, with I, J two subsets of R. We say that I, J are W-equivalent if there exists $w \in W$ such that $I^w = J$.

In the next result, we use the notation w_I for the longest element of a finite standard parabolic subgroup W_I.

Theorem 1.4.2 (Richardson [25])**.** *Let W be an arbitrary Coxeter group, with R the set of fundamental reflections. Let $g \in W$ be an involution. Then there exists $I \subseteq R$ such that w_I is central in W_I, and g is conjugate to w_I. In addition, for $I, J \subseteq R$, w_I is conjugate to w_J if and only if I and J are W-equivalent.*

Let W be a finite Weyl group with root system Φ, and let $\tilde{W}$ be the corresponding affine Weyl group, whose set of reflections is $\{s_{\alpha,l},\ \alpha \in \Phi^+ \text{ and } l \in \mathbb{Z}\}$. The next two lemmas give conditions under which reflections w and w' of $\tilde{W}$ commute.

Lemma 1.4.3. *Let $\alpha, \beta \in \Phi$. Then, s_α commutes with s_β if and only if $\langle \alpha, \beta \rangle = 0$ or $\alpha = \pm\beta$.*

Proof Consider $\mathbf{v} \in V$ and $\alpha, \beta \in \Phi$. Then, we have

$$\begin{aligned}
s_\alpha(\mathbf{v}) &= \mathbf{v} - \frac{2\langle \mathbf{v}, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \\
s_\beta s_\alpha(\mathbf{v}) &= \mathbf{v} - \frac{2\langle \mathbf{v}, \beta\rangle}{\langle \beta, \beta\rangle}\beta - \frac{2\langle \mathbf{v}, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha + \frac{4\langle \mathbf{v}, \alpha\rangle\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle}\beta, \\
s_\alpha s_\beta(\mathbf{v}) &= \mathbf{v} - \frac{2\langle \mathbf{v}, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha - \frac{2\langle \mathbf{v}, \beta\rangle}{\langle \beta, \beta\rangle}\beta + \frac{4\langle \mathbf{v}, \beta\rangle\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle\langle \alpha, \alpha\rangle}\alpha.
\end{aligned}$$

Hence, by using the calculations above, $s_\alpha s_\beta(\mathbf{v}) = s_\beta s_\alpha(\mathbf{v})$ when $\frac{4\langle \mathbf{v},\beta\rangle\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle\langle\alpha,\alpha\rangle}\alpha = \frac{4\langle \mathbf{v},\alpha\rangle\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}\beta$. Then, $\alpha = \pm\beta$ or $\langle\alpha,\beta\rangle = 0$. □

Lemma 1.4.4. *For all $\alpha,\beta \in \Phi^+$, $s_{\alpha,l}$ commutes with $s_{\beta,l'}$ where $l,l' \in \mathbb{Z}$, if and only if we have $\alpha = \beta$ and $l = l'$, or $\langle\alpha,\beta\rangle = 0$.*

Proof. Let α and β be elements of the root system Φ^+ and $l,l' \in \mathbb{Z}$. Then we have

$$\begin{aligned} s_{\alpha,l} &= s_\alpha(\mathbf{v}) + 2l\frac{\alpha}{\langle\alpha,\alpha\rangle} = s_\alpha(\mathbf{v}) + l\alpha^\vee, \\ s_{\alpha,l}s_{\beta,l'}(\mathbf{v}) &= s_\alpha s_\beta(\mathbf{v}) + s_\beta(l\alpha^\vee) + l'\beta^\vee, \\ s_{\beta,l'}s_{\alpha,l}(\mathbf{v}) &= s_\beta s_\alpha(\mathbf{v}) + s_\alpha(l'\beta^\vee) + l\alpha^\vee. \end{aligned}$$

Hence, $s_{\alpha,l}s_{\beta,l'} = s_{\beta,l'}s_{\alpha,l}$ precisely when, for all $\mathbf{v} \in V$, we have

$$\begin{aligned} s_\alpha s_\beta(\mathbf{v}) + s_\beta(l\alpha^\vee) + l'\beta^\vee &= s_\beta s_\alpha(\mathbf{v}) + s_\alpha(l'\beta^\vee) + l\alpha^\vee, \\ s_\alpha s_\beta(\mathbf{v}) + l\alpha^\vee - l\langle\alpha^\vee,\beta\rangle\beta^\vee + l'\beta^\vee &= s_\beta s_\alpha(\mathbf{v}) + l'\beta^\vee - l'\langle\beta^\vee,\alpha\rangle\alpha^\vee + l\alpha^\vee, \\ s_\alpha s_\beta(\mathbf{v}) - l\langle\alpha^\vee,\beta\rangle\beta^\vee &= s_\beta s_\alpha(\mathbf{v}) - l'\langle\beta^\vee,\alpha\rangle\alpha^\vee. \end{aligned}$$

In particular, setting $\mathbf{v} = 0$ we get $l\langle\alpha^\vee,\beta\rangle\beta^\vee = l'\langle\beta^\vee,\alpha\rangle\alpha^\vee$. Hence, either $\alpha = \beta$ or $\langle\alpha,\beta\rangle = 0$. If $\alpha = \beta$, we get $s_{\alpha,l}s_{\alpha,l'}(\mathbf{v}) = s_\alpha s_\alpha(\mathbf{v}) + s_\alpha(l\alpha^\vee) + l'\alpha^\vee = \mathbf{v} + (l - l')\alpha^\vee$, whereas $s_{\alpha,l'}s_{\alpha,l}(\mathbf{v}) = (l'-l)\alpha^\vee$. Therefore, $s_{\alpha,l}$ commutes with $s_{\alpha,l'}$ if and only if $l = l'$. Now, if $\alpha \neq \beta$ then $\langle\alpha,\beta\rangle = 0$, which implies $s_\alpha s_\beta = s_\beta s_\alpha$ and so $s_{\alpha,l}s_{\beta,l'} = s_{\beta,l'}s_{\alpha,l}$. The proof is completed. □

1.5 Preliminary Results

Many results in this thesis are based on the research of Bates, Bundy, Rowley and Hart: in particular when G is a finite Coxeter group of type B_n or D_n and $\mathcal{C}(G,X)$ is a commuting involution graph. They proved this main theorem.

Theorem 1.5.1 (Theorem 1.1 of [7]). *Suppose that W is of type B_n, and let*

$$\sigma = \overset{+}{(1\,2)}\cdots\overset{+}{(2m-1\ \ 2m)}\overset{+}{(2m+1)}\cdots\overset{+}{(2m+l)}\overset{-}{(2m+l+1)}\cdots\overset{-}{(2m+l+t)}.$$

Set $X = a^G$ and $k := \max\{l,t\}$. Then the following hold.

(i) If $m = 0$, then $\mathcal{C}(G,X)$ is a complete graph.

(ii) If $k = 0$, then $\operatorname{Diam}\mathcal{C}(G,X) \leq 2$.

(iii) If $k = 1$ and $m > 0$, then $\mathcal{C}(G,X)$ is disconnected.

(iv) If $k \geq 2$ and $n > 5$, then $\operatorname{Diam}\mathcal{C}(G,X) \leq 4$.

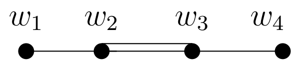
(v) If $n = 5$, $m = 1$ and $k = 2$ then Diam $\mathcal{C}(G, X) = 5$. *If $n = 5$, $m = 1$ and $k = 3$ then* Diam $\mathcal{C}(G, X) = 2$. *Finally if $n = 4$, $m = 1$ and $k = 2$ then $\mathcal{C}(G, X)$ is disconnected.*

Here, we present part of their result which is related to our proof in particular in Chapter 4: when G is an exceptional finite Coxeter group and X a conjugacy class of involutions (Theorem 1.2 of [7]). Let $a \in X$.

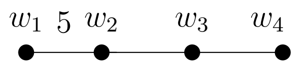
(i) If G is of type E_6 , then Diam $\mathcal{C}(G, X) \leq 5$.

(ii) If G is of type E_7 or E_8 , then Diam $\mathcal{C}(G, X) \leq 4$.

(iii) If G is of type F_4 and $|X| > 1$, then either $\mathcal{C}(G, X)$ is disconnected or Diam $\mathcal{C}(G, X) = 3$ and $(w_2w_3)^2 \in X$. The Coxeter graph of G of type F_4 is as follows.



(iv) If G is of type H_3 or H_4 and $|X| > 1$, then either $\mathcal{C}(G, X)$ is disconnected or G is of type H_4, Diam $\mathcal{C}(G, X) = 2$ and $w_1w_3 \in X$. The Coxeter graph of G of type H_4 is as follows.



Since $W(\tilde{A}_n)$ is a subgroup of $W(\tilde{D}_n)$, the result proved by Hart in [23] was useful when we investigated the diameter of $\mathcal{C}(G, X)$ when G is an affine Coxeter group. The main results are stated below.

Theorem 1.5.2. *(*Theorem 1.1 of [23]) Let $G \cong W(\tilde{A}_{n-1})$ and X a conjugacy class of involutions with m transpositions. Then $\mathcal{C}(G, X)$ is disconnected if and only if either $n = 2m + 1$, or $m = 1$ and $n \in \{2, 4\}$.

For an element $(g, \mathbf{v})$ in a class X of involutions of $\tilde{W}$, let $\hat{g} = g$ and $\hat{X} = \hat{g}^W$.

Theorem 1.5.3. (Theorem 1.2 of [23]) Let $G \cong W(\tilde{A}_{n-1})$, and $W \cong W(A_{n-1})$. Suppose that $\mathcal{C}(G, X)$ is connected where X is a class of involutions having m transpositions. If $n > 2m$ or m is even, then

$$\text{Diam}\ \mathcal{C}(G, X) \leq \text{Diam}\ \mathcal{C}(\hat{G}, \hat{X}) + 2.$$

If $n = 2m$ and m is odd, then

$$\text{Diam}\ \mathcal{C}(G, X) \leq \text{Diam}\ \mathcal{C}(\hat{G}, \hat{X}) + 3.$$

The next three results give criteria under which labelled 1-cycles, transpositions and double-transpositions commute. They will be used repeatedly in the proofs in the following chapters and in particular in Chapter 2 and Chapter 3.

Lemma 1.5.4. *Let $\alpha \in \{1, \ldots, n\}$ and $\lambda, \mu \in \mathbb{Z}$. Then $\overset{\overset{\lambda}{-}}{(\alpha)}$ commutes with $\overset{\overset{0}{+}}{(\alpha)}$ for all λ, whereas $\overset{\overset{\lambda}{-}}{(\alpha)}$ commutes with $\overset{\overset{\mu}{-}}{(\alpha)}$ if and only if $\lambda = \mu$.*

Proof. For convenience, we assume without loss of generality that $\alpha = 1$. Note that two involutions commute if and only if their product is an involution. We have that $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{0}{+}}{(1)}=\overset{\overset{\lambda}{-}}{(1)}$, which is an involution, so $\overset{\overset{\lambda}{-}}{(1)}$ commutes with $\overset{\overset{0}{+}}{(1)}$. In the case of two negative 1-cycles we have $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\mu}{-}}{(1)}=\overset{\overset{\mu-\lambda}{+}}{(1)}$, which is only an involution if $\mu - \lambda = 0$. So $\overset{\overset{\lambda}{-}}{(1)}$ commutes with $\overset{\overset{\mu}{-}}{(1)}$ if and only if $\lambda = \mu$. $\square$

Lemma 1.5.5. *Let α, β be distinct elements of $\{1, \ldots, n\}$ and let λ, μ and ν be integers.*

(i) $\overset{\overset{\lambda}{\pm}}{(\alpha\,\beta)}$ and $\overset{\overset{\mu}{\pm}}{(\alpha\,\beta)}$ commute if and only if $\lambda = \mu$. But $\overset{\overset{\lambda}{+}}{(\alpha\,\beta)}$ and $\overset{\overset{\mu}{-}}{(\alpha\,\beta)}$ commute for all λ and μ.

(ii) $\overset{\overset{\lambda}{\pm}}{(\alpha\,\beta)}$ and $\overset{\overset{0}{+}}{(\alpha)}\overset{\overset{0}{+}}{(\beta)}$ commute for all λ, but there is no value of μ or λ for which $\overset{\overset{\lambda}{\pm}}{(\alpha\,\beta)}$ and $\overset{\overset{0}{+}}{(\alpha)}\overset{\overset{\mu}{-}}{(\beta)}$ or $\overset{\overset{\mu}{-}}{(\alpha)}\overset{\overset{0}{+}}{(\beta)}$ commute.

(iii) $\overset{\overset{\lambda}{+}}{(\alpha\,\beta)}$ and $\overset{\overset{\mu}{-}}{(\alpha)}\overset{\overset{\nu}{-}}{(\beta)}$ commute if and only if $\mu - \nu = 2\lambda$, whereas $\overset{\overset{\lambda}{-}}{(\alpha\,\beta)}$ and $\overset{\overset{\mu}{-}}{(\alpha)}\overset{\overset{\nu}{-}}{(\beta)}$ commute if and only if $\mu + \nu = 2\lambda$.

Proof. We lose no generality by assuming, for ease of notation, that $\alpha = 1$, $\beta = 2$ and $n = 2$. Recall that $\overset{\overset{\lambda}{+}}{(1\,2)}$ and $\overset{\overset{\lambda}{-}}{(1\,2)}$ are shorthand for $(\overset{+}{(1\,2)}, (\lambda, -\lambda))$ and $(\overset{-}{(1\,2)}, (\lambda, \lambda))$ respectively. Involutions commute precisely when their product is an involution (or the identity), which we can check using Lemma 1.3.1.

(i) We calculate

$$\overset{\overset{\lambda}{+}}{(1\,2)}\overset{\overset{\mu}{+}}{(1\,2)} = (\overset{+}{(1\,2)}, (\lambda, -\lambda))(\overset{+}{(1\,2)}, (\mu, -\mu)) = (1, (\lambda, -\lambda)^{\overset{+}{(1\,2)}} + (\mu, -\mu)) = (1, (\mu-\lambda, \lambda-\mu)).$$

This means $\overset{\overset{\lambda}{+}}{(1\,2)}$ and $\overset{\overset{\mu}{+}}{(1\,2)}$ commute if and only if $\lambda = \mu$. We also have

$$\overset{\overset{\lambda}{-}}{(1\,2)}\overset{\overset{\mu}{-}}{(1\,2)} = (\overset{-}{(1\,2)}, (\lambda, \lambda))(\overset{-}{(1\,2)}, (\mu, \mu)) = (1, (\lambda, \lambda)^{\overset{-}{(1\,2)}} + (\mu, \mu)) = (1, (\mu - \lambda, \mu - \lambda)),$$

so again $\overset{\overset{\lambda}{-}}{(1\,2)}$ and $\overset{\overset{\mu}{-}}{(1\,2)}$ commute if and only if $\lambda = \mu$. Finally a similar calculation shows that $\overset{\overset{\lambda}{+}}{(1\,2)}\overset{\overset{\mu}{-}}{(1\,2)} = \overset{\overset{\mu+\lambda}{-}}{(1)}\ \overset{\overset{\mu-\lambda}{-}}{(2)}$, which is an involution for all values of λ and μ, so $\overset{\overset{\lambda}{+}}{(1\,2)}$ and $\overset{\overset{\mu}{-}}{(1\,2)}$ always commute.

(ii) Certainly $\overset{\overset{\lambda}{\pm}}{(1\,2)}$ and $\overset{\overset{0}{+}}{(1)}\overset{\overset{0}{+}}{(2)}$ commute for all λ; after all, $\overset{\overset{0}{+}}{(1)}\overset{\overset{0}{+}}{(2)}$ is just the identity element when $n = 2$. Moreover, already in the underlying group W of type B_2 we observe that $\overset{\pm}{(1\,2)}$ does not commute with $\overset{+}{(1)}\overset{-}{(2)}$ or $\overset{-}{(1)}\overset{+}{(2)}$, so there is no value of μ or λ for which $\overset{\overset{\lambda}{\pm}}{(1\,2)}$ and $\overset{\overset{0}{+}}{(1)}\overset{\overset{\mu}{-}}{(2)}$ or $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$ commute.

(iii) We have $\overset{\overset{\lambda}{+}}{(1\,2)}\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\nu}{-}}{(2)} = (\overset{\overset{\mu-\lambda}{-}}{1}\ \ \overset{\overset{\nu+\lambda}{-}}{2})$; this is an involution if and only if $\nu+\lambda = \mu-\lambda$. Thus $\overset{\overset{\lambda}{+}}{(1\,2)}$ and $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\nu}{-}}{(2)}$ commute if and only if $\mu - \nu = 2\lambda$. Similarly $\overset{\overset{\lambda}{-}}{(1\,2)}\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\nu}{-}}{(2)} = (\overset{\overset{\mu-\lambda}{+}}{1}\ \ \overset{\overset{\nu-\lambda}{+}}{2})$; this is an involution if and only if $\nu - \lambda = -(\mu-\lambda)$. Therefore $\overset{\overset{\lambda}{-}}{(1\,2)}$ and $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\nu}{-}}{(2)}$ commute if and only if $\mu+\nu = 2\lambda$. $\square$

Lemma 1.5.6. *Let* $g_1 = \overset{\overset{\lambda_1}{+}}{(\alpha\,\beta)}\overset{\overset{\lambda_2}{+}}{(\gamma\,\delta)}$, $g_2 = \overset{\overset{\lambda_1}{+}}{(\alpha\,\beta)}\overset{\overset{\lambda_2}{-}}{(\gamma\,\delta)}$, $g_3 = \overset{\overset{\lambda_1}{-}}{(\alpha\,\beta)}\overset{\overset{\lambda_2}{-}}{(\gamma\,\delta)}$, $h_1 = \overset{\overset{\mu_1}{+}}{(\alpha\,\gamma)}\overset{\overset{\mu_2}{+}}{(\beta\,\delta)}$, $h_2 = \overset{\overset{\mu_1}{+}}{(\alpha\,\gamma)}\overset{\overset{\mu_2}{-}}{(\beta\,\delta)}$ *and* $h_3 = \overset{\overset{\mu_1}{-}}{(\alpha\,\gamma)}\overset{\overset{\mu_2}{-}}{(\beta\,\delta)}$, *for distinct* $\alpha,\beta,\gamma,\delta$ *in* $\{1,\ldots,n\}$ *and integers* λ_i, μ_i. *Then*

(i) g_1 *commutes with* h_1 *if and only if* $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$*;*

(ii) g_1 *does not commute with* h_2*;*

(iii) g_1 *commutes with* h_3 *if and only if* $\mu_1 - \lambda_1 = \mu_2 + \lambda_2$*;*

(iv) g_2 *commutes with* h_2 *if and only if* $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$*;*

(v) g_2 *does not commute with* h_3*;*

(vi) g_3 *commutes with* h_3 *if and only if* $\mu_1 - \lambda_1 = \lambda_2 - \mu_2$.

Proof. For ease of notation we may assume that $n = 4$ and $g_1 = \overset{\overset{\lambda_1}{+}}{(1\,2)}\overset{\overset{\lambda_2}{+}}{(3\,4)}$.

(i) $g_1 = (\overset{+}{(1\,2)}\overset{+}{(3\,4)}, (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2))$ and $h_1 = (\overset{+}{(1\,3)}\overset{+}{(2\,4)}, (\mu_1, \mu_2, -\mu_1, -\mu_2))$. Hence

$$\begin{aligned} g_1h_1 &= (\overset{+}{(1\,4)}\overset{+}{(2\,3)}, (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)^{(\overset{+}{(1\,3)}\overset{+}{(2\,4)})} + (\mu_1, \mu_2, -\mu_1, -\mu_2)) \\ &= (\overset{+}{(1\,4)}\overset{+}{(2\,3)}, (\mu_1+\lambda_2, \mu_2-\lambda_2, -\mu_1+\lambda_1, -\mu_2-\lambda_1)). \end{aligned}$$

Now g_1 and h_1 commute if and only if g_1h_1 is an involution. This occurs if and only if $\mu_1 + \lambda_2 = -(\mu_2 - \lambda_1)$ and $\mu_2 - \lambda_2 = -(-\mu_1 + \lambda_1)$. Rearranging gives $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$, as required for part (i).

(ii) Since $\overset{+}{(1\,2)}\overset{+}{(3\,4)}$ and $\overset{+}{(1\,3)}\overset{-}{(2\,4)}$ do not commute, it is impossible for g_1 to commute with h_2.

(iii) We have $g_1 = (\overset{+}{(1\,2)}\overset{+}{(3\,4)}, (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2))$ and $h_3 = (\overset{-}{(1\,3)}\overset{-}{(2\,4)}, (\mu_1, \mu_2, \mu_1, \mu_2))$, and

$$\begin{aligned} g_1 g_3 &= (\overset{-}{(1\,4)}\overset{-}{(2\,3)}, (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)^{\overset{-}{(1\,3)}\overset{-}{(2\,4)}} + (\mu_1, \mu_2, \mu_1, \mu_2)) \\ &= (\overset{-}{(1\,4)}\overset{-}{(2\,3)}, (\mu_1 - \lambda_2, \mu_2 + \lambda_2, \mu_1 - \lambda_1, \mu_2 + \lambda_1)). \end{aligned}$$

Now g_1 and h_3 commute if and only if $g_1 h_3$ is an involution. From the above calculation this occurs if and only if $\mu_1 - \lambda_2 = \mu_2 + \lambda_1$ and $\mu_2 + \lambda_2 = \mu_1 - \lambda_1$. That is, if and only if $\mu_2 + \lambda_2 = \mu_1 - \lambda_1$.

(iv) We calculate $g_2 h_2 = (\overset{-}{(1\,4)}\overset{+}{(2\,3)}, (\mu_1 + \lambda_2, \mu_2 - \lambda_2, -\mu_1 + \lambda_1, \mu_2 + \lambda_1))$. Thus g_2 commutes with h_2 if and only if $\mu_1 - \lambda_1 = \mu_2 - \lambda_2$.

(v) Since $\overset{+}{(1\,2)}\overset{-}{(3\,4)}$ and $\overset{-}{(1\,3)}\overset{-}{(2\,4)}$ do not commute, g_2 cannot commute with h_3.

(vi) This follows because $g_3 h_3 = (\overset{+}{(1\,4)}\overset{+}{(2\,3)}, (\mu_1 - \lambda_2, \mu_2 - \lambda_2, \mu_1 - \lambda_1, \mu_2 - \lambda_1)$. $\square$

We conclude with an observation about connectedness. For an element $g = (\sigma, \mathbf{v})$ in a conjugacy class X of $G \cong \tilde{W}$, we define $\hat{g} = \sigma$. Then let $\hat{X}$ be the conjugacy class of $\hat{g}$ in W. Clearly if $g, h \in X$, then $\hat{g}, \hat{h} \in \hat{X}$.

Lemma 1.5.7. *Suppose $g, h \in X$. If $d(\hat{g}, \hat{h}) = k$, then $d(g, h) \geq k$. If $\mathcal{C}(W, \hat{X})$ is disconnected, then $\mathcal{C}(\tilde{W}, X)$ is disconnected.*

Proof. The result follows immediately from the observation that if g commutes with h in G, then $\hat{g}$ commutes with $\hat{h}$ in W. $\square$

Chapter 2

Coxeter Groups of Type $\tilde{C}_n$

2.1 Introduction

In this chapter, we investigate the commuting involution graphs $\mathcal{C}(G, X)$ for the affine Weyl groups of type $\tilde{C}_n$. Let G_n be an affine Weyl group of type $\tilde{C}_n$, for some $n \geq 2$, writing G when n is not specified, and let X be a conjugacy class of involutions of G. We write Diam $\mathcal{C}(G, X)$ for the diameter of $\mathcal{C}(G, X)$ when $\mathcal{C}(G, X)$ is a connected graph, in other words the maximum distance $d(x, y)$ between any $x, y \in X$ in the graph. Since conjugation by any group element induces a graph automorphism, we can determine the diameter by fixing any $a \in X$, and then Diam $\mathcal{C}(G, X) = \max\{d(x, a) : x \in X\}$. Our main result is the following.

Theorem 2.1.1. *Let X be a conjugacy class of involutions in the group G_n of type $\tilde{C}_n$. If the commuting graph $\mathcal{C}(G, X)$ is connected, then its diameter is at most $n + 2$.*

The proof of Theorem 2.1.1 is broken into several cases depending on the type of the conjugacy class. To describe this further, and in order to state our results about connectedness, we need to describe a parametrisation of the involution conjugacy class. We refer to Lemma 1.3.1 and to Notation 1.3.1 to explain in Section 2 that involutions in G can be written as 'labelled permutations'. These are permutations expressed as products of disjoint cycles in which every cycle has a sign and an integer written above it. For example in G_{10} one of the involutions is $\overset{\overset{0}{+}}{(12)}\overset{\overset{0}{+}}{(34)}\ \overset{\overset{3}{-}}{(5)}\overset{\overset{4}{-}}{(6)}\overset{\overset{3}{-}}{(7)}\overset{\overset{1}{-}}{(8)}\overset{\overset{0}{-}}{(9)}\overset{\overset{0}{+}}{(10)}$. A cycle is positive if it has a plus sign, and negative if it has a minus sign. The *labelled cycle type* of an involution σ will be a quadruple (m, k_e, k_o, l) where m is the number of transpositions, k_e is the number of negative 1-cycles with an even number above them, k_o is the number of negative 1-cycles with an odd number above them, and l is the number of positive 1-cycles. For the example in G_{10}, its labelled cycle type is $(2, 2, 3, 1)$.

We will show (Theorem 2.2.1) that involution conjugacy classes in G are parametrised by labelled cycle type. We may now give the conditions under which the graph $\mathcal{C}(G, X)$

is connected.

Theorem 2.1.2. *Let X be a conjugacy class of involutions in G_n (where $n \geq 2$) with labelled cycle type (m, k_e, k_o, l). Then $\mathcal{C}(G, X)$ is disconnected in each of the following cases.*

(i) $m = 0$ *and* $l = 0$;

(ii) $m > 0$, $l = 0$ *and either* $k_e = 1$ *or* $k_o = 1$;

(iii) $m > 0$ *and* $\max(k_e, k_o, l) = 1$;

(iv) $n = 4$ *and* $m = 1$;

(v) $n = 6$, $m = 1$ *and* $k_e = k_o = 2$.

In all other cases, $\mathcal{C}(G, X)$ is connected.

We note that the bound on diameter given in Theorem 2.1.1 is best possible. For example we have verified that when $n = 8$, $m = 1$ and $k_e = k_o = 3$, the commuting involution graph has diameter 10.

To construct the proofs needed for this result, we rely on the structure of the Coxeter group G as a semi-direct product of a finite Coxeter group W of type C_n (whose elements can be thought of as signed permutations) with a $\mathbb{Z}^n$ lattice. In Section 2.2 by using Richardson [25] Theorem 1.4.2, we show that the involution conjugacy classes are parameterised by labelled cycle type. Section 2.3 is dedicated to proving the main theorems. Throughout we make use of the labelled cycle form along with the criteria for commuting developed in Lemmas 1.5.4, 1.5.5 and 1.5.6 in Chapter 1. In Section 2.4 we give examples of selected commuting involution graphs.

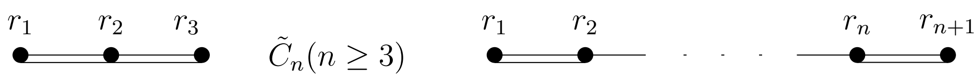
2.2 Involution Conjugacy Classes in G

It turns out that involution conjugacy classes are characterised by labelled cycle type. To see this, we use a well-known result, due to Richardson, that gives a description of involution conjugacy classes in Coxeter groups.

In the next result, we use the notation w_I for the longest element of a finite standard parabolic subgroup W_I.

Recall from Chapter 1 the Coxeter graphs of G_2 (which is a Coxeter group of type $\tilde{C}_2$) and G_n for $n \geq 3$ (which is a Coxeter group of type $\tilde{C}_n$) are as follows.

$\tilde{C}_2$ $\tilde{C}_n (n \geq 3)$



We may set $r_1 = (\overset{\overset{0}{-}}{1})$, $r_i = (\overset{\overset{0}{+}}{i-1}\ \overset{\overset{0}{+}}{i})$ for $2 \leq i \leq n$, and $r_{n+1} = (\overset{\overset{1}{-}}{n})$.

It is well known that in the finite Coxeter group W of type C_n, elements are conjugate if and only if they have the same signed cycle type. In particular two involutions are conjugate when they have the same number of transpositions, the same number of negative 1-cycles and the same number of positive 1-cycles.

Theorem 2.2.1. *Involutions in G are conjugate if and only if they have the same labelled cycle type. In particular, every involution is conjugate to exactly one element $a = a_{m,k_e,k_o,l}$ of the form*

$$\overset{\overset{0}{+}}{(1\ 2)} \cdots \overset{\overset{0}{+}}{(2m-1\ 2m)}\overset{\overset{0}{-}}{(2m+1)} \cdots \overset{\overset{0}{-}}{(2m+k_e)}\overset{\overset{1}{-}}{(2m+k_e+1)} \cdots \overset{\overset{1}{-}}{(n-l)}\overset{\overset{0}{+}}{(n-l+1)} \cdots \overset{\overset{0}{+}}{(n)}.$$

Proof. Let x be an involution of G. Since G is a Coxeter group (of type $\tilde{C}_n$ for some n), we may apply Theorem 1.4.2 to see that x is conjugate to w_I for some finite standard parabolic subgroup W_I of G in which w_I is central. Therefore, the connected components of the Coxeter graph for W_I are of types A_1 or B_i for some i (including, by a slight abuse of notation, B_1, where we have connected components with just the vertex r_1 or r_{n+1}). Thus $I = \{r_1, r_2, \ldots, r_i\} \cup J \cup \{r_{j+1}, r_{j+2}, \ldots, r_{n+1}\}$ for some i, j with $0 \leq i < j \leq n+1$, where J is a subset of $\{r_{i+2}, \ldots, r_{j-1}\}$ no two elements of which are adjacent vertices in the Coxeter graph. By conjugation in $\langle r_{i+2}, \ldots, r_{j-1} \rangle$ (which after all is isomorphic to the symmetric group $\mathrm{Sym}(j-i-1)$), we can assume that for some i, j and m with $0 \leq i \leq i + 2m < j < n+1$ we have

$$I = \{r_1, r_2, \ldots, r_i\} \cup \{r_{i+2}, r_{i+4}, \ldots, r_{i+2m}\} \cup \{r_{j+1}, r_{j+2}, \ldots, r_{n+1}\}.$$

This gives that x is conjugate to w_I, where

$$w_I = \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(i)}\overset{\overset{0}{+}}{(i+1\ \ i+2)} \cdots \overset{\overset{0}{+}}{(i+2m-1\ \ i+2m)}\overset{\overset{0}{+}}{(i+2m+1)} \cdots \overset{\overset{0}{+}}{(j-1)}\overset{\overset{1}{-}}{(j)} \cdots \overset{\overset{1}{-}}{(n)}.$$

Let $c = (h, \mathbf{0})$, where

$$h = (\overset{+}{1}\ \ \overset{+}{i+1})(\overset{+}{2}\ \ \overset{+}{i+2}) \cdots (\overset{+}{2m}\ \ \overset{+}{i+2m})(\overset{+}{i+2m+1}\ \ \overset{+}{j})\ (\overset{+}{i+2m+2}\ \ \overset{+}{j+1}) \cdots (\overset{+}{i+2m+1+n-j}\ \ \overset{+}{n}).$$

From Equation 1.3page 18, we see that

$$w_I^c = \overset{\overset{0}{+}}{(1\ 2)} \cdots \overset{\overset{0}{+}}{(2m-1\ 2m)}\overset{\overset{0}{-}}{(2m+1)} \cdots \overset{\overset{0}{-}}{(2m+k_e)}\overset{\overset{1}{-}}{(2m+k_e+1)} \cdots \overset{\overset{1}{-}}{(n-l)}\overset{\overset{0}{+}}{(n-l+1)} \cdots \overset{\overset{0}{+}}{(n)}.$$

Therefore, by setting $a = w_I^c$ we see that each involution in G is conjugate to at least one element of the required form.

Now consider an involution x in G with labelled cycle type (m, k_e, k_o, l), and suppose $y = r_i x r_i^{-1}$ for some simple reflection r_i. Write $x = (\sigma, \mathbf{v})$ and $y = (\tau, \mathbf{u})$. By Equation (1.3), τ is conjugate to σ in the underlying Weyl group W. Hence τ and σ have the same number of transpositions, negative 1-cycles and positive 1-cycles as each other. In other words, the labelled cycle type of y is (m, k'_e, k'_o, l) for some k'_e, k'_o satisfying $k'_e + k'_o = k_e + k_o$. Now, from Equation (1.3) again, $(\overset{\lambda}{\bar{i}})$ is a labelled 1-cycle of x if and only if $(\overset{\mu}{\bar{i^g}})$ is a labelled 1-cycle of y, where $\lambda = v_i$ and $\mu = u_{i^g}$. In particular this means $(\bar{i^g})$ is a signed cycle of τ, so that $(w^\tau)_{i^g} = -w_{i^g}$. Now $\mathbf{u} = \mathbf{v}^g + \mathbf{w} - \mathbf{w}^\tau$. So $\mu = u_{i^g} = (v^g)_{i^g} + w_{i^g} - (w^\tau)_{i^g} = v_i + 2w_{i^g} \equiv \lambda \mod 2$. Therefore $k_o = k'_o$. Hence $k_e = k'_e$ and so x and y have the same labelled cycle type. In particular x is conjugate to at most one element a of the form stated in the theorem. Conversely, any two involutions of the same labelled cycle type (m, k_e, k_o, l) are both conjugate to $a_{m,k_e,k_o,l}$, and hence to each other. Thus conjugacy is parameterised by labelled cycle type, and the set of elements $\{a_{m,k_e,k_o,l} : 2m + k_e + k_o + l = n\}$ contains exactly one representative of each conjugacy class of involutions in G_n. $\square$

2.3 Proofs of Main Results

We can now prove Theorem 2.1.2 (which gives necessary and sufficient conditions for $\mathcal{C}(G, X)$ to be disconnected) in one direction. The proof in the other direction will arise from bounding the diameters of graphs not shown in Theorem 2.3.1 to be disconnected.

Theorem 2.3.1. *Let X be a conjugacy class of involutions in G_n (where $n \geq 2$) with labelled cycle type (m, k_e, k_o, l). Then $\mathcal{C}(G, X)$ is disconnected in each of the following cases.*

(i) $m = 0$ *and* $l = 0$*;*

(ii) $m > 0$, $l = 0$ *and either* $k_e = 1$ *or* $k_o = 1$*;*

(iii) $m > 0$ *and* $\max(k_e, k_o, l) = 1$*;*

(iv) $n = 4$ *and* $m = 1$*;*

(v) $n = 6$, $m = 1$ *and* $k_e = k_o = 2$.

Proof. Let X be a conjugacy class of involutions in G_n, with labelled cycle type (m, k_e, k_o, l), and $a = a_{m,k_e,k_o,l} \in X$ as defined in Theorem 2.2.1. We deal with each case in turn.

(i) Suppose $l = 0$ and $m = 0$. Then any x in X is of the form $(\overset{\lambda_1}{\bar{1}})(\overset{\lambda_2}{\bar{2}}) \cdots (\overset{\lambda_n}{\bar{n}})$ for some λ_i (where k_e of the λ_i are even and k_o are odd). By Lemma 1.5.4, x does not commute with any other element of X. So in fact $\mathcal{C}(G, X)$ is completely disconnected in this case.

(ii) Suppose $l = 0$ and $k_o = 1$ (the case $k_e = 1$ is similar). Then

$$a = (\overset{\overset{0}{+}}{1\;2}) \cdots (\overset{\overset{0}{+}}{2m-1\;\;2m})(\overset{\overset{0}{-}}{2m+1}) \cdots (\overset{\overset{0}{-}}{n-1})(\overset{\overset{1}{-}}{n})\;.$$

Suppose $b \in X$ such that a commutes with b. Consider the cycle of b that contains n. This must be a negative 1-cycle or a 2-cycle, because b has the same labelled cycle type as a. If it is a 2-cycle, then b cannot commute with a, by Lemma 1.5.5(iii)because $\mu \pm \lambda = 1 \neq 2\lambda$. Therefore it is a negative 1-cycle $(\overset{\overset{\lambda}{-}}{n})$ where λ is odd, and then by Lemma 1.5.4, b contains $(\overset{\overset{1}{-}}{n})$. The same argument shows that any element c of X which commutes with b must also contain $(\overset{\overset{1}{-}}{n})$, and inductively all elements in the connected component of $\mathcal{C}(G, X)$ containing a must contain $(\overset{\overset{1}{-}}{n})$. Therefore $\mathcal{C}(G, X)$ is disconnected.

(iii) Suppose $m > 0$ and $\max(k_e, k_o, l) = 1$. If $l = 0$ then $\mathcal{C}(G, X)$ is disconnected by (ii). So we can assume $l = 1$. If either of k_e or k_o is zero, then Theorem 1.5.1(iii) and Lemma 1.5.7 imply that $\mathcal{C}(G, X)$ is disconnected. It remains to consider the case $l = k_e = k_o = 1$. Here, $2m = n - 3$ and

$$a = (\overset{\overset{0}{+}}{1\;2}) \cdots (\overset{\overset{0}{+}}{2m-1\;\;2m})(\overset{\overset{0}{-}}{n-2})(\overset{\overset{1}{-}}{n-1})(\overset{\overset{0}{+}}{n})\;.$$

Suppose $b \in X$ such that a commutes with b, and suppose $(\overset{\overset{\mu}{\pm}}{t})$ is a 1-cycle of b. If $t \in \{1, \ldots, 2m\}$ then as a contains a transposition $(\overset{\overset{0}{+}}{tt'})$ for some t', Lemma 1.5.5 (ii) and (iii) show that the only way a and b could commute is if b contained $(\overset{\overset{0}{+}}{t})(\overset{\overset{0}{+}}{t'})$ or $(\overset{\overset{\mu}{-}}{t})(\overset{\overset{\nu}{-}}{t})$ where $\mu \equiv \nu \pmod 2$, contradicting our assumptions about the labelled cycle type of b. Therefore the elements appearing in 1-cycles of b are $n-2$, $n-1$ and n. Inductively this holds for all elements in the connected component of $\mathcal{C}(G, X)$ containing a. Therefore $\mathcal{C}(G, X)$ is disconnected.

(iv) Suppose $n = 4$ and $m = 1$. If $\max(k_e, k_o, l) = 1$ then $\mathcal{C}(G, X)$ is disconnected by (iii). If this doesn't happen, then one of k_e, k_o or l is 2. By Theorem 1.5.1(v) and Lemma 1.5.7, $\mathcal{C}(G, X)$ is again disconnected.

(v) Suppose $n = 6$, $m = 1$ and $k_e = k_o = 2$. For any x in X, we have

$$x = (\overset{\overset{\lambda}{+}}{\alpha\;\alpha'})(\overset{\overset{\mu}{-}}{\beta})(\overset{\overset{\mu'}{-}}{\beta'})(\overset{\overset{\nu}{-}}{\gamma})(\overset{\overset{\nu'}{-}}{\gamma'}),$$

where $\{\alpha, \alpha', \beta, \beta', \gamma, \gamma'\} = \{1, 2, 3, 4, 5, 6\}$, μ and μ' are even, and ν, ν' are odd. Associate a set $T(x) = \{\{\alpha, \alpha'\}, \{\beta, \beta'\}, \{\gamma, \gamma'\}\}$ to x. Given that, by Lemma 1.5.5 (iii), a transposition can only commute with a pair of negative 1-cycles if either both cycles are odd or both are even, and also with reference to Lemma 1.5.4, we see that if y in X commutes with x, then $T(x) = T(y)$. Therefore, for example, $\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(5)}\overset{\overset{1}{-}}{(6)}$ and $\overset{\overset{0}{+}}{(4\,5)}\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{1}{-}}{(6)}$ are not adjacent in $\mathcal{C}(G, X)$. □

Our first result bounding diameters is when $m = 0$.

Theorem 2.3.2. *If $m = 0$ and $l \geq 1$, then*

$$\operatorname{Diam} \mathcal{C}(G, X) = \begin{cases} 2 & \text{if } 2l \geq n \\ \lceil \frac{n}{l} \rceil & \text{if } 0 \in \{k_e, k_o\} \\ n+1 & \text{if } l = 1 \text{ and } 0 \notin \{k_e, k_o\}. \end{cases}$$

In all other cases $\lceil \frac{n}{l} \rceil \leq \operatorname{Diam} \mathcal{C}(G, X) \leq \lceil \frac{n}{l} \rceil + 2$.

Proof. In this case we have

$$a = \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(k_e)}\overset{\overset{1}{-}}{(k_e+1)} \cdots \overset{\overset{1}{-}}{(k_e+k_o)}\overset{\overset{0}{+}}{(k_e+k_o+1)} \cdots \overset{\overset{0}{+}}{(n)}.$$

Let $g \in X$. Then for appropriate $\varepsilon_1, \ldots, \varepsilon_{k_e+k_o}, \rho_1, \ldots, \rho_l$ and $\lambda_1, \ldots, \lambda_{k_e+k_o}$ we have that $g = \prod_{i=1}^{k_e+k_o} \overset{\overset{\lambda_i}{-}}{(\varepsilon_i)} \prod_{i=1}^{l} \overset{\overset{0}{+}}{(\rho_i)}$. Since $g \in X$, exactly k_e of the λ_i must be even. If $2l \geq n$, that is, $l \geq k_e + k_o$, then g commutes with every element h in X which contains the cycles $\overset{\overset{0}{+}}{(1)} \cdots \overset{\overset{0}{+}}{(k_e+k_e)}$. Now h certainly commutes with a, and so $\operatorname{Diam} \mathcal{C}(G, X) = 2$.

Now we suppose that $2l < n$, and that one of k_e and k_o is zero. Without loss of generality, we may choose $k_o = 0$. Then, writing $k = k_e$, we have $a = \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(k)}\overset{\overset{0}{+}}{(k+1)} \cdots \overset{\overset{0}{+}}{(n)}$. Since conjugation by elements of the centralizer of a preserves distance in $\mathcal{C}(G, X)$, without loss of generality we may take g to be of the following form for some integer r with $0 \leq r \leq l$ and even integers λ_i:

$$g = \overset{\overset{0}{+}}{(1)} \cdots \overset{\overset{0}{+}}{(r)}\overset{\overset{\lambda_{r+1}}{-}}{(r+1)} \cdots \overset{\overset{\lambda_{r+k}}{-}}{(r+k)}\overset{\overset{0}{+}}{(r+k+1)} \cdots \overset{\overset{0}{+}}{(n)}.$$

Now consider the following sequence, where $p = \lfloor \frac{k}{l} \rfloor$ and $\lambda_{r+k+1}, \ldots, \lambda_n$ are arbitrary even integers:

$$\begin{aligned}
g_0 &= \overset{\overset{0}{+}}{(1)} \cdots \overset{\overset{0}{+}}{(l)}\overset{\overset{\lambda_{l+1}}{-}}{(l+1)} \cdots \overset{\overset{\lambda_n}{-}}{(n)},\\
g_1 &= \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(l)}\overset{\overset{0}{+}}{(l+1)} \cdots \overset{\overset{0}{+}}{(2l)}\overset{\overset{\lambda_{2l+1}}{-}}{(2l+1)} \cdots \overset{\overset{\lambda_n}{-}}{(n)},\\
g_2 &= \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(2l)}\overset{\overset{0}{+}}{(2l+1)} \cdots \overset{\overset{0}{+}}{(3l)}\overset{\overset{\lambda_{3l+1}}{-}}{(3l+1)} \cdots \overset{\overset{\lambda_n}{-}}{(n)},\\
&\vdots\\
g_{p-1} &= \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{((p-1)l)}\overset{\overset{0}{+}}{((p-1)l+1)} \cdots \overset{\overset{0}{+}}{(pl)}\overset{\overset{\lambda_{pl+1}}{-}}{(pl+1)} \cdots \overset{\overset{\lambda_n}{-}}{(n)},\\
g_p &= \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(pl)}\overset{\overset{0}{+}}{(pl+1)} \cdots \overset{\overset{0}{+}}{((p+1)l)}\overset{\overset{\lambda_{(p+1)l+1}}{-}}{((p+1)l+1)} \cdots \overset{\overset{\lambda_n}{-}}{(n)}.
\end{aligned}$$

It is clear that g_i commutes with g_{i+1} for $0 \le i < p$. Moreover g commutes with g_0, and g_p commutes with a. If l divides k, then $g_p = a$, which implies that $d(g,a) \le p+1 = \frac{k}{l}+1 = \lceil \frac{n}{l} \rceil$. If l does not divide k, then $d(g,a) \le p+2 = \lfloor \frac{k}{l} \rfloor + 2 = \lceil \frac{n}{l} \rceil$. For any $x \in X$ define $c(x)$ to be the number of 'correct' negative 1-cycles in x. That is, cycles $\overset{\overset{0}{-}}{(\alpha)}$ where $1 \le \alpha \le k_e$ or $\overset{\overset{1}{-}}{(\beta)}$ where $k_e < \beta \le k_e + k_o$. Thus, for example, $c(a) = k_e + k_o$. We say that other negative 1-cycles are 'incorrect'. The diameter of the graph in each case does equal this bound because at each stage of a path from g to a we can add at most l to the number of correct negative 1-cycles, but this requires the element being considered to share no fixed points with a. Hence, for example the element

$$g = \overset{\overset{2}{-}}{(1)} \cdots \overset{\overset{2}{-}}{(k)}\overset{\overset{0}{+}}{(k_e+k_o+1)} \cdots \overset{\overset{0}{+}}{(n)}$$

must be distance at least $\lceil \frac{n-l}{l} \rceil + 1 = \lceil \frac{n}{l} \rceil$ from a.

The remaining case is when k_e and k_o are both non-zero, and $2l < n$.

Let $g \in X$. Then g commutes with some element x_0 whose positive 1-cycles are $\overset{\overset{0}{+}}{(1)} \cdots \overset{\overset{0}{+}}{(l)}$. We will describe a sequence $x_0, x_1, \ldots$ where at each stage x_i is an element of X such that $c(x_i) \ge li$ and the positive 1-cycles of x_i are $\overset{\overset{0}{+}}{(\alpha_1)} \cdots \overset{\overset{0}{+}}{(\alpha_r)}\overset{\overset{0}{+}}{(\beta_1)} \cdots \overset{\overset{0}{+}}{(\beta_s)}$ for some α_j, β_j where $1 \le \alpha_j \le k_e$ and $k_e < \beta_j \le k_e + k_o$, with $r + s = l$. Moreover, for $i > 0$, x_i will commute with x_{i-1}.

Observe that the positive 1-cycles of x_0 have the required form, and $c(x_0) \ge 0 \times l = 0$. Assume that we have $x_0, \ldots, x_i$ and let the positive 1-cycles of x_i be

$\overset{\overset{0}{+}}{(\alpha_1)}\cdots\overset{\overset{0}{+}}{(\alpha_r)}\overset{\overset{0}{+}}{(\beta_1)}\cdots\overset{\overset{0}{+}}{(\beta_s)}$. To form x_{i+1} we look for incorrect cycles $\overset{\overset{\lambda_1}{-}}{(\gamma_1)}\cdots\overset{\overset{\lambda_r}{-}}{(\gamma_r)}\overset{\overset{\mu_1}{-}}{(\delta_1)}\cdots\overset{\overset{\mu_s}{-}}{(\delta_s)}$ of x_i, where each λ_j is even, each μ_j is odd, $\gamma_j \leq n-l$ and $\delta_j \leq n-l$. If such cycles can be found, then x_i commutes with x_{i+1} where x_{i+1} is given by replacing the cycles $\overset{\overset{0}{+}}{(\alpha_1)}\cdots\overset{\overset{0}{+}}{(\alpha_r)}\overset{\overset{0}{+}}{(\beta_1)}\cdots\overset{\overset{0}{+}}{(\beta_s)}\overset{\overset{\lambda_1}{-}}{(\gamma_1)}\cdots\overset{\overset{\lambda_r}{-}}{(\gamma_r)}\overset{\overset{\mu_1}{-}}{(\delta_1)}\cdots\overset{\overset{\mu_s}{-}}{(\delta_s)}$ of x_i with

$$\overset{\overset{0}{-}}{(\alpha_1)}\cdots\overset{\overset{0}{-}}{(\alpha_r)}\overset{\overset{1}{-}}{(\beta_1)}\cdots\overset{\overset{1}{-}}{(\beta_s)}\overset{\overset{0}{+}}{(\gamma_1)}\cdots\overset{\overset{0}{+}}{(\gamma_r)}\overset{\overset{0}{+}}{(\delta_1)}\cdots\overset{\overset{0}{+}}{(\delta_s)},$$

and leaving all other cycles unchanged. It is clear that x_{i+1} commutes with x_i; moreover $x_{i+1} \in X$, has the appropriate positive 1-cycles and $c(x_{i+1}) \geq (i+1)l$.

This sequence can continue until we have some x_i with cycles $\overset{\overset{0}{+}}{(\alpha_1)}\cdots\overset{\overset{0}{+}}{(\alpha_r)}\overset{\overset{0}{+}}{(\beta_1)}\cdots\overset{\overset{0}{+}}{(\beta_s)}$ but (without loss of generality) fewer than r incorrect cycles $\overset{\overset{\lambda}{-}}{(\gamma)}$ where λ is even and $\gamma \leq n-l$. Suppose there are exactly t such cycles (with $t < r$). So x_i has the cycles $\overset{\overset{\lambda_1}{-}}{(\gamma_1)}\cdots\overset{\overset{\lambda_t}{-}}{(\gamma_t)}$. Since $\alpha_j < k_e$ for each j there must be $\varepsilon_{t+1},\ldots,\varepsilon_n$ where $n-l < \varepsilon_j \leq n$ for each ε_j, and even numbers λ_j such that x_i has the cycles $\overset{\overset{\lambda_1}{-}}{(\gamma_1)}\cdots\overset{\overset{\lambda_t}{-}}{(\gamma_t)}\overset{\overset{\lambda_{t+1}}{-}}{(\varepsilon_{t+1})}\cdots\overset{\overset{\lambda_r}{-}}{(\varepsilon_r)}$. Similarly x_i has cycles $\overset{\overset{\mu_1}{-}}{(\delta_1)}\cdots\overset{\overset{\mu_t'}{-}}{(\delta_t')}\overset{\overset{\mu_{t'+1}}{-}}{(\varepsilon_{t'+1}')}\cdots\overset{\overset{\mu_r}{-}}{(\varepsilon_s')}$ for odd μ, some $t' \leq s$, $\delta_j \leq n-l$ and $\varepsilon_j' > n-l$.

We now define y to be x_i with

$$\overset{\overset{0}{+}}{(\alpha_1)}\cdots\overset{\overset{0}{+}}{(\alpha_r)}\overset{\overset{0}{+}}{(\beta_1)}\cdots\overset{\overset{0}{+}}{(\beta_s)}\overset{\overset{\lambda_1}{-}}{(\gamma_1)}\cdots\overset{\overset{\lambda_r}{-}}{(\gamma_t)}\overset{\overset{\lambda_{t+1}}{-}}{(\varepsilon_{t+1})}\cdots\overset{\overset{\lambda_r}{-}}{(\varepsilon_r)}\overset{\overset{\mu_1}{-}}{(\delta_1)}\cdots\overset{\overset{\mu_t'}{-}}{(\delta_t')}\overset{\overset{\mu_{t'+1}}{-}}{(\varepsilon_{t'+1}')}\cdots\overset{\overset{\mu_r}{-}}{(\varepsilon_s')}$$

replaced with

$$\overset{\overset{0}{-}}{(\alpha_1)}\cdots\overset{\overset{0}{-}}{(\alpha_r)}\overset{\overset{1}{-}}{(\beta_1)}\cdots\overset{\overset{1}{-}}{(\beta_s)}\overset{\overset{0}{+}}{(\gamma_1)}\cdots\overset{\overset{0}{+}}{(\gamma_t)}\overset{\overset{0}{+}}{(\varepsilon_{t+1})}\cdots\overset{\overset{0}{+}}{(\varepsilon_r)}\overset{\overset{0}{+}}{(\delta_1)}\cdots\overset{\overset{0}{+}}{(\delta_t')}\overset{\overset{0}{+}}{(\varepsilon_{t'+1}')}\cdots\overset{\overset{0}{+}}{(\varepsilon_s')}.$$

Now x_i commutes with y, and $c(y) \geq (i+1)l$. We observe that $d(g,y) \leq i+2$. Notice that every β with $k_e < \beta \leq k_e + k_o$ is either a fixed point of y or appears in a cycle $\overset{\overset{\mu}{-}}{(\beta)}$ with μ odd. Every incorrect even negative 1-cycle of y features a fixed point of a, so for some $q \leq l$, and conjugating y by a suitable element of the centralizer of a if necessary, we can assume that the even negative 1-cycles of y are

$$\overset{\overset{\lambda_1}{-}}{(n-l+1)}\cdots\overset{\overset{\lambda_q}{-}}{(n-l+q)}\overset{\overset{0}{-}}{(q+1)}\cdots\overset{\overset{0}{-}}{(k_e)}.$$

Now y has at least $(i+1)l$ correct negative 1-cycles. If we ignore the even negative

1-cycles and the correct odd negative 1-cycles, then $n-(i+1)l-q$ cycles remain (including l fixed points). We can use the result for $k_e=0$ on this remaining part of y to see that y is distance at most $\lceil\frac{n-(i+1)l-q}{l}\rceil$ from the element z of X given by

$$\overset{\lambda_1}{\overset{-}{(n-l+1)}}\cdots\overset{\lambda_q}{\overset{-}{(n-l+q)}}\overset{0}{\overset{-}{(q+1)}}\cdots\overset{0}{\overset{-}{(k_e)}}\overset{0}{\overset{+}{(1)}}\cdots\overset{0}{\overset{+}{(q)}}\overset{0}{\overset{+}{(n-l+q+1)}}\cdots\overset{0}{\overset{+}{(n)}}\overset{1}{\overset{-}{(k_e+1)}}\cdots\overset{1}{\overset{-}{(k_e+k_o)}}.$$

Now z commutes with a, and so $d(g,a)\leq d(g,y)+d(y,z)+1=i+2+\lceil\frac{n-q-(i+1)l}{l}\rceil+1=2+\lceil\frac{n-q}{l}\rceil$. If $l=1$ and $q=0$, then in fact $z=a$ so $d(g,a)=n+1$. If $q=1$ then again $d(g,a)=n+1$. If $l>1$ then we have $d(g,a)\leq\lceil\frac{n}{l}\rceil+2$. To give a lower bound on the diameter consider

$$g=\overset{2}{\overset{-}{(1)}}\cdots\overset{2}{\overset{-}{(k_e)}}\overset{3}{\overset{-}{(k_e+1)}}\cdots\overset{3}{\overset{-}{(k_e+k_o)}}\overset{0}{\overset{+}{(k_e+k_o+1)}}\cdots\overset{0}{\overset{+}{(n)}}.$$

To create l additional correct negative 1-cycles at each stage of a path from g to a one requires each fixed point to be a point not fixed by a; moreover in this case completion of the process for, say, the even negative 1-cycles requires the recreation of at least one fixed point between $n-l+1$ and n and hence fewer than l correct negative 1-cycles being created at the next stage. Thus when $l=1$ we have $d(g,a)\geq n+1$, and when $l>1$ we have $d(g,a)\geq\lceil\frac{n}{l}\rceil$. This completes the proof. □

From now on, assume that $m>0$. Then we can take

$$a=\overset{0}{\overset{+}{(1\,2)}}\cdots\overset{0}{\overset{+}{(2m-1\;2m)}}\overset{0}{\overset{-}{(2m+1)}}\cdots\overset{0}{\overset{-}{(2m+k_e)}}\overset{1}{\overset{-}{(2m+k_e+1)}}\cdots\overset{1}{\overset{-}{(n-l)}}\overset{0}{\overset{+}{(n-l+1)}}\cdots\overset{0}{\overset{+}{(n)}},$$

where $2m+k_e+k_o+l=n$.

Proposition 2.3.3. *Suppose $k_e=k_o=0$. If $2m=n$, then* Diam $\mathcal{C}(G,X)\leq 3$. *If $n\geq 5$ and $l\geq 2$ then* Diam $\mathcal{C}(G,X)\leq 5$.

Proof. Let $x\in X$. We can write $a=(g,\mathbf{0})$ and $x=(h,\mathbf{v})$, where g,h are conjugate elements of the underlying Weyl group W. Define an element $y=(h',\mathbf{0})$, by $h'=hw_0$, where w_0 is the unique central involution of W. The result is that every minus sign in h corresponds to a plus sign in h', and every plus sign corresponds to a minus sign. Since $k=0$, h and h' are conjugate in W. So, by Theorem 1.5.1(ii), $d(g,h')\leq 2$ when $2m=n$ and $d(g,h')\leq 4$ otherwise. Now $d(a,y)=d(g,h')$ and y commutes with x by Lemmas 1.5.4 and 1.5.5. The result follows immediately. □

Proposition 2.3.4. *Suppose $m\geq 1$, $l=0$, one of k_e and k_o is zero, and either $m>1$ or* $\max\{k_e,k_0\}\geq 3$ *(or both). Then $\mathcal{C}(G,X)$ is connected with diameter at most n. If $x\in X$ and $d(x,a)=n$ then $m=1$ and the transposition of x is $\overset{\lambda}{\overset{*}{(1\,2)}}$ for some λ with $*$ is $+$ or $-$.*

Proof. Let $x \in X$. Without loss of generality $k_o = 0$. We start with the case $m = 1$. We will show that if the transposition of x is $\overset{\overset{\lambda}{*}}{(1\ 2)}$ for some λ, then $d(x,a) \leq n$. Otherwise $d(x,a) \leq n-1$. The graph for $n = 5$ (and hence $m = 1$) is Figure 2.1 page 47 where $a = \overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{0}{-}}{(5)}$ and $x = \overset{\overset{\sigma}{\pm}}{(1\ 2)}\overset{\overset{\lambda}{-}}{(3)}\overset{\overset{\lambda'}{-}}{(4)}\overset{\overset{\lambda''}{-}}{(5)}$. We can see from this graph that the inductive hypothesis holds for $n = 5$, so assume $n \geq 6$. Suppose the transposition of x contains some α with $\alpha > 2$. Then by Lemma 1.5.5 (iii) x commutes with some $y \in X$ such that y has the 1-cycle $\overset{\overset{0}{-}}{(\alpha)}$ and such that the transposition of y is not $\overset{\overset{\lambda}{*}}{(1\ 2)}$. Ignoring the cycle $\overset{\overset{0}{-}}{(\alpha)}$ we can work within $G_{\{1,\ldots,\alpha-1,\alpha+1,\ldots,n\}}$, to see that inductively $d(y,a) \leq n-2$. Hence $d(x,a) \leq n-1$. If the transposition of x is $\overset{\overset{\lambda}{*}}{(1\ 2)}$ then x certainly commutes with an element of X which does not have this transposition. So $d(x,a) \leq n$ as required.

Now we assume $m \geq 2$ and proceed by induction on k_e to show that Diam $\mathcal{C}(G,X) \leq n-1$. Suppose $k_e = 2$. Then x is distance at most 2 from an element y of X which has the transposition $\overset{\overset{0}{+/-}}{(n-1\ \ n)}$. To see this, note that if both $n-1$ and n appear in transpositions of x, or if both appear in 1-cycles of x, then Lemma 1.5.5 or Lemma 1.5.6, as appropriate, implies that x commutes with some x' in X which contains a transposition of the form $\overset{\overset{\lambda}{-/+}}{(n-1\ \ n)}$ for some λ. Then x' commutes with y. If on the other hand, x contains (for example) the 1-cycle $\overset{\overset{\mu}{-}}{(n)}$ and $n-1$ appears in a transposition $\overset{\overset{\sigma}{+/-}}{(\varepsilon\ \ n-1)}$ for some ε less than $n-1$, then x commutes with x'' in X containing the transpositions $\overset{\overset{\lambda}{-}}{(\varepsilon\ \ n-1)}$ and $\overset{\overset{\sigma}{-}}{(\varepsilon'\ \ n)}$ for some λ and ε'. Lemma 1.5.6 now implies that x commutes with an appropriate y, in particular one containing the transpositions $\overset{\overset{\sigma+\lambda}{-}}{(\varepsilon\ \varepsilon')}$ and $\overset{\overset{0}{-}}{(n-1\ \ n)}$. Now y in turn commutes with some z in X with the 1-cycles $\overset{\overset{0}{-}}{(n-1)}$ and $\overset{\overset{0}{-}}{(n)}$. If we ignore these cycles and work in G_{n-2}, then Table 2.2 implies that when $n = 6$, $d(z,a) \leq 2$, and when $n > 6$ Proposition 2.3.3 tells us that $d(z,a) \leq 3$. Therefore Diam $\mathcal{C}(G,X) \leq n-1$.

Finally, suppose $m \geq 2$ and $k_e > 2$. Suppose there is some transposition of x containing an element α with $\alpha > 2m$. Then by Lemma 1.5.5(iii) x commutes with some $y \in X$ such that y has the 1-cycle $\overset{\overset{0}{-}}{(\alpha)}$. By induction $d(y,a) \leq n-2$. Hence $d(x,a) \leq n-1$. The final possibility is that the elements of the transpositions of x are $\{1, 2, \ldots, 2m\}$. Since $m > 1$ we can use Lemma 1.5.6 to show that x commutes with some y in X containing the transposition $\overset{\overset{0}{*}}{(1\,2)}$. Working in $G_{\{3,4,\ldots,n\}}$ (using the case $m = 1$ and induction on m) we see that $d(y, a) \leq n - 2$. Hence $d(x, a) \leq n - 1$, which completes the proof of Proposition 2.3.4. $\square$

Lemma 2.3.5. *Suppose $m = 1$, $l = 0$, $k_o = 2$, $k_e \geq 3$ and $x \in X$. Then $\mathcal{C}(G, X)$ is connected with diameter at most $n + 1$.*

Proof. Let $x \in X$. The distance of x from a will largely hinge on the whereabouts of n and $n - 1$. But first we deal with the case where the transposition of x is $\overset{\overset{\lambda}{+/-}}{(12)}$. Here x commutes with an element y of X having cycles $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\mu'}{-}}{(2)}$ with μ and μ' odd. Using Proposition 2.3.4 on the remaining cycles of y, we see that y is distance at most $n - 2$ from the element b of X whose cycles are the same as a except that we have $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\mu'}{-}}{(2)}\overset{\overset{0}{+}}{(n-1\;n)}$ instead of $\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}\overset{\overset{0}{+}}{(1\,2)}$. Clearly $d(b, a) = 2$. Hence $d(x, a) \leq n + 1$. This also shows that any element with odd negative 1-cycles $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\mu'}{-}}{(2)}$ is distance at most n from a. Assume from now on that the transposition of x is not $\overset{\overset{\lambda}{+/-}}{(12)}$, and that its odd negative 1-cycles are not $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\mu'}{-}}{(2)}$.

If the transposition of x is $\overset{\overset{\lambda}{+/-}}{(n-1\;n)}$ then x is distance 2 from some $y \in X$ with cycles $\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}$. Ignoring these cycles we use Proposition 2.3.4 in G_{n-2} to see that $d(y, a) \leq n - 2$ if the transposition of y is $\overset{\overset{\lambda}{+/-}}{(12)}$ and $d(y, a) \leq n - 3$ otherwise. Thus $d(x, a) \leq n$.

If x has cycles $\overset{\overset{\mu}{-}}{(n-1)}\overset{\overset{\mu'}{-}}{(n)}$ where $\mu \equiv \mu' \mod 2$, then x is distance 3 from some y in X with cycles $\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}$. Thus $d(x, a) \leq n + 1$ if the transposition of x is $\overset{\overset{\lambda}{+/-}}{(12)}$ and $d(x, a) \leq n$ otherwise.

If one of $n - 1$ and n, say n, appears in the transposition of x and the other appears in an even negative 1-cycle $\overset{\overset{\lambda}{-}}{(n-1)}$, then x commutes with some $x' \in X$ with the cycles $\overset{\overset{\lambda}{-}}{(n-1)}\overset{\overset{\lambda}{-}}{(n)}$. Then x' is distance 2 from some $y \in X$ with the cycles $\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}$, and such that the transposition of y is not $\overset{\overset{\sigma}{+/-}}{(12)}$ for any σ. By Proposition 2.3.4 again, $d(x, a) \leq n$.

If x contains $\overset{\overset{\lambda}{-}}{(n-1)}\overset{\overset{\mu}{-}}{(n)}\overset{\overset{2\sigma-\mu}{-}}{(\alpha)}$ where λ is even, μ is odd, σ is an integer and $\alpha < n-1$, then x commutes with an element of X containing $\overset{\overset{\lambda}{-}}{(n-1)}\overset{\overset{\sigma}{-}}{(n\;\alpha)}$, which commutes with an element of X containing $(\overset{\overset{\lambda}{-}}{n-1})(\overset{\overset{\lambda}{-}}{n})(\overset{\overset{2\sigma-\lambda}{-}}{\alpha})$, which commutes with an element of X containing $(\overset{\overset{0}{-}}{n-1\ n})(\overset{\overset{2\sigma-\lambda}{-}}{\alpha})$, which finally commutes with an element y of X containing $(\overset{\overset{1}{-}}{n-1})(\overset{\overset{1}{-}}{n})(\overset{\overset{2\sigma-\lambda}{-}}{\alpha})$, such that the transposition of y is the same as the transposition of x, in particular not $(\overset{\overset{\lambda}{+/-}}{1\ 2})$. By Proposition 2.3.4, $d(y,a) \leq n-3$. Hence $d(x,a) \leq n+1$.

The final case to consider is where n is contained in the transposition of x and $n-1$ is in an odd negative 1-cycle. So x contains $(\overset{\overset{\lambda}{-}}{\alpha\ n})(\overset{\overset{\mu}{-}}{\beta})(\overset{\overset{2\sigma-\mu}{-}}{n-1})$ for some α, β and integers λ, μ, σ with μ odd. If $\beta \notin \{1,2\}$ then subject to appropriate conjugation we can set $\beta = 3$. Using Proposition 2.3.4 in $G_{\{1,\dots,n\}\setminus\{3,n-1\}}$ we see that x is distance at most $n-3$ from the element y where $y = (\overset{\overset{0}{+}}{12})(\overset{\overset{2\sigma}{-}}{n})(\overset{\overset{0}{-}}{4})\cdots(\overset{\overset{0}{-}}{n-2})(\overset{\overset{\mu}{-}}{3})(\overset{\overset{2\sigma-\mu}{-}}{n-1})$. Then y commutes with

$$(\overset{\overset{\sigma}{-}}{3\ n-1})(\overset{\overset{2\sigma}{-}}{n})(\overset{\overset{0}{-}}{4})\cdots(\overset{\overset{0}{-}}{n-2})(\overset{\overset{1}{-}}{1})(\overset{\overset{1}{-}}{2}),$$

which commutes with $(\overset{\overset{0}{+}}{45})(\overset{\overset{2\sigma}{-}}{n})(\overset{\overset{2\sigma}{-}}{n-1})(\overset{\overset{0}{-}}{3})(\overset{\overset{0}{-}}{6})\cdots(\overset{\overset{0}{-}}{n-2})(\overset{\overset{1}{-}}{1})(\overset{\overset{1}{-}}{2})$ which is distance 2 from a. Thus $d(x,a) \leq n+1$. The last case is where without loss of generality $\beta = 1$ and we can assume α is 2 or 5. Let α' be the other element of $\{2,5\}$. Then x is distance 2 from some x' in X containing $(\overset{\overset{\sigma}{-}}{1\ n-1})(\overset{\overset{\tau}{-}}{\alpha})(\overset{\overset{\tau'}{-}}{\alpha'})(\overset{\overset{\nu}{-}}{3})(\overset{\overset{2\kappa-\nu}{-}}{4})$ where τ' is determined by x but we may choose τ arbitrarily, and ν, κ are integers with ν odd. Now let $y = (\overset{\overset{0}{+}}{n-1\ n})(\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{-}}{5})\cdots(\overset{\overset{0}{-}}{n-2})(\overset{\overset{\nu}{-}}{3})(\overset{\overset{2\kappa-\nu}{-}}{4})$. Then $d(x,x') = 2$ and $d(y,a) = 3$. What is $d(x',y)$? If $k_e > 3$ then $n \geq 8$. Set $\tau = 0$. Now working in $G_{\{1,\dots,n\}\setminus\{\alpha,3,4\}}$ we see from Proposition 2.3.4 that $d(x',y) \leq n-4$. If $k_e = 3$ then $n = 7$. This time set $\tau = \tau'$. Then x' commutes with $(\overset{\overset{0}{+}}{25})(\overset{\overset{0}{-}}{1})(\overset{\overset{2\sigma}{-}}{6})(\overset{\overset{2\sigma'}{-}}{7})(\overset{\overset{\nu}{-}}{3})(\overset{\overset{2\kappa-\nu}{-}}{4})$ for some σ', which commutes with $(\overset{\overset{\sigma+\sigma'}{-}}{67})(\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{-}}{5})(\overset{\overset{\nu}{-}}{3})(\overset{\overset{2\kappa-\nu}{-}}{4})$ which commutes with y, so $d(x',y) = 3$. Hence in all cases $d(x,a) \leq n+1$, which completes the proof of Lemma 2.3.5. □

Theorem 2.3.6. *Suppose $n \geq 7$, $m \geq 1$, $l = 0$ and k_e and k_o are both at least 2. Then $\mathcal{C}(G,X)$ is connected with diameter at most $n+2$. If $m \geq 2$, then* Diam $\mathcal{C}(G,X) \leq n$.

Proof. Assume that $k_e \geq k_o$. Suppose first that $m = 1$ and let $x \in X$. We use induction on k_o to show that if the transposition of x is not $(\overset{\overset{\lambda}{+/-}}{1\ 2})$, then $d(x,a) \leq n+1$. Otherwise $d(x,a) \leq n+2$. If $k_o = 2$ the result holds by Lemma 2.3.5. If $k_o > 2$ and the transposition of x is not $(\overset{\overset{\lambda}{+/-}}{1\ 2})$, then x contains cycles $(\overset{\overset{\nu}{+/-}}{\alpha\beta})(\overset{\overset{\lambda}{-}}{\gamma})(\overset{\overset{\lambda'}{-}}{\delta})$ where $\alpha > 2$, $\gamma > 2$, $(\overset{\overset{\sigma}{-}}{\alpha})$ is a cycle of a and λ, λ' and σ are all congruent modulo 2. Now x commutes with some $x' \in X$ containing the cycles $\overset{\overset{\sigma}{-}}{(\alpha)}\overset{\overset{\nu'}{-}}{(\gamma\ \delta)}$ for appropriate ν'. If we ignore $\overset{\overset{\sigma}{-}}{(\alpha)}$ and work in $G_{\{1,\ldots,n\}\setminus\{\alpha\}}$, then inductively $d(x',a) \leq n$. Hence $d(x,a) \leq n+1$. Suppose the transposition of x is $\overset{\overset{\lambda}{+/-}}{(1\ 2)}$. Then x commutes with some y in X that does not contain this transposition, and we have seen that $d(y,a) \leq n+1$. Hence $d(x,a) \leq n+2$. This completes the case $m=1$.

If $m > 1$ and $k_o = 2$ then it is easy to see that x is distance at most 2 from an element of X containing the transposition $\overset{\overset{0}{+}}{(n-1\ \ n)}$. Thus x is distance at most 3 from an element y of X containing $\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}$. Ignoring these 1-cycles we may work in G_{n-2} and apply Proposition 2.3.4 to see that $d(z,a) \leq n-3$. Hence $d(x,a) \leq n$.

Now suppose $m > 1$ and $k_o > 2$. If x has a transposition containing an element α with $\alpha > 2m$, then x commutes with an element y containing $\overset{\overset{0}{-}}{(\alpha)}$ or $\overset{\overset{1}{-}}{(\alpha)}$ (choose whichever of these is a cycle of a). Then we can ignore this cycle and work in $G_{\{1,\ldots,n\}\setminus\{\alpha\}}$. Inductively, using the base case $k_o = 2$, we see that $d(y,a) \leq n-1$. Hence $d(x,a) \leq n$. Finally we deal with the case that every transposition of x is of the form $\overset{\overset{\lambda}{+/-}}{(\alpha\ \beta)}$ where $\alpha < \beta \leq 2m$. Because $k_e \geq k_o \geq 3$, it must be the case that x contains cycles: $\overset{\overset{\sigma_1}{+/-}}{(\alpha_1\ \alpha_2)}\overset{\overset{\sigma_2}{+/-}}{(\alpha_3\ \alpha_4)}\overset{\overset{\lambda_1}{-}}{(\beta_1)}\overset{\overset{\lambda_2}{-}}{(\beta_2)}\overset{\overset{\mu_1}{-}}{(\gamma_1)}\overset{\overset{\mu_2}{-}}{(\gamma_2)}$ where $\lambda_1 \equiv \lambda_2 \mod 2$, $\mu_1 \equiv \mu_2 \mod 2$, $\{\alpha_1,\ldots,\alpha_4\} \subseteq \{1,\ldots,2m\}$, $2m < \beta_1 < \beta_2 \leq 2m+k_e$ and $2m+k_e < \gamma_1 < \gamma_2 \leq n$. Then x is distance 2 from an element y with the cycles $\overset{\overset{0}{-}}{(\beta)}$ and $\overset{\overset{1}{-}}{(\gamma)}$. Now, working inductively in $G_{\{1,\ldots,n\}\setminus\{\beta,\gamma\}}$ we see that $d(y,a) \leq n-2$. Hence $d(x,a) \leq n$. $\square$

Lemma 2.3.7. *If $n=5$, $m=1$, $l=1$ and $k_e=2$, then* Diam $\mathcal{C}(G,X) = 5$.

Proof. Let $x \in X$. If the transposition is $\overset{\overset{\lambda}{+/-}}{(12)}$ then using Table 2.1 for $G_{\{3,4,5\}}$ we see that $d(x,a) \leq 3$. Suppose the transposition of x is $\overset{\overset{\lambda}{+/-}}{(\alpha\ \beta)}$ where $\{\alpha,\beta\} \subseteq \{3,4,5\}$, and let γ be the remaining element of $\{3,4,5\}$. Then x commutes with $x' = \overset{\overset{0}{+/-}}{(\alpha\ \beta)}\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(2)}\overset{\overset{0}{+}}{(\gamma)}$ for some even integers λ and λ'. Now x' commutes with $\overset{\overset{(\lambda+\lambda')/2}{-}}{(12)}\ \overset{\overset{0}{-}}{(\alpha)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{0}{+}}{(\gamma)}$, which commutes with a. So $d(x,a) \leq 3$. The remaining cases are (interchanging 1 and 2 if necessary) when $x = \overset{\overset{\sigma}{+/-}}{(1\ \alpha)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{0}{+}}{(\gamma)}$ or $\overset{\overset{\sigma}{+/-}}{(1\ \alpha)}\overset{\overset{0}{+}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}$ for appropriate σ, λ and λ'. The following is a path of length at most 5 from either of these to a: $\quad x\ ,\ \overset{\overset{0}{-/+}}{(1\ \alpha)}\overset{\overset{0}{+}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}$,

$\overset{\overset{(\lambda'+\lambda)/2}{-}}{(\beta\ \gamma)}\,\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{+}}{(2)}\overset{\overset{0}{-}}{(\alpha)}$, $\overset{\overset{0}{+}}{(\beta\ \gamma)}\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{+}}{(\alpha)}$, $\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{0}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\alpha)}$, a. Hence in all cases $d(x, a) \leq 5$, which completes the proof. □

We observe, because we will need it for Lemma 2.3.9 later, that the proof of Lemma 2.3.7 shows that $d(x, a) \leq 4$ in all cases except where (modulo interchanging 1 and 2, or 3 and 4) the transposition of x is $\overset{\overset{0}{+}}{(1\ 3)}$.

Theorem 2.3.8. *Suppose $m \geq 1$, $l \geq 1$ and $\max\{k_e, k_o, l\} \geq 2$. Then $\mathcal{C}(G, X)$ is connected with diameter at most n.*

Proof. Suppose n is minimal such that $\mathcal{C}(G, X)$ is a counterexample, and let $x \in X$ such that $d(x, a) > n$. By Lemma 2.3.7 we can assume $n \geq 6$. If $l \geq 2$ then $x \in X$ commutes with some $y \in X$ containing $\overset{\overset{0}{+}}{(n)}$. Ignoring this 1-cycle we can work in G_{n-1} to find a path to a, which inductively is of length at most $n-1$, which implies $d(x, a) \leq n$, contrary to our choice of x. Hence $l = 1$.

If elements α and β lying between $2m+1$ and $2m+k_e$ are contained in transpositions of x, then x commutes with some $x' \in X$ having the transposition $\overset{\overset{\lambda}{+/-}}{(\alpha\ \beta)}$ for some λ (if $m = 1$ then we can set $x = x'$). If we ignore this transposition of x' we can work in $G_{\{1,\ldots,n\}\setminus\{\alpha,\beta\}}$, which is either the case $l = 1$ with a smaller m, so inductively the graph has diameter at most $n-2$, or (if $m = 1$) we can use Theorem 2.3.2, in which case the graph has diameter $n-1$. In either case, we see that x is distance at most $n-1$ from the element b of X whose cycles are the same as a except that b has $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{+}}{(\alpha\ \beta)}$ instead of $\overset{\overset{0}{-}}{(12)}\overset{\overset{0}{-}}{(\alpha)}\overset{\overset{0}{-}}{(\beta)}$. Since b commutes with a we have $d(x, a) \leq n$. The same argument holds if $2m + k_e < \alpha < \beta < n$. If $1 \leq \alpha < \beta \leq 2m$ then similar reasoning shows again that x is distance at most $n-1$ from an element b containing $\overset{\overset{0}{+}}{(\alpha\ \beta)}$ that commutes with a. So $d(x, a) \leq n$. Thus none of these pairs α, β exist in transpositions of x. This implies $m \leq 2$. Moreover if $m = 2$ then $k_0 \neq 0$ and without loss of generality the transpositions of x contain $1, 5, n-1$ and n.

If there is some β in a transposition of x with $2m + k_e < \beta < n$ and if $k_o \geq 2$, then x commutes with some $y \in X$ containing $\overset{\overset{1}{-}}{(\beta)}$. Inductively we can work in $G_{\{1,\ldots,n\}\setminus\{\beta\}}$ to see that $d(y, a) \leq n-1$. Hence $d(x, a) \leq n$. Similarly, as long as either $k_e > 2$ or $k_o \geq 2$ (or both), if there is some α in a transposition of x with $2m < \alpha \leq 2m + k_e$ then inductively $d(x, a) \leq n$. This means that if $m = 2$ then $k_e = 2$ and $k_o = 1$.

Suppose that $m = 2$, $k_e = 2$ and $k_o = 1$, so that $n = 8$. We have observed that the transpositions of x must contain $1, 5, 7$ and 8. If 6 is not contained in an even negative 1-cycle of x, then x commutes with some y containing a transposition $\overset{\overset{\lambda}{+/-}}{(\alpha\ \beta)}$ for some λ, where $\{\alpha, \beta\} \subset \{2, 3, 4\}$. As at the start of this proof, $d(y, a) \leq 7$. More explicitly, inductively y is distance at most 6 from an element b containing $\overset{\overset{0}{+}}{(\alpha\ \beta)}$ that commutes with a. Hence $d(x, a) \leq 8$, a contradiction. Thus $m = 1$.

If x contains $\overset{\overset{0}{+}}{(\alpha)}$ where $2+k_e < \alpha < n$ then x commutes with some $x' \in X$ containing $\overset{\overset{1}{-}}{(\alpha)}$. So, using the result for $G_{\{1,\ldots,n\}\setminus\{\alpha\}}$, we get $d(x', a) \leq n-1$. Thus $d(x, a) \leq n$. The same reasoning holds if $k_e > 2$ and x contains $\overset{\overset{0}{+}}{(\alpha)}$ for some α where $2m < \alpha \leq 2m+k_e$. Suppose first that $k_o \neq 1$. Since $n \geq 6$ we must have $k_o \geq 2$ or $k_o = 0$ and $k_e \geq 3$. Thus the transposition of x cannot contain any α with $2m < \alpha < n$. Hence without loss of generality x contains $\overset{\overset{\lambda}{*}}{(1\ n)}\overset{\overset{0}{+}}{(2)}$. Suppose that x contains $\overset{\overset{\sigma}{-}}{(\beta)}\overset{\overset{\sigma'}{-}}{(\beta')}$ where $\sigma \equiv \sigma' \mod 2$ and either $3 \leq \beta < \beta' \leq k_e + 2$ or $k_e + 3 \leq \beta < \beta' < n$. Then, x commutes with some $y \in X$ where y contains $\overset{\overset{\lambda}{-}}{(\beta\ \beta')}\overset{\overset{\sigma}{-}}{(1)}$ for some λ. Now a commutes with some element z containing $\overset{\overset{\lambda}{-}}{(\beta\ \beta')}\overset{\overset{\sigma}{-}}{(1)}\overset{\overset{\sigma}{-}}{(2)}$. Using Theorem 2.3.2 on $G_{\{1,\ldots,n\}\setminus\{1,\beta,\beta'\}}$ (that is, removing $\overset{\overset{\lambda}{-}}{(\beta\ \beta')}\overset{\overset{\alpha}{-}}{(1)}$, from y and z) we see that $d(y, z) \leq n-2$. Therefore, $d(x, a) \leq n$. Finally, we are reduced to the possibility that x contains $\overset{\overset{\lambda}{*}}{(1\ n)}\overset{\overset{0}{+}}{(2)}$ and any pair $\overset{\overset{\sigma}{-}}{(\beta)}\overset{\overset{\sigma'}{-}}{(\beta')}$ where $\beta < \beta'$ and $\sigma \equiv \sigma' \mod 2$ satisfies $3 \leq \beta \leq k_e + 2 < \beta' < n$. Since $n > 5$ and $k_o \neq 1$, the only way this can occur is when $k_e = k_o = 2$ and $n = 7$. Conjugating by an element of the centraliser of a if necessary, we can assume that $x = \overset{\overset{\lambda}{*}}{(1\ 7)}\overset{\overset{\sigma}{-}}{(3)}\overset{\overset{\sigma'}{-}}{(5)}\overset{\overset{\mu}{-}}{(4)}\overset{\overset{\mu'}{-}}{(6)}\overset{\overset{0}{+}}{(2)}$ where σ and σ' are even, and μ, μ' are odd. The following is a path from x to a in the graph: x, $\overset{\overset{\sigma-\sigma'}{+}}{(3\ 5)}\overset{\overset{0}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(7)}\overset{\overset{\mu}{-}}{(4)}\overset{\overset{\mu'}{-}}{(6)}\overset{\overset{0}{+}}{(2)}$, $\overset{\overset{0}{-}}{(3\ 5)}\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{\mu}{-}}{(4)}\overset{\overset{\mu'}{-}}{(6)}\overset{\overset{0}{+}}{(7)}$, $\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(5)}\overset{\overset{\mu'}{-}}{(6)}\overset{\overset{1}{-}}{(7)}\overset{\overset{0}{+}}{(4)}$, $\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{\mu'}{-}}{(6)}\overset{\overset{1}{-}}{(7)}\overset{\overset{0}{+}}{(5)}$, $\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(5)}\overset{\overset{1}{-}}{(7)}\overset{\overset{0}{+}}{(6)}$, a. So $d(x, a) \leq 6$, another contradiction.

The final case to consider is where $m = 1$ and $k_o = 1$. Assume first that $n \geq 7$. Suppose that x contains $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(2)}$ with λ, λ' both even. Then x commutes with some $y \in X$ containing $\overset{\overset{\nu}{-}}{(1\ 2)}$ for some ν. As observed at the start of this proof, $d(y, a) \leq n-1$. Hence $d(x, a) \leq n$. Now suppose x does not contain $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(2)}$. Then there is a set $A = \{\alpha_1, \ldots, \alpha_6\}$ containing 1, 2 and $n-1$ such that $x = \overset{\overset{\sigma}{+/-}}{(\alpha_1\ \alpha_2)}\overset{\overset{\lambda}{-}}{(\alpha_3)}\overset{\overset{\lambda'}{-}}{(\alpha_4)}\overset{\overset{\mu}{-}}{(\alpha_5)}\overset{\overset{0}{+}}{(\alpha_6)}\bar{x}$

where λ and λ' are both even and μ is odd. By replacing x with a conjugate under the centraliser of a, we can further assume that $A = \{1,2,3,4,n-1,\beta\}$ for some β. Let $z = \overset{\overset{0}{+}}{(1\;2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(n-1)}\overset{\overset{0}{+}}{(\beta)}\bar{x}$. Now, using the case $k_o = 1, n = 6$ on $G_{\{1,2,3,4,n-1,\beta\}}$ we see that $d(x,z) \leq 6$. Next we apply Theorem 2.3.2 to $\overset{\overset{0}{+}}{(\beta)}x'$ and $\overset{\overset{0}{-}}{(5)}\overset{\overset{0}{-}}{(6)}\ldots\overset{\overset{0}{-}}{(n-2)}\overset{\overset{0}{+}}{(n)} \in G_{\{5,\ldots,n-2,n\}}$ (noting that here there are no odd negative 1-cycles) to see that $d(z,a) \leq n-5$. Hence $d(x,a) \leq 6+n-5 = n+1$, another contradiction.

The remaining possibility is that $n = 6$, $m = 1$, $k_e = 2, k_o = 1$. But Lemma 2.3.9 immediately after this proof shows that this graph has diameter at most 6, which is the final contradiction completing the proof of Theorem 2.3.8. $\square$

Lemma 2.3.9. *If* $n = 6$*,* $m = 1$*,* $k_e = 2$ *and* $k_o = 1$ *then* $\operatorname{Diam} \mathcal{C}(G,X) \leq 6$.

Proof. We have $a = \overset{\overset{0}{+}}{(1\;2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(6)}$. Let $x \in X$. Suppose x contains $\overset{\overset{\sigma}{+/-}}{(1\;2)}$ for some σ. By Theorem 2.3.2, the graph of $\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(6)}$ in $G_{\{3,4,5,6\}}$ has diameter 5. Hence $d(x,a) \leq 5$. If x contains $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(2)}$ where λ, λ' are even, then x commutes with some x' containing $\overset{\overset{\sigma}{+/-}}{(1\;2)}$ for some σ, and we have just seen that $d(x',a) \leq 5$. Hence $d(x,a) \leq 6$. We next consider the cases where 1 and 2 are in different types of 1-cycle of x. Because interchanging 1 and 2 does not affect the distance of x from a, there are just three cases to consider here: x contains $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$, $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\mu}{-}}{(2)}$ or $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$, where λ is even and μ is odd. In the first case we have $x = \overset{\overset{\sigma}{+/-}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(\gamma)}\overset{\overset{\mu}{-}}{(\delta)}\overset{\overset{0}{+}}{(2)}$ where $\{\alpha,\beta,\gamma,\delta\} = \{3,4,5,6\}$, λ, λ' are even and μ is odd. Here x commutes with $\overset{\overset{0}{-/+}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\mu}{-}}{(\delta)}\overset{\overset{0}{+}}{(\gamma)}$, which commutes with the element b given by $b = \overset{\overset{0}{+}}{(1\;2)}\overset{\overset{0}{-}}{(\alpha)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{1}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}$. Glancing at Table 2.2 we observe that in $G_{\{3,4,5,6\}}$ elements at distance 4 or 5 from $\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(6)}$ require at least one even label to be non-zero So $d(b,a) \leq 3$. hence $d(x,a) \leq 5$. For the second case, where x contains $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\mu}{-}}{(2)}$, note that x commutes with some x' containing $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$. Hence, using the first case we get $d(x,a) \leq 6$. The third case is where x contains $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$. Here, x commutes with some x' of the form $\overset{\overset{\sigma}{+/-}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}\overset{\overset{\lambda'}{-}}{(5)}\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$. The following is a path from x' to a: $x' = \overset{\overset{\sigma}{+/-}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}\overset{\overset{\lambda'}{-}}{(5)}\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$, $\overset{\overset{0}{-/+}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}\overset{\overset{0}{-}}{(2)}\overset{\overset{\mu}{-}}{(1)}\overset{\overset{0}{+}}{(5)}$, $\overset{\overset{0}{+}}{(\alpha\;\beta)}\overset{\overset{\lambda}{-}}{(\gamma)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(1)}$, $\overset{\overset{0}{+}}{(\alpha\;\beta)}\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(\gamma)}$, $\overset{\overset{0}{+}}{(1\;2)}\overset{\overset{0}{-}}{(\alpha)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(\gamma)}$, a. So $d(x,a) \leq 6$.

It remains to take care of the possibility that exactly one of 1 and 2 lies in the transposition of x. Here, without loss of generality, we can assume x contains $\overset{\overset{\sigma}{+/-}}{(1\ \alpha)}$ for $\alpha > 2$. If x contains $\overset{\overset{\mu}{-}}{(2)}$ for μ odd, or $\overset{\overset{0}{+}}{(2)}$, then x commutes with some x' containing $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{0}{+}}{(2)}$, for even λ, and as shown earlier in this proof, $d(x', a) \leq 5$. Hence $d(x, a) \leq 6$. We may thus assume that $x = \overset{\overset{\sigma}{+/-}}{(1\ \alpha)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}$. If $\delta = 5$ then x commutes with x' given by $\overset{\overset{\sigma}{+/-}}{(1\ \alpha)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{1}{-}}{(5)}\overset{\overset{0}{+}}{(\gamma)}$. By Theorem 2.3.7 applied to $G_{\{1,2,3,4,6\}}$ we see that $d(x', a) \leq 5$ and so $d(x, a) \leq 6$. So we can assume that $\delta \neq 5$. Now x is distance 4 from the element b given by $b = \overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{0}{-}}{(\delta)}\overset{\overset{1}{-}}{(\alpha)}\overset{\overset{0}{+}}{(\gamma)}$, as shown by the following path. $x = \overset{\overset{\sigma}{+/-}}{(1\ \alpha)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}, \overset{\overset{0}{-/+}}{(1\ \alpha)}\overset{\overset{\lambda'}{-}}{(\delta)}\overset{\overset{\lambda'}{-}}{(\beta)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(2)}, \overset{\overset{0}{+}}{(\beta\ \delta)}\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(\alpha)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(2)}, \overset{\overset{0}{+}}{(\beta\ \delta)}\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\alpha)}$, $\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{-}}{(\beta)}\overset{\overset{0}{-}}{(\delta)}\overset{\overset{1}{-}}{(\alpha)}\overset{\overset{0}{+}}{(\gamma)} = b$. If $\alpha = 5$, then $d(b, a) = 1$ so $d(x, a) \leq 5$. If $\alpha = 6$, then $d(b, a) \leq 2$, so $d(x, a) \leq 6$. If $\gamma = 5$, then $d(b, a) \leq 2$ so $d(x, a) \leq 6$. Since interchanging 3 and 4 does not affect the distance from a, there is only one case left to deal with: $\alpha = 3$ and $\beta = 5$. So $x = \overset{\overset{\sigma}{+/-}}{(1\ 3)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda'}{-}}{(5)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}$. But x is the same distance from a as $y = \overset{\overset{\sigma}{+/-}}{(2\ 3)}\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{\lambda'}{-}}{(5)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}$, and y commutes with z given by $z = \overset{\overset{\sigma'}{+/-}}{(1\ 5)}\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\lambda''}{-}}{(3)}\overset{\overset{\mu}{-}}{(\gamma)}\overset{\overset{0}{+}}{(\delta)}$ for appropriate λ'' and σ'. By the '$\alpha = 5$' case above, $d(z, a) \leq 5$. Therefore $d(x, a) = d(y, a) \leq d(z, a) + 1 \leq 6$. This completes the proof. $\square$

Proof of Theorems 2.1.1 and 2.1.2 Suppose $m = 0$. If $l = 0$ then $\mathcal{C}(G, X)$ is disconnected by Theorem 2.3.1(i). Otherwise Diam $\mathcal{C}(G, X) \leq n + 1$ by Theorem 2.3.2. Suppose $m = 1$ and $l = 0$. If one of k_e and k_o is 1 or the largest of k_e and k_o is 2, then $\mathcal{C}(G, X)$ is disconnected by Theorem 2.3.1 (ii), (iv) and (v). Otherwise Diam $\mathcal{C}(G, X) \leq n + 2$ by Propositions 2.3.3, 2.3.4, Lemma 2.3.5 and Theorem 2.3.6. Suppose $m > 1$ and $l = 0$. If either of k_e or k_o is 1, then $\mathcal{C}(G, X)$ is disconnected by Theorem 2.3.1(ii). Otherwise Diam $\mathcal{C}(G, X) \leq n$ by Propositions 2.3.3, 2.3.4 and Theorem 2.3.6. Finally suppose $m \geq 1$ and $l > 0$. If $\max\{k_e, k_o, l\} = 1$ then $\mathcal{C}(G, X)$ is disconnected by Theorem 2.3.1(iii). Otherwise Diam $\mathcal{C}(G, X) \leq n$ by Lemma 2.3.7, Theorem 2.3.8 and Lemma 2.3.9. $\square$

2.4 Examples

We first summarise the information on commuting involution graphs for G_2, G_3 and G_4. For fixed $a \in X$, the i^{th} disc $\Delta_i(a)$ is the set of elements of X which are distance i from a. Since a length preserving graph automorphism interchanges classes with $k_e \geq k_o$

and those with $k_e \leq k_o$, we list here only those classes with $k_e \geq k_o$. In describing the elements we omit positive 1-cycles. In G_2 there are two connected graphs, both of diameter 2. If $a = \overset{\overset{0}{-}}{(1)}$ then $\Delta_1(a) = \{\overset{\overset{\lambda}{-}}{(2)} : \lambda \text{ even}\}$ with the remaining elements of X comprising the second disc. If $a = \overset{\overset{0}{+}}{(12)}$ then $\Delta_1(a) = \{\overset{\overset{\sigma}{-}}{(12)} : \sigma \in \mathbb{Z}\}$, with all other elements of X lying in the second disc. Tables 2.1 and 2.2 give, for each connected graph and each disc, a list of orbit representatives under the action of $C_G(a)$. We use λ, λ' and so on to represent arbitrary even numbers, with $[\lambda]$ being an arbitrary nonzero even number. We use μ, μ' and so on for arbitrary odd numbers, with $[\mu]$ being any odd number other than 1. Finally σ, σ' and so on will be arbitrary integers with $[\sigma]$ an arbitrary nonzero integer.

a	Δ_1	Δ_2	Δ_3	Δ_4
$\overset{\overset{0}{-}}{(1)}$	$\overset{\overset{\lambda}{-}}{(2)}$	$\overset{\overset{[\lambda]}{-}}{(1)}$		
$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{\lambda}{-}}{(3)}$	$\overset{\overset{\lambda'}{-}}{(2)}\overset{\overset{\lambda}{-}}{(3)}$, $\overset{\overset{0}{-}}{(1)}\overset{\overset{[\lambda']}{-}}{(2)}$	$\overset{\overset{[\lambda]}{-}}{(1)}\overset{\overset{[\lambda']}{-}}{(2)}$	
$\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2)}$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{\mu}{-}}{(3)}$, $\overset{\overset{1}{-}}{(2)}\overset{\overset{\lambda}{-}}{(3)}$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{\mu}{-}}{(2)}$, $\overset{\overset{\lambda}{-}}{(2)}\overset{\overset{\mu}{-}}{(3)}$, $\overset{\overset{\lambda}{-}}{(1)}\overset{\overset{1}{-}}{(2)}$, $\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\lambda}{-}}{(3)}$	$\overset{\overset{\mu}{-}}{(1)}\overset{\overset{\lambda}{-}}{(2)}$, $\overset{\overset{[\lambda]}{-}}{(1)}\overset{\overset{\mu}{-}}{(3)}$, $\overset{\overset{[\mu]}{-}}{(2)}\overset{\overset{\lambda}{-}}{(3)}$	$\overset{\overset{[\lambda]}{-}}{(1)}\overset{\overset{[\mu]}{-}}{(2)}$

Table 2.1: Connected Graphs for G_3 (with $k_e \geq k_o$)

Figure 2.1 is the collapsed adjacency graph for $a = \overset{\overset{0}{+}}{(12)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}\overset{\overset{0}{-}}{(5)}$ in G_5, which has diameter 5. In the graph σ is an arbitrary integer, $[\sigma]$ is an arbitrary non-zero integer and λ, λ' and λ'' are arbitrary even integers. For simplicity we have only included shortest paths – that is, we have omitted edges between nodes in the same disc.

a	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
$\overset{\underline{0}}{(1)}$	$\overset{\underline{\lambda}}{(2)}$	$\overset{\underline{\lceil\lambda\rceil}}{(1)}$			
$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\lambda}}{(3)}$, $\overset{\underline{\lambda}}{(3)}\overset{\underline{\lambda'}}{(4)}$	$X \setminus (\Delta_1(a) \cup \{a\})$			
$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{0}}{(3)}$	$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{\lambda}}{(4)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\lambda'}}{(3)}\overset{\underline{\lambda}}{(4)}$ $\overset{\underline{o}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{\lceil\lambda\rceil}}{(3)}$	$\overset{\underline{\lambda}}{(2)}\overset{\underline{\lambda'}}{(3)}\overset{\underline{\lambda''}}{(4)}$ $\overset{\underline{0}}{(1)}\overset{\underline{\lceil\lambda\rceil}}{(2)}\overset{\underline{\lceil\lambda'\rceil}}{(3)}$	$\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lceil\lambda'\rceil}}{(2)}\overset{\underline{\lceil\lambda''\rceil}}{(3)}$	
$\overset{\underline{0}}{(1)}\overset{\underline{1}}{(2)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\mu}}{(3)}$, $\overset{\underline{\lambda}}{(3)}\overset{\underline{\mu}}{(4)}$ $\overset{\underline{1}}{(2)}\overset{\underline{\lambda}}{(3)}$	$X \setminus (\Delta_1(a) \cup \{a\})$			
$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{1}}{(3)}$	$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{\mu}}{(4)}$ $\overset{\underline{0}}{(1)}\overset{\underline{1}}{(3)}\overset{\underline{\lambda}}{(4)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\lambda}}{(3)}\overset{\underline{\mu}}{(4)}$,$\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{\lceil\mu\rceil}}{(3)}$ $\overset{\underline{0}}{(1)}\overset{\underline{\mu}}{(2)}\overset{\underline{\lambda}}{(4)}$, $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{1}}{(3)}\overset{\underline{\lambda'}}{(4)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{1}}{(3)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\mu}}{(2)}\overset{\underline{\lambda}}{(3)}$, $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lceil\lambda'\rceil}}{(2)}\overset{\underline{1}}{(3)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\mu}}{(2)}\overset{\underline{\lambda'}}{(4)}$, $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{\mu}}{(4)}$ $\overset{\underline{0}}{(1)}\overset{\underline{\lceil\mu\rceil}}{(3)}\overset{\underline{\lambda}}{(4)}$,$\overset{\underline{\mu}}{(1)}\overset{\underline{\lambda}}{(3)}\overset{\underline{\lambda'}}{(4)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lambda'}}{(3)}\overset{\underline{\mu}}{(4)}$	$\overset{\underline{0}}{(1)}\overset{\underline{\lceil\lambda\rceil}}{(2)}\overset{\underline{\lceil\mu\rceil}}{(3)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\mu}}{(2)}\overset{\underline{\lambda'}}{(3)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lceil\lambda'\rceil}}{(2)}\overset{\underline{\mu}}{(4)}$ $\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lceil\mu\rceil}}{(3)}\overset{\underline{\lambda'}}{(4)}$	$\overset{\underline{\lceil\lambda\rceil}}{(1)}\overset{\underline{\lceil\lambda'\rceil}}{(2)}\overset{\underline{\lceil\mu\rceil}}{(3)}$
$\overset{\underset{+}{0}}{(12)}\overset{\underset{+}{0}}{(34)}$	$\overset{\underline{\sigma}}{(12)}\overset{\underset{+}{0}}{(34)}$ $\overset{\underline{\sigma}}{(12)}\overset{\underline{\sigma'}}{(34)}$ $\overset{\underset{+}{\sigma}}{(13)}\overset{\underset{+}{\sigma}}{(24)}$ $\overset{\underline{\sigma'}}{(13)}\overset{\underline{\sigma''}}{(24)}$	$X \setminus (\Delta_1(a) \cup \{a\})$			

Table 2.2: Connected Graphs for G_4 (with $k_e \geq k_o$)

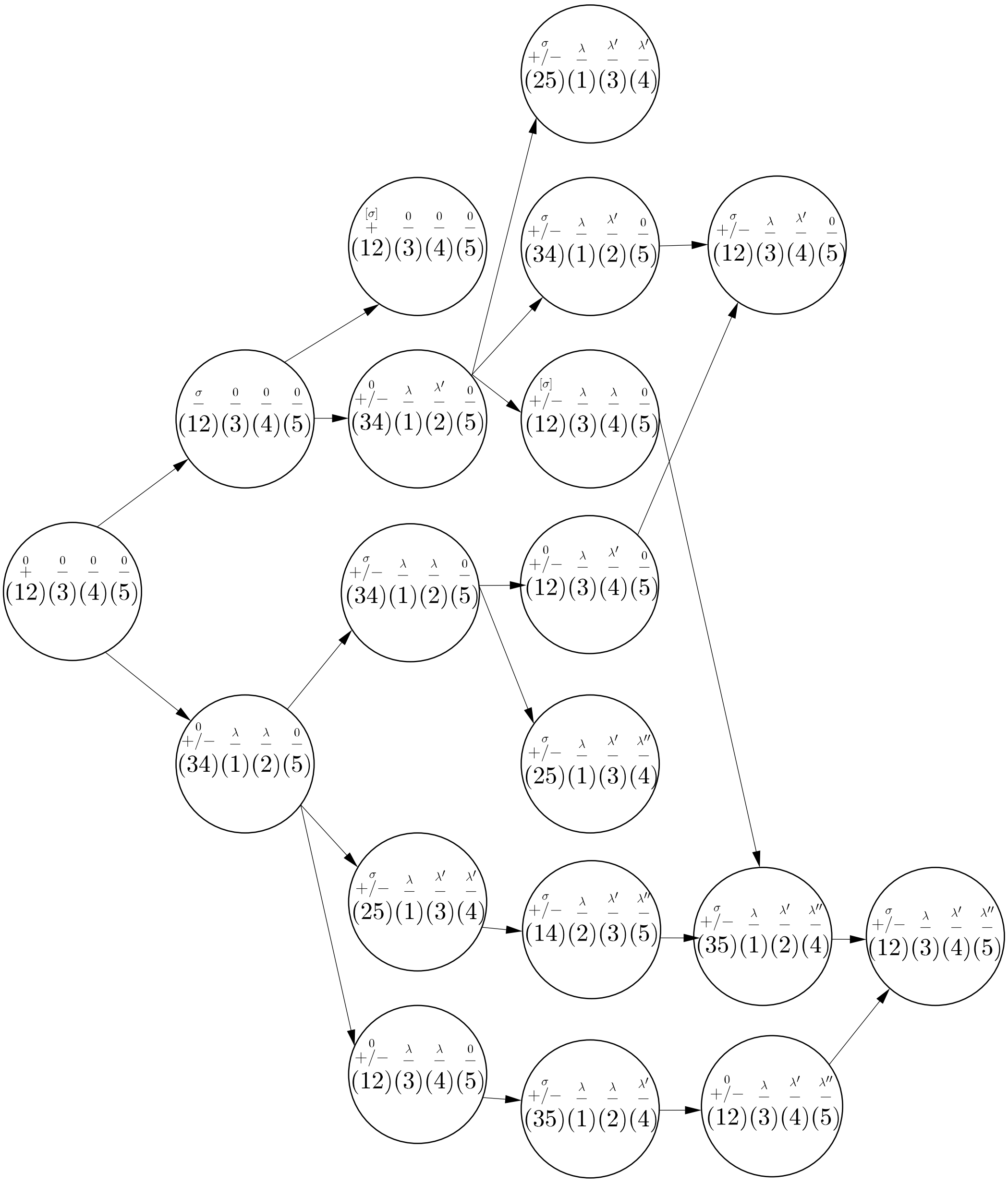


Figure 2.1: $n = 5$, $k = 3$ and $m = 1$

Chapter 3

Coxeter Groups of Type $\tilde{B}_n$ or $\tilde{D}_n$

3.1 Introduction

In this chapter, we explore the commuting involution graphs $\mathcal{C}(G, X)$ in the affine Coxeter groups $\tilde{B}_n$ and $\tilde{D}_n$ by taking into consideration results for the corresponding groups ([7], [23]) and for $W(\tilde{C}_n)$, as derived in Chapter 2. We prove that $\mathcal{C}(W(\tilde{C}_n), X)$, $\mathcal{C}(W(\tilde{B}_n), X)$ and $\mathcal{C}(W(\tilde{D}_n), X)$ have the same diameter in most cases.
Let G_n denote $W(\tilde{B}_n)$ for some $n \geq 2$ or $W(\tilde{D}_n)$ for some $n \geq 4$, for the rest of this chapter, writing G when n is not specified and let X be a conjugacy class of involutions of G.
We will show that conjugacy classes in G are parametrised by the number m of transpositions, the number k_o of negative 1-cycles with an odd number above them, the number k_e of negative 1-cycles with an even number above them, and the number l of positive 1-cycles (the fixed points). We prove that if X is a conjugacy class of involutions, then either the graph is disconnected or it has a diameter at most $(n+2)$.
Our main results are as follows:

Theorem 3.1.1. *Suppose that X is a conjugacy class of involutions in $W(\tilde{B}_n)$ (where $n \geq 2$) or $W(\tilde{D}_n)$ (where $n \geq 4$), such that the elements of X have labelled cycle type (m, k_e, k_o, l). Then $\mathcal{C}(G, X)$ is disconnected in each of the following cases.*

(i) $m = 0$ *and* $l = 0$;

(ii) $m > 0$, $l = 0$, $k_e = 1$;

(iii) $m > 0$ *and* $\max\{k_e, k_o, l\} = 1$;

(iv) $n = 4$ *and* $m = 1$;

(v) $n = 6$, $m = 1$, $k_o = k_e = 2$.

Theorem 3.1.2. *If $m = 0$ and $l \geq 1$, then*

$$\text{Diam}\,\mathcal{C}(G,X) = \begin{cases} 2 & \text{if } 2l \geq n \\ \lceil \frac{n}{l} \rceil & \text{if } 0 \in \{k_e, k_o\} \\ n+1 & \text{if } l = 1 \text{ and } 0 \notin \{k_e, k_o\}. \end{cases}$$

In all other cases $\lceil \frac{n}{l} \rceil \leq \text{Diam}\,\mathcal{C}(G,X) \leq \lceil \frac{n}{l} \rceil + 2$.

Theorem 3.1.3. *Suppose $m \geq 1$.*

(i) If $2m = n$, then $\text{Diam}\,\mathcal{C}(G,X) \leq 4$.

(ii) If $l = 0$, $0 \in \{k_e, k_o\}$, and either $m > 1$ or $\max\{k_e, k_o\} \geq 3$, *then* $\text{Diam}\,\mathcal{C}(G,X) \leq n+1$.

(iii) If $l = 0$ and $\min\{k_e, k_o\} \geq 2$, *then* $\text{Diam}\,\mathcal{C}(G,X) \leq n+2$.

(iv) If $l \geq 1$ and $\max\{k_e, k_o, l\} > 1$, *then* $\text{Diam}\,\mathcal{C}(G,X) \leq n$.

Corollary 3.1.4. *Let X be a conjugacy class of involutions in the Coxeter group G of type $\tilde{B}_n$ or $\tilde{D}_n$. If $\mathcal{C}(G,X)$ is a connected graph, then its diameter is at most $n+2$.*

For the remainder of this chapter, a is a fixed involution with X its conjugacy class in the affine irreducible Coxeter group G. For $x \in X$ and $i \in \mathbb{N}$ we define

$$\Delta_i(x) = \{y \in X |\ d(x,y) = i\}$$

where $d(\cdot,\cdot)$ is the usual graph theoretic distance.

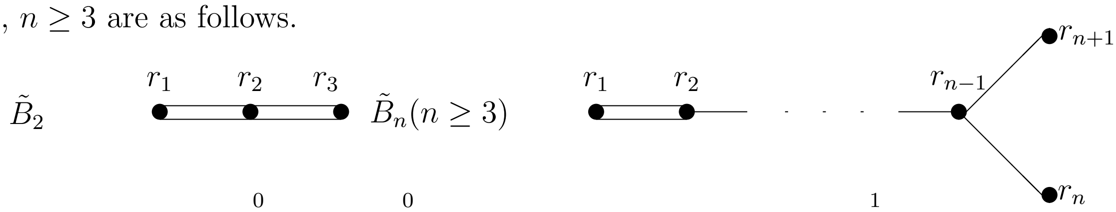
In Sections 3.2 and 3.3 we will establish notation, describe the conjugacy classes of involutions in G and prove results we will require. Section 3.4 is dedicated to the proofs of Theorems 3.1.1, 3.1.2 and 3.1.3. In Section 3.5 we determine the involution conjugacy classes in G for $|R| \leq 5$. We state in Section 3.6 examples of commuting involution graphs when $n \leq 6$ to illustrate our results.

3.2 The Group G_n

Referring to [21], [23] and to Section 1.2.2, let W be a finite Weyl group with root system Φ and $\Phi^\vee$ the set of coroots. The semi direct product of W with translation group Z of the coroot lattice $\Phi^\vee$ of W is the affine Weyl group $\tilde{\mathrm{W}}$. Let $(w,z) \in W \times Z$. Let G_n be the affine Weyl group of type $\tilde{B}_n$. Then, we fix $W \cong Sym(n)\!: 2^n, n \geq 2$; if $w \in Sym(n)$ then w acts on $R_n = \langle e_1, \ldots .\ e_n \rangle$ by permuting the subscripts of the basis vectors and if $w \in (\frac{\mathbb{Z}}{2\mathbb{Z}})^n$, it sends the e_i to $-e_i$.
The root system Φ of W is the set $\{\pm(\ e_i \pm e_j) : 1 \leq i < j \leq n\} \bigcup \{\pm\ e_i : 1 \leq i \leq n\}$ with coroots $\alpha^\vee = \alpha$ if $\alpha = \pm(e_i \pm e_j)$ and $\alpha^\vee = 2\alpha$ if $\alpha = \pm e_i$.
If G_n is the Weyl group of type $\tilde{D}_n$, then we fix $W \cong Sym(n)\!: 2^{n-1}, n \geq 4$; if $w \in$

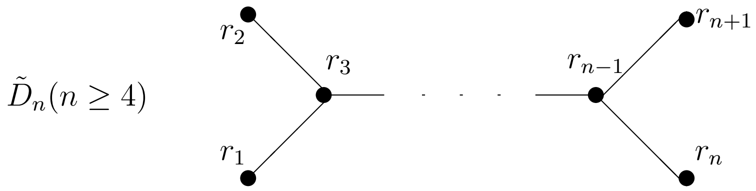
$Sym(n)$ then w acts on $R_n = \langle e_1, \ldots. e_n \rangle$ by permuting the subscripts of the basis vectors and if $w \in (\frac{\mathbb{Z}}{2\mathbb{Z}})^{n-1}$, it sends the e_i to $-e_i$ .The root system Φ of W is the set $\{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\}$ with coroots $\alpha^\vee = \alpha$. Writing a translation z by $\sum_1^n \lambda_i e_i$ as $(\lambda_1, \ldots, \lambda_n)$, we see that $z \in Z$ if and only if $\sum_1^n \lambda_i \in 2\mathbb{Z}$. This holds both for $\tilde{B}_n$ and $\tilde{D}_n$.

Referring to Section 1.3.1, we use the notation $\overset{\overset{\lambda_i}{-}}{(a_i\ b_i)}$, $\overset{\overset{\lambda_i}{+}}{(a_i\ b_i)}$ to represent respectively $(\overset{\overset{v_i}{-}}{a_i}\ \overset{\overset{v_i}{-}}{b_i})$, $(\overset{\overset{v_i}{+}}{a_i}\ \overset{\overset{v_i}{+}}{b_i})$, and we adopt the convention that cycles $(\overset{\overset{0}{+}}{a})$ are omitted, as these fix both a and v_a.
As stated in Section 1.2.4, the Coxeter graphs of G_2 of type $\tilde{C}_2 \cong \tilde{B}_2$ and G_n of type $\tilde{B}_n$, $n \geq 3$ are as follows.



We may set $r_1 = \overset{\overset{0}{-}}{(1)}$, $r_i = \overset{\overset{0}{+}}{(i-1\ i)}$ for $2 \leq i \leq n$ and $r_{n+1} = \overset{\overset{1}{-}}{(n-1\ n)}$.

The Coxeter graph of G_n of type $\tilde{D}_n$, $n \geq 4$ is as follows.



We may set here $r_1 = \overset{\overset{0}{+}}{(1\ 2)}$ $r_2 = \overset{\overset{0}{-}}{(1\ 2)}$ $r_i = \overset{\overset{0}{+}}{(i-1\ i)}$ for $3 \leq i \leq n$ and $r_{n+1} = \overset{\overset{1}{-}}{(n-1\ n)}$.

3.3 Involution Conjugacy Classes in G

There does not appear to be any uniform way to describe or parametrise the conjugacy classes of Weyl groups. By Lemma 1.3.1, $a \in G_n$ is an involution if and only if a is written as a product of disjoint signed cycles of the form

$$a = \overset{\overset{\lambda_1}{+}}{(a_1\ b_1)} \cdots \overset{\overset{\lambda_2}{+}}{(a_t\ b_t)}\overset{\overset{\lambda_{t+1}}{-}}{(a_{t+1}\ b_{t+1})} \cdots \overset{\overset{\lambda_m}{-}}{(a_m\ b_m)}\overset{\overset{\lambda_{2m+1}}{-}}{(c_{2m+1})} \cdots \overset{\overset{\lambda_{n-l}}{-}}{(c_{n-l})}\overset{\overset{0}{+}}{(d_{n-l+1})} \cdots \overset{\overset{0}{+}}{(d_n)}$$

for appropriate a_i, b_i, c_i, t, m and l where $\lambda_i \in \mathbb{Z}$ for $1 < i \leq n-l$.
The *labelled cycle type* of a is the tuple (m, k_e, k_o, l) where m is the number of transpositions, k_e is the number of negative one-cycles with an even number above them, k_o is the number of negative one-cycles with an odd number above them, and l is the number

of positive one-cycles (fixed points), in the labelled cycle form of a. For example, the labelled cycle type of $(\overset{\overset{0}{+}}{1}\,2)(\overset{\overset{2}{-}}{3})(\overset{\overset{6}{-}}{4})(\overset{\overset{1}{+}}{5})(\overset{\overset{0}{+}}{6})(\overset{\overset{0}{+}}{7})$ is $(1,2,1,2)$.

From now on, let $W(\tilde{C_n})$ be the group of type $\tilde{C_n}$ consisting of labelled permutations $(w, \mathbf{v})$ as described in Chapter 2. Let $W(\tilde{B_n})$ be the subgroup of $W(\tilde{C_n})$ such that the coordinate sum of $\mathbf{v}$ is even, and let $W(\tilde{D_n})$ be the subgroup of $W(\tilde{B_n})$ whose elements have an even number of minus signs in their signed permutation part. By Theorem 1.4.2, each involution is conjugate to the central involution w_I of some standard parabolic subgroup W_I. This means that in $W(\tilde{B_n})$ the connected components of the Coxeter graph of W_I are of type B_i including 'B_1', given by the set $\{r_1, r_2, ..., r_i\}$ which contributes $(\overset{\overset{0}{-}}{1}) \cdots (\overset{\overset{0}{-}}{i})$ to w_I; type A_1 contributing a transposition; and finally type D_j, where j is even and $j \geq 2$, given by the set $\{r_{n-j+2}, ..., r_n, r_{n+1}\}$, contributing $(\overset{\overset{1}{-}}{n-j+2}) \cdots (\overset{\overset{1}{-}}{n})$ to w_I. Thus, for such w_I we have $k_e = i$ only if I has a B_i component, and $k_o = j$ only if I has a D_j component. In the case of $W(D_n)$ there is another type D component instead of B_i components; if we have a connected component containing r_1 and r_2, then it is $\{r_1, r_2, ..., r_i\}$ for some i greater than 1. We can call this a D_i' component, with again $k_o = j$ when I has a D_j component.

Lemma 3.3.1. *If involutions in $W(\tilde{B_n})$ or $W(\tilde{D_n})$ are conjugate, then they have the same labelled cycle type.*

Proof. These elements are in subgroups of $W(\tilde{C_n})$; if they are conjugate in the subgroup G of type $\tilde{B}_n$ or $\tilde{D}_n$, they must be conjugate in $W(\tilde{C_n})$, and hence by Theorem 2.2.1 they have the same labelled cycle type (m, k_e, k_o, l). □

Theorem 3.3.2. *Let x be an involution in $W(\tilde{B_n})$ or $W(\tilde{D_n})$ with labelled cycle type (m, k_e, k_o, l). Suppose $n > 2m$ and that both k_e and k_o are non zero. Then x is conjugate to exactly one element $a = a_{(m,k_e,k_o,l)}$ of the form*

$$(\overset{\overset{0}{+}}{1}\,2) \cdots (\overset{\overset{0}{+}}{2m-1}\,2m)\ (\overset{\overset{0}{-}}{2m+1}) \cdots (\overset{\overset{0}{-}}{2m+k_e})(\overset{\overset{1}{-}}{2m+k_e+1}) \cdots (\overset{\overset{1}{-}}{n-l})(\overset{\overset{0}{+}}{n-l+1}) \cdots (\overset{\overset{0}{+}}{n})$$

.

Proof. Let x be an involution in $W(\tilde{B_n})$ or in $W(\tilde{D_n})$. By Theorem 1.4.2, x is conjugate to w_I for some finite standard parabolic subgroup W_I of $W(\tilde{B_n})$ or of $W(\tilde{D_n})$ in which w_I is central.

Hence, in $W(\tilde{B_n})$, the connected components of the Coxeter graph for W_I are of types B_i or A_1 or D_j for some $i \in \mathbb{N}$ and $j \in 2\mathbb{N}$ (including, by slight abuse of notation, B_1, where we have connected components with just vertex r_1) where $k_e = i$ and $k_o = j$. In $W(\tilde{D}_n)$, the connected components of the Coxeter graph for W_I are of types D_j or A_1 or D'_i for some $i, j \in 2\mathbb{N}$ with $k_e = i$ and $k_o = j$. Hence, since k_e and k_o are nonzero by hypothesis, we have $I = \{r_1, r_2, ..., r_i\} \cup J \cup \{r_{j+1}, r_{j+2}, ..., r_n, r_{n+1}\}$ for some i, j with $1 \leq i < j < n$, where $J \subset \{r_{i+2}, ..., r_{j-1}\}$ no two elements of which are adjacent vertices in the Coxeter graph. By conjugation in $\langle r_{i+2}, ..., r_{j-1}\rangle$ (which after all is isomorphic to the symmetric group $\mathrm{Sym}(j-i-1)$), we can assume that for some i, j and m with $1 \leq i \leq i+2m < j < n$ in $W(\tilde{B}_n)$ or $2 \leq i \leq i+2m < j < n$ in $W(\tilde{D}_n)$, we have

$$I = \{r_1, r_2, ..., r_i\} \cup \{r_{i+2}, r_{i+4}, ..., r_{i+2m}\} \cup \{r_{j+1}, r_{j+2}, ..., r_n, r_{n+1}\}.$$

This gives that x is conjugate to w_i, where $w_I =$

$$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\cdots\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{+}}{(i+1\ i+2)}\cdots\overset{\overset{0}{+}}{(i+2m-1\ i+2m)}\overset{\overset{0}{+}}{(i+2m+1)}\cdots\overset{\overset{0}{+}}{(j-1)}\overset{\overset{1}{-}}{(j)}\cdots\overset{\overset{1}{-}}{(n)}\ .$$

Let $c = (h, \mathbf{0})$, where $h =$

$$\begin{gathered}\overset{+}{(1\ i+1)}\overset{+}{(2\ i+2)}\cdots\overset{+}{(2m\ i+2m)}\overset{+}{(i+2m+1\ j)}\\ \overset{+}{(i+2m+2\ j+1)}\cdots\overset{+}{(i+2m+1+n-j\ n)}.\end{gathered}$$

In G (which is of type $\tilde{B}_n$ or $\tilde{D}_n$), the element $(\sigma, \mathbf{v})$ is conjugate to $(\tau, \mathbf{u})$ via some $(g, \mathbf{w})$ if and only if:

$$(\tau, \mathbf{u}) = (\sigma, \mathbf{v})^{(g,\mathbf{w})} = (g^{-1}\sigma g, \mathbf{v}^g + \mathbf{w} - \mathbf{w}^{g^{-1}\sigma g}). \tag{3.1}$$

Hence, we see that $w_i^c =$

$$\overset{\overset{0}{+}}{(1\ 2)}\cdots\overset{\overset{0}{+}}{(2m-1\ 2m)}\ \overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(2m+k_e)}\overset{\overset{1}{-}}{(2m+k_e+1)}\cdots\overset{\overset{1}{-}}{(n-l)}\overset{\overset{0}{+}}{(n-l+1)}\cdots\overset{\overset{0}{+}}{(n)}.$$

Therefore, by Lemma 3.3.1, x is conjugate to exactly one element $a_{m,k_e,k_o,l}$. □

In order to define the other conjugacy classes of involutions in $W(\tilde{B}_n)$ and in $W(\tilde{D}_n)$, we need the next lemma.

Lemma 3.3.3. *Let F be a group, N a subgroup of index 2 in F, and $x \in F$.*
(i) If $C_F(x)$ is not contained in N, then $x^N = x^F$.
(ii) If $C_F(x)$ is contained in N, then $x^F = x^N \cup z^N$, where $z = x^s$ for any $s \in F \backslash N$.

Proof. Let F be a group and N a subgroup of index 2 in F and $x \in N$. Suppose that $C_G(x) \not\subset N$. We will show that $x^N = x^G$. As $C(G, X) \not\subset N$, there must be some $s \in C_G(x) \backslash N$. Since N has an index 2 in F, $F = N \cup sN$. Suppose that $z \in x^F$. Then $z = x^f$ for some $f \in F$. If $f \in N$, then clearly $z \in x^N$. If $f \notin N$, then $f = st$ for some $t \in N$. This implies that $z = x^f = x^{st} = x^{s^t} = x^t$ where $x^s = x$. Hence, we have $z \in x^N$. Therefore $x^F = x^N$ for all the $x \in F$.
Now, let $C_G(x) \subset N$. Then $F = N \cup sN$ for some $s \in F \backslash N$. This implies that for all $x \in F$, we have x of the form t or st where $t \in N$. Let $z = x^s$. We notice that $z \in N$ where N is a normal subgroup in F. Therefore we have $x^f = \{x^t : t \in N\} \cup \{x^{st} : t \in N\} = x^N \cup z^N$. The proof is completed. $\square$

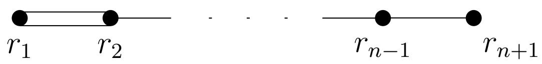
Proposition 3.3.4. *Let G be $W(\tilde{B_n})$ or $W(\tilde{D_n})$. Let x be an involution in G with $l > 0$ (l is the number of positive 1-cycles). Then the conjugacy class of x in G is equal to the conjugacy class of x in $W(\tilde{C_n})$. That is, the conjugacy class of x in G is precisely the set of involutions with the same labelled cycle type as x.*

Proof. Let x be an involution in $W(\tilde{B_n})$ or $W(\tilde{D_n})$. Since $l > 0$, then an involution x contains the cycle $\overset{\overset{0}{+}}{(i)}$ for some $i \in \{1, 2, .., n\}$. This means that x is centralised by both $\overset{\overset{1}{-}}{(i)}$ (which is in $W(\tilde{C_n}) \backslash W(\tilde{B_n})$) and $\overset{\overset{0}{-}}{(i)}$ (which is in $W(\tilde{B_n}) \backslash W(\tilde{D_n})$). Then since $C_{W(\tilde{C_n})}(x)$ is not contained in $W(\tilde{B_n})$ and $C_{W(\tilde{B_n})}(x)$ is not contained in $W(\tilde{D_n})$, by Lemma 3.3.3, $x^{W(\tilde{C_n})} = x^{W(\tilde{B_n})}$ and $x^{W(\tilde{B_n})} = x^{W(\tilde{D_n})}$. Thus $x^{W(\tilde{C_n})} = x^{W(\tilde{B_n})} = x^{W(\tilde{D_n})}$ by transitivity. The proof is completed. $\square$

Proposition 3.3.5. *Let x be an involution in $W(\tilde{B_n})$ with labelled cycle type (m, k_e, k_o, l), where $n > 2m > 0$, $l = 0$ and exactly one of k_o and k_e is nonzero. Then x is conjugate to exactly one of*

$$\begin{aligned}
w_1 =& (1 \overset{\overset{1}{-}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m-1 \overset{\overset{0}{+}}{\ } 2m)\overset{\overset{0}{-}}{(2m+1)} \cdots \overset{\overset{0}{-}}{(n)},\\
w_2 =& (1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m-1 \overset{\overset{0}{+}}{\ } 2m)\overset{\overset{0}{-}}{(2m+1)} \cdots \overset{\overset{0}{-}}{(n)},\\
w_3 =& (1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m-1 \overset{\overset{0}{+}}{\ } 2m)\overset{\overset{1}{-}}{(2m+1)} \cdots \overset{\overset{1}{-}}{(n)}\ .
\end{aligned}$$

Proof. Let x be an involution in $W(\tilde{B_n}), k_e \neq 0$ and $k_o = 0$, and suppose x is conjugate in $W(\tilde{B_n})$ to w_I. Then since $k_o = 0$ and $k_e \neq 0$, one of r_n and r_{n+1} is not in I and r_1 must be in I. Hence, if r_n is not in I, then $I \subset R \backslash \{r_n\}$ and its graph is a subgraph of the following:



Now, the connected components of the Coxeter graph for W_I are of types A_1 or B_i for some $i \in \mathbb{N}$. Since $l = 0$, we have no positive 1-cycles. Therefore $I = \{r_1, r_2, ..., r_{i-1}, r_i\} \cup J$ with $i = k_e < n$ and J is either $\{r_{i+2}, r_{i+4}, ..., r_{n-3}, r_{n-1}\}$ or $\{r_{i+2}, r_{i+4}, ..., r_{n-2}, r_{n+1}\}$. The first case is impossible as $l = 0$. Thus $n \equiv i \pmod 2$ and

$$I = \{r_1, r_2, ..., r_i\} \cup \{r_{i+2}, r_{i+4}, ..., r_{n-2}, r_{n+1}\}.$$

This implies that x is conjugate to w_I, where

$$w_I = \overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(i)}(i+1 \overset{\overset{0}{+}}{\;} i+2) \cdots (2m+i-3 \overset{\overset{0}{+}}{\;} 2m+i-2)(n-\overset{\overset{1}{-}}{\;}1\ n).$$

Let $c = (h, \mathbf{0})$, where

$$h = (1 \overset{+}{\;} n-1)(2 \overset{+}{\;} n)(3 \overset{+}{\;} n-3) \cdots (i\ n - \overset{+}{\;} 2m+2).$$

Now, x is conjugate to w_I^c and w_I^c is precisely w_1. If r_{n+1} is not in I, then $r_n \in I$ and $I = \{r_1, ..., r_i\} \cup \{r_{i+2}, ..., r_{n-2}, r_n\}$. A similar calculation shows that x is conjugate to w_2.

Suppose $k_e = 0$ and $k_o \neq 0$. Then r_1 is not an element of I and we have $I \subset R - \{r_1\}$. Then the connected components for W_I are of type A_1 or D_j for some $j \in 2\mathbb{N}$. Since x has no fixed points we must have

$$I = \{r_2, r_4, ..., r_{2m}\} \cup \{r_{2m+2}, r_{2m+3}, ..., r_n, r_{n+1}\}$$

Hence, x is conjugate to w_3, since

$$w_3 = w_I = (1 \overset{\overset{0}{+}}{\;} 2)(3 \overset{\overset{0}{+}}{\;} 4) \cdots (2m - \overset{\overset{0}{+}}{\;} 1\ 2m)(2m \overset{\overset{1}{-}}{+} 1) \cdots \overset{\overset{1}{-}}{(n)}.$$

We must show that none of w_1, w_2 and w_3 are conjugate. Lemma 3.3.1 tells us that w_1 and w_3 are not conjugate, and w_2 and w_3 are not conjugate, because they have different labelled cycle types. Now, for w_1 and w_2 we have $\overset{\overset{1}{-}}{(1)}\ w_1\ \overset{\overset{1}{-}}{(1)} = w_2$, so $\overset{\overset{1}{-}}{(1)}$ does not centralize w_1. Setting $w_2 = (\tau, \mathbf{u}) = (\sigma, \mathbf{v})$ in Equation 3.1, if w_2 is centralized by

$(g, \mathbf{w})$, then

$$\begin{aligned}
(\tau, \mathbf{0}) &= ((\overset{+}{(1\,2)}\overset{+}{(3\,4)} \cdots \overset{+}{(2m-1\ 2m)}\overset{-}{(2m+1)} \cdots \overset{-}{(n)}, (0, ..., 0)) \\
&=(\overset{+}{(1\,2)}\overset{+}{(3\,4)} \cdots \overset{+}{(2m-1\ 2m)}\overset{-}{(2m+1)} \cdots \overset{-}{(n)}, (0, 0, ..., 0))^{(g,\mathbf{w})} \\
&= (((\overset{+}{(1\,2)}\overset{+}{(3\,4)} \cdots \overset{+}{(2m-1\ 2m)}\overset{-}{(2m+1)} \cdots \overset{-}{(n)})^g, (0, 0, ..., 0)^g + \mathbf{w} - \mathbf{w}^{g^{-1}\sigma g}) \\
&= (\tau, \mathbf{w} - \mathbf{w}^{\tau}).
\end{aligned}$$

Let $\mathbf{w} = (\lambda_1, \lambda_3, ..., \lambda_n)$. We have $(0, 0, ..., 0) = \mathbf{w} - \mathbf{w}^{((\overset{+}{(1\,2)}\overset{+}{(3\,4)} \cdots \overset{+}{(2m-1\ 2m)}\overset{-}{(2m+1)} \cdots \overset{-}{(n)})}$. That is, $(0, 0, ..., 0) = (\lambda_2 - \lambda_1, \lambda_1 - \lambda_2, \lambda_3 - \lambda_4, \lambda_4 - \lambda_3, ..., \lambda_{2m} - \lambda_{2m-1}, 2\lambda_{2m+1}, ..., 2\lambda_n)$. This means that $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, ..., \lambda_{2m} = \lambda_{2m-1}$ and $\lambda_i = 0$ for $i \in \{2m+1, ..., n\}$. Hence, the coordinate sum of $\mathbf{w}$ is even. Therefore, $(g, \mathbf{w}) \in W(\tilde{B}_n)$, meaning $C_{W(\tilde{C}_n)}(x)$ is contained in $W(\tilde{B}_n)$. Therefore, by Lemma 3.3.3, w_1 and w_2 represent two different conjugacy classes. Note that $(w_3)^{W(\tilde{B}_n)} = (w_3)^{W(\tilde{C}_n)}$. □

Proposition 3.3.6. *Let x be an involution in $W(\tilde{D}_n)$, $n > 2m > 0$, $l = 0$ and exactly one of k_o and k_e is nonzero. Then x is conjugate to exactly one of*

$$\begin{aligned}
w_1 =&\overset{1}{\overset{-}{(1\,2)}}\overset{0}{\overset{+}{(3\,4)}} \cdots \overset{0}{\overset{+}{(2m-1\ 2m)}}\overset{0}{\overset{-}{(2m+1)}} \cdots \overset{0}{\overset{-}{(n)}}, \\
w_2 =&\overset{0}{\overset{+}{(1\,2)}}\overset{0}{\overset{+}{(3\,4)}} \cdots \overset{0}{\overset{+}{(2m-1\ 2m)}}\overset{0}{\overset{-}{(2m+1)}} \cdots \overset{0}{\overset{-}{(n)}}, \\
w_3 =&\overset{0}{\overset{+}{(1\,2)}}\overset{0}{\overset{+}{(3\,4)}} \cdots \overset{0}{\overset{+}{(2m-1\ 2m)}}\overset{1}{\overset{-}{(2m+1)}} \cdots \overset{1}{\overset{-}{(n)}}, \\
w_4 =&\overset{0}{\overset{-}{(1\,2)}}\overset{0}{\overset{+}{(3\,4)}} \cdots \overset{0}{\overset{+}{(2m-1\ 2m)}}\overset{1}{\overset{-}{(2m+1)}} \cdots \overset{1}{\overset{-}{(n)}}
\end{aligned}$$

Proof. Let x be an involution in $W(\tilde{D}_n)$, $l = 0$, $k_e \neq 0$ and $k_o = 0$. Since $k_e \neq 0$, x has a cycle $\overset{2\lambda}{\overset{-}{(i)}}$ for some i. As $\overset{2\lambda}{\overset{-}{(i)}} \in W(\tilde{B}_n) \backslash W(\tilde{D}_n)$, then we see that $W(\tilde{D}_n)$ does not contain $C_{W(\tilde{B}_n)}(x)$. Therefore, by Lemma 3.3.3 and by Proposition 3.3.5, x is conjugate to exactly one of w_1 and w_2 in $W(\tilde{D}_n)$. Now, let $k_e = 0$ and $k_o \neq 0$ and suppose x is conjugate in $W(\tilde{D}_n)$ to w_I. Because $k_e = 0$, at most one of r_1 and r_2 is in I. Because $l = 0$, at least one of r_1 and r_2 is in I. So I contains exactly one of r_1 and r_2. Because $k_o \neq 0$, both r_n and r_{n+1} are in I. Assume first that $r_1 \in I$ and $r_2 \notin I$, then the connected components of the Coxeter graph for W_I are of types D_j or A_1 for some $j \in 2\mathbb{N}$. Thus

$$I = \{r_1, r_4, r_6, ..., r_{2m}\} \cup \{r_{2m+2}, r_{2m+3}, ..., r_n, r_{n+1}\}.$$

This implies that x is conjugate to w_3, where

$$w_3 = (\overset{\overset{0}{+}}{1}\,2)\cdots(\overset{\overset{0}{+}}{2m-1}\,2m)(\overset{\overset{1}{-}}{2m+1})\cdots(\overset{\overset{1}{-}}{n}) .$$

Now, if r_1 is not in I and r_2 is in I, then we have

$$I = \{r_2, r_4, ..., r_{2m}\} \cup \{r_{2m+2}, ..., r_n, r_{n+1}\}.$$

Hence, x is conjugate to w_4. Write $w_3 = (\tau, (0, ..., 0, 1, ..., 1))$. Suppose $(g, \mathbf{w}) \in W(\tilde{B}_n)$ with $w_3^{(g,\mathbf{w})} = w_3$. Then, by Equation (3.1), we have

$$(\tau, (0, ..., 0, 1, ..., 1)) = (\tau^g, (0, ..., 0, 1, ..., 1)^g + \mathbf{w} - \mathbf{w}^\tau). \tag{3.2}$$

Now g is an element of $W(B_n)$ that centralises τ. So g permutes $\{1, ..., 2m\}$ and $\{2m+1, ..., n\}$. That is, $g \in \mathcal{C}_{W(B_{2m})}((\overset{+}{1}\,2)(\overset{+}{3}\,4)\cdots(\overset{+}{2m-1}\,2m)) \times W(B_{\{2m+1,...,n\}})$. Now $\mathcal{C}_{W(B_{2m})}((\overset{+}{1}\,2)(\overset{+}{3}\,4)\cdots(\overset{+}{2m-1}\,2m)) \subseteq W(D_{2m})$: we know this from the fact that the conjugacy classes of elements with this signed cycle type split in $W(D_{2m})$. Write $g = g_1 g_2$, where $g_1 \in \mathcal{C}_{W(B_{2m})}((\overset{+}{1}\,2)(\overset{+}{3}\,4)\cdots(\overset{+}{2m-1}\,2m))$ and $g_2 \in W(B_{\{2m+1,...,n\}})$. We note that the number of minus signs in g_1 is even. Let the number of minus signs in g_2 be μ. Then $g \in W(D_n)$ if and only if μ is even.
Now, setting $\mathbf{w} = (\lambda_1, ..., \lambda_n)$, we consider the vector part of Equation (3.2).

$$\begin{aligned}(0, ..., 0, 1, ..., 1) &= (0, ..., 0, 1, ..., 1)^g + \\ &(\lambda_1 - \lambda_2, \lambda_2 - \lambda_1, ..., \lambda_{2m-1} - \lambda_{2m}, \lambda_{2m} - \lambda_{2m-1}, 2\lambda_{2m+1}, ..., 2\lambda_n) \\ &= (0, ..., 0, 1, ..., 1)^{g_2} + \\ &(\lambda_1 - \lambda_2, \lambda_2 - \lambda_1, ..., \lambda_{2m-1} - \lambda_{2m}, \lambda_{2m} - \lambda_{2m-1}, 2\lambda_{2m+1}, ..., 2\lambda_n)\end{aligned}$$

Thus $\lambda_1 = ... = \lambda_{2m} = \lambda$. Observe that the coordinate sum of $(0, ..., 0, 1, ..., 1)^{g_2}$ is $k_o - 2\mu$. Taking coordinate sums of both sides, we obtain

$$\begin{aligned} k_o &= k_o - 2\mu + 2\sum_{i=2m+1}^{n} \lambda_i, \\ \mu &= \sum_{i=2m+1}^{n} \lambda_i \equiv 2m\lambda + \sum_{i=2m+1}^{n} \lambda_i \equiv (\text{mod } 2), \\ &\equiv \sum_{i=1}^{n} \lambda_i \equiv (\text{mod } 2). \end{aligned}$$

Since $(g, \mathbf{w}) \in W(\tilde{B}_n)$, this means μ is even. Therefore $(g, \mathbf{w}) \in W(\tilde{D}_n)$.
Hence, By Lemma 3.3.3, we have that $\mathcal{C}_{W(\tilde{B}_n)}(x)$ is contained in $W(\tilde{D}_n)$. Then, we have $x^{W(\tilde{B}_n)} = x^{W(\tilde{D}_n)} \cup y^{W(\tilde{D}_n)}$ with $y \in W(\tilde{D}_n)$. (In $W(\tilde{B}_n)$ we have $(\overset{\overset{0}{-}}{1})\, w_3\, (\overset{\overset{0}{-}}{1}) = w_4$. Therefore, w_3 and w_4 represent one conjugacy class in $W(\tilde{B}_n)$).

Then $x^{W(\tilde{D_n})}$ is the set of elements which are conjugate to w_3, and $y^{W(\tilde{D_n})}$ is the set of elements which are conjugate to w_4. Hence, in $W(\tilde{D_n})$, if $n > 2m, l = 0$ and one of k_e and k_o is non zero, then we have 4 different conjugacy classes. $\square$

Proposition 3.3.7. *Let x be an involution in $W(\tilde{B_n})$, $n = 2m$ and $k_e = k_o = l = 0$. Then, x is conjugate to exactly one of these forms below*

$$w_5 = (1 \overset{\overset{0}{+}}{} 2) \cdots (2m-3 \overset{\overset{0}{+}}{} 2m-2)\ (2m-1 \overset{\overset{0}{+}}{} 2m),$$
$$w_6 = (1 \overset{\overset{0}{+}}{} 2) \cdots (2m-3 \overset{\overset{0}{+}}{} 2m-2)(2m-1 \overset{\overset{1}{-}}{} 2m).$$

Proof. Let x be an involution in $W(\tilde{B_n})$ with $l = k_e = k_o = 0$. This means r_1 is not in I and at most one of r_{n+1} and r_n is in I. Then the connected components of the Coxeter graph for W_I are of type A_1. Since $l = 0$, we have $I = \{r_2, r_4, ..., r_{n-2}, r_n\}$ or $I = \{r_2, r_4, ..., r_{n-2}, r_{n+1}\}$. Then x is conjugate to w_5 or w_6. Now, suppose w_5 is centralized by $(g, \mathbf{w})$ in $W(\tilde{C_n})$. By Equation (3.1), we have $(0, .., 0) = (0, ..., 0)^g + \mathbf{w} - (\mathbf{w})^{(1\overset{+}{}2)(3\overset{+}{}4)\cdots(2m-3\overset{+}{}2m-2)(2m-1\overset{+}{}2m)}$. This implies that $(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) - (\lambda_2, \lambda_1, ..., \lambda_n, \lambda_{n-1}) = 0$, so $\lambda_1 = \lambda_2, ..., \lambda_n = \lambda_{n-1}$. Thus the coordinate sum of $\mathbf{w}$ is even.

Hence, $(g, \mathbf{w}) \in W(\tilde{B_n})$. By Lemma 3.3.3, we have two different conjugacy classes. Therefore, x is conjugate to exactly one of w_5 and w_6. $\square$

Proposition 3.3.8. *Let x be an involution in $W(\tilde{D_n})$, $n = 2m$ and $k_e = k_o = l = 0$. Then, x is conjugate to exactly one of*

$$w_5 = (1 \overset{\overset{0}{+}}{} 2)(3 \overset{\overset{0}{+}}{} 4) \cdots (2m-3 \overset{\overset{0}{+}}{} 2m-2)(2m-1 \overset{\overset{0}{+}}{} 2m),$$
$$w_6 = (1 \overset{\overset{0}{+}}{} 2)(3 \overset{\overset{0}{+}}{} 4) \cdots (2m-3 \overset{\overset{0}{+}}{} 2m-2)(2m-1 \overset{\overset{1}{-}}{} 2m),$$
$$w_7 = (1 \overset{\overset{0}{-}}{} 2)(3 \overset{\overset{0}{+}}{} 4) \cdots (2m-3 \overset{\overset{0}{+}}{} 2m-2)(2m-1 \overset{\overset{0}{+}}{} 2m),$$
$$w_8 = (1 \overset{\overset{0}{-}}{} 2)(3 \overset{\overset{0}{+}}{} 4) \cdots (2m-1 \overset{\overset{0}{+}}{} 2m-2)(2m-1 \overset{\overset{1}{-}}{} 2m).$$

Proof. Let x be an involution in $W(\tilde{D_n})$, $n = 2m$ and $k_e = k_o = l = 0$. Hence, x does not have negative 1-cycles. This means that at most one of r_1 and r_2 is in I and at most one of r_{n-1} and r_n is in I. Hence the connected components of the Coxeter graph for W_I are of type A_1. Since $l = 0$, we must have one of the following for I:

$$I = \{r_1, r_4, ..., r_{n-2}, r_n\},$$
$$I = \{r_1, r_4, ..., r_{n-2}, r_{n+1}\},$$
$$I = \{r_2, r_4, ..., r_{n-2}, r_n\},$$
$$I = \{r_2, r_4, ..., r_{n-2}, r_{n+1}\}.$$

Observe that $\mathcal{C}_{W(B_n)}((\overset{+}{1}\,2)(\overset{+}{3}\,4)\cdots(\overset{+}{2m-3}\,2m-2)(\overset{+}{2m-1}\,2m)) \subseteq W(D_n)$. Hence if $(g,\mathbf{w})$ centralises w_5, we have $g \in W(D_n)$. This is true similarly for w_6, w_7 and w_8.

Thus for each w_i we have $\mathcal{C}_{W(\tilde{B}_n)}(w_I) = \mathcal{C}_{W(\tilde{D}_n)}(w_I)$. Moreover $\overset{\underline{0}}{(i)} \in W(\tilde{B}_n) \diagdown W(\tilde{D}_n)$, and $\overset{\underline{0}}{(i)}\ w_5\ \overset{\underline{0}}{(i)} = w_7$, $\overset{\underline{0}}{(i)}\ w_6\ \overset{\underline{0}}{(i)} = w_8$. Thus $w_5^{W(\tilde{B}_n)} = w_5^{W(\tilde{D}_n)} \cup w_7^{W(\tilde{D}_n)}$ and $w_6^{W(\tilde{B}_n)} = w_6^{W(\tilde{D}_n)} \cup w_8^{W(\tilde{D}_n)}$.
Hence in $W(\tilde{D}_n)$, if $n = 2m$, then we have four conjugacy classes. $\square$

Proposition 3.3.9. *Let x be an involution in $W(\tilde{B}_n)$ and $m = l = 0$. Then x is conjugate to exactly one of*

$$\begin{aligned}
w_9 &= \overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\cdots\overset{\underline{0}}{(n-1)}\overset{\underline{0}}{(n)},\\
w_{10} &= \overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\cdots\overset{\underline{0}}{(i)}\overset{\underline{1}}{(i+1)}\overset{\underline{1}}{(n)} \ \ (\text{for } 1 \le i \le n-1),\\
w_{11} &= \overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\cdots\overset{\underline{0}}{(n-1)}\overset{\underline{2}}{(n)},\\
w_{12} &= \overset{\underline{1}}{(1)}\overset{\underline{1}}{(2)}\cdots\overset{\underline{1}}{(n-1)}\overset{\underline{1}}{(n)}\ .
\end{aligned}$$

Proof. Let x be an involution in $W(\tilde{B_n})$ conjugate to w_I. There are four cases when w_I does not have transpositions. Namely when r_1, r_n are in I and r_{n+1} is not in I, or r_1, r_n, r_{n+1} are in I or r_1, r_{n+1} are in I, and r_n is not in I, or r_2, r_n and r_{n+1} are in I and r_1 is not in I. We have $m = l = 0$. Then the connected components for W_I are of type B_i or D_j for some $i \in \mathbb{N}$ and $j \in 2\mathbb{N}$ where $i + j = n$. Therefore, we have

$$\begin{aligned}
I &= \{r_1, r_2, ..., r_{n-1}, r_n\} \text{ or},\\
I &= \{r_1, r_2, ..., r_i, r_{i+2}, ..., r_n, r_{n+1}\} \text{ for some } i, \text{or}\\
I &= \{r_1, r_2, ..., r_{n-1}, r_{n+1}\} \text{ or},\\
I &= \{r_2, ..., r_n, r_{n+1}\}.
\end{aligned}$$

In this order, we obtain w_9 to w_{12} above. Hence x is conjugate to at least one of $w_9, w_{10}, w_{11}, w_{12}$. Suppose x has labelled cycle type $(0, k_e, k_o, 0)$. If $k_o > 0$, then x contains $\overset{\underline{\lambda}}{(j)}$ for some odd λ, and so $\overset{\underline{\lambda}}{(j)} \in \mathcal{C}_{W(\tilde{C}_n)} \backslash \mathcal{C}_{W(\tilde{B}_n)}(x)$. Thus $x^{W(\tilde{C}_n)} = x^{W(\tilde{B}_n)}$ and x is conjugate to exactly one of $w_{10} = \overset{\underline{0}}{(1)}\cdots\overset{\underline{0}}{(k_e)}\overset{\underline{1}}{(k_e+1)}\cdots\overset{\underline{1}}{(n)}$ (if $k_o < n$) or w_{12} (if $k_o = n$). Moreover, w_{10} or w_{12} are not conjugate to each other, nor to w_9 or w_{11}, as they have different labelled cycle types.

Suppose x has labelled cycle type $(0,n,0,0)$, and let $(g,\mathbf{w}) \in W(\tilde{B}_n)$. Now

$$\begin{aligned} {w_9}^{(g,\mathbf{w})} &= ((\bar{1}) \cdots (\bar{n}), \mathbf{0}^g + \mathbf{w} - \mathbf{w}^{(\bar{1})\cdots(\bar{n})}) \\ &= ((\bar{1}) \cdots (\bar{n}), 2\mathbf{w}). \end{aligned}$$

Since $(g,\mathbf{w}) \in W(\tilde{B}_n)$, the coordinate sum of $\mathbf{w}$ is even. Thus x is conjugate to w_9 if and only if the coordinate sum of x is congruent to 0 modulo 4, and to w_{11} otherwise. Therefore, when $m = l = 0$ we have four conjugacy classes in $W(\tilde{B}_n)$. $\square$

Proposition 3.3.10. *Let x be an involution in $W(\tilde{D_n})$ and $m = l = 0$. Then x is conjugate to exactly one of*

$$\begin{aligned} w_9 &= \overset{0}{(\bar{1})}\overset{0}{(\bar{2})} \cdots \overset{0}{(\overline{n-1})}\overset{0}{(\bar{n})}, \\ w_{10} &= \overset{0}{(\bar{1})} \cdots \overset{0}{(\bar{i})}\overset{1}{(\overline{i+1})} \cdots \overset{1}{(\bar{n})},\ 1 \le i \le n-1, \\ w_{11} &= \overset{0}{(\bar{1})}\overset{0}{(\bar{2})} \cdots \overset{0}{(\overline{n-1})}\overset{2}{(\bar{n})}, \\ w_{12} &= \overset{1}{(\bar{1})}\overset{1}{(\bar{2})} \cdots \overset{1}{(\overline{n-1})}\overset{1}{(\bar{n})}, \\ w_{13} &= \overset{-1}{(\bar{1})}\overset{1}{(\bar{2})} \cdots \overset{1}{(\overline{n-1})}\overset{1}{(\bar{n})}\ . \end{aligned}$$

Proof. Let x be an involution in $W(\tilde{D_n})$ with $m = l = 0$. This implies that x does not have transpositions. The connected components of the Coxeter graph for W_I are of type D_i or D_i' (another type D component when $I = \{r_1, r_2, ..., r_i\}$) where i, j are even and $i + j = n$. We have

$$\begin{aligned} I &= \{r_1, r_2 ..., r_{n-1}, r_n\}, \text{or} \\ I &= \{r_1, r_2, r_3, ..., r_i, r_{i+2}, ..., r_n, r_{n+1}\}, \text{or} \\ I &= \{r_1, r_2, ..., r_{n-1}, r_{n+1}\}, \text{or} \\ I &= \{r_1, r_3, r_4 ..., r_n, r_{n+1}\}, \text{or} \\ I &= \{r_2, r_3, ..., r_n, r_{n+1}\}. \end{aligned}$$

We have x is conjugate to w_9 or w_{10} or respectively w_{12}.

Now, $\overset{0}{(\bar{\mathbf{0}})}$ centralizes w_I and $\overset{0}{(\bar{\mathbf{0}})} \in C_{W(\tilde{B_n})}(w_I) \backslash C_{W(\tilde{D_n})}(w_I)$ wherever w_I is w_9, w_{10} or w_{11}. Therefore, by Lemma 3.3.3 we have $w_9^{W(\tilde{B_n})} = w_9^{W(\tilde{D_n})}$, $w_{10}^{W(\tilde{B_n})} = w_{10}^{W(\tilde{D_n})}$ and $w_{11}^{W(\tilde{B_n})} = w_{11}^{W(\tilde{D_n})}$.

Thus only when x has $k_e = 0$. This occurs when we have

$$
\begin{aligned}
I &= \{r_1, r_3, r_4 ..., r_n, r_{n+1}\} \text{ or} \\
I &= \{r_2, r_3 ..., r_n, r_{n+1}\}
\end{aligned}
$$

Therefore, x is conjugate to w_{12} or w_{13}.
Suppose $(g, \mathbf{w}) \in \mathcal{C}_{W(\tilde{B}_n)}(w_{12})$. Then by Equation (3.1), we have

$$(1, ..., 1) = (1, ..., 1)^g + 2\mathbf{w}.$$

If g has μ minus signs, then taking the coordinate sum on both sides gives (recalling that $\mathbf{w}$ has even coordinate sum)

$$n \equiv n - 2\mu \pmod 4.$$

Hence μ is even, which means $(g, \mathbf{w}) \in W(\tilde{D}_n)$ and by Lemma 3.3.3, w_{12} splits into two conjugacy classes in $W(\tilde{D}_n)$. Hence $w_{12}^{W(\tilde{B}_n)} = w_{12}^{W(\tilde{D}_n)} \cup w_{13}^{W(\tilde{D}_n)}$. Thus, when $m = l = 0$ we have 5 conjugacy classes in $W(\tilde{D}_n)$. The proof is completed. $\square$

Corollary 3.3.11. *Let x be an involution in $W(\tilde{B}_n)$ with labelled cycle type (m, k_e, k_o, l). We have $x^{w(\tilde{B}_n)} = x^{w(\tilde{C}_n)}$ if and only if either $l > 0$ or $k_o > 0$ (or both); in all other cases there are at least two classes for each labelled cycle type. In particular:*

(a) If $m = l = 0$, then x is conjugate to exactly one of

$$
\begin{aligned}
&\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)} \cdots \overset{\overset{0}{-}}{(n-1)}\overset{\overset{0}{-}}{(n)}, \\
&\overset{\overset{0}{-}}{(1)} \cdots \overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(i+1)} \cdots \overset{\overset{1}{-}}{(n)} \ with 0 \leq i < n, \\
&\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)} \cdots \overset{\overset{0}{-}}{(n-1)}\overset{\overset{2}{-}}{(n)}, \\
&\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)} \cdots \overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}\ .
\end{aligned}
$$

(b) If $n = 2m$ and $k_e = k_o = l = 0$, then x is conjugate to exactly one of

$$
\begin{aligned}
&\overset{\overset{0}{+}}{(1\ 2)} \cdots \overset{\overset{0}{+}}{(2m-3\ 2m-2)}\overset{\overset{0}{+}}{(2m-1\ 2m)}, \\
&\overset{\overset{0}{+}}{(1\ 2)} \cdots \overset{\overset{0}{+}}{(2m-3\ 2m-2)}\overset{\overset{1}{-}}{(2m-1\ 2m)}.
\end{aligned}
$$

(c) If $n > 2m > 0$, $l = 0$ and exactly one of k_e and $k_o \neq 0$, then x is conjugate to exactly one of

$$\overset{\overset{1}{-}}{(1\ 2)}\overset{\overset{0}{+}}{(3\ 4)} \cdots \overset{\overset{0}{+}}{(2m-1\ 2m)}\overset{\overset{0}{-}}{(2m+1)} \cdots \overset{\overset{0}{-}}{(n)},$$

$$\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)}\overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(n)},$$
$$\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)}\overset{\overset{1}{-}}{(2m+1)}\cdots\overset{\overset{1}{-}}{(n)}\ .$$

(d) *In all the other cases x is conjugate to exactly one element $a = a_{(m,k_e,k_o,l)}$ of the form*

$$\overset{\overset{0}{+}}{(1\,2)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)}\ \overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(2m+k_e)}\overset{\overset{1}{-}}{(2m+k_e+1)}\cdots\overset{\overset{1}{-}}{(n-l)}\overset{\overset{0}{+}}{(n-l+1)}\cdots\overset{\overset{0}{+}}{(n)}$$

.

Now, let x be an involution in $W(\tilde{D_n})$ with labelled cycle type (m, k_e, k_o, l). Then $x^{W(\tilde{D_n})} = x^{W(\tilde{C_n})}$ if $l > 0$ or both $k_e > 0$ and $k_o > 0$. Otherwise there are at least four classes for each cycle type. In particular:

(a) *If $m = l = 0$, then x is conjugate to exactly one of*

$$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\cdots\overset{\overset{0}{-}}{(n-1)}\overset{\overset{0}{-}}{(n)},$$
$$\overset{\overset{0}{-}}{(1)}\cdots\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(i+1)}\cdots\overset{\overset{1}{-}}{(n)}, where 1 \leq i < n,$$
$$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\cdots\overset{\overset{0}{-}}{(n-1)}\overset{\overset{2}{-}}{(n)},$$
$$\overset{\overset{-1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\cdots\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)},$$
$$\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\cdots\overset{\overset{1}{-}}{(n-1)}\overset{\overset{1}{-}}{(n)}\ .$$

(b) *If $n = 2m$ and $k_e = k_o = l = 0$, then an involution x is conjugate to exactly one of*

$$\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)},$$
$$\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{1}{-}}{(2m-1\,2m)}.$$
$$\overset{\overset{0}{-}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)},$$
$$\overset{\overset{0}{-}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{1}{-}}{(2m-1\,2m)}.$$

(c) *If $n > 2m$, $l = 0$ and exactly one of k_e and $k_o \neq 0$, then x is conjugate to exactly one of*

$$\overset{\overset{1}{-}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)}\overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(n)},$$
$$\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)}\cdots\overset{\overset{0}{+}}{(2m-1\,2m)}\overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(n)},$$

$$\overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{+}}{(3\ 4)}\cdots\overset{\overset{0}{+}}{(2m-1\ 2m)}\overset{\overset{1}{-}}{(2m+1)}\cdots\overset{\overset{1}{-}}{(n)}\ .$$
$$\overset{\overset{0}{-}}{(1\ 2)}\overset{\overset{0}{+}}{(3\ 4)}\cdots\overset{\overset{0}{+}}{(2m-1\ 2m)}\overset{\overset{1}{-}}{(2m+1)}\cdots\overset{\overset{1}{-}}{(n)}\ .$$

(d) *In all the other cases x is conjugate to exactly one element $a = a_{(k_o,m,l,k_e)}$ of the form*

$$\overset{\overset{0}{+}}{(1\ 2)}\cdots\overset{\overset{0}{+}}{(2m-1\ 2m)}\ \overset{\overset{0}{-}}{(2m+1)}\cdots\overset{\overset{0}{-}}{(2m+k_e)}\overset{\overset{1}{-}}{(2m+k_e+1)}\cdots\overset{\overset{1}{-}}{(n-l)}\overset{\overset{0}{+}}{(n-l+1)}\cdots\overset{\overset{0}{+}}{(n)}\ .$$

In Section 3.4 we use results about commuting involution graphs from Chapters 1 and 2 and the result from [7] about commuting involution graphs in $W(B_n)$ to derive the results about commuting involution graphs in $W(\tilde{B}_n)$ and $W(\tilde{D}_n)$.

3.4 Proofs of Main Results

We begin this section by obtaining more precision about how conjugacy classes split. Before the next result we need a definition.
Suppose $x \in W(\tilde{C}_n)$ with labelled cycle type $(m, k_e, 0, 0)$. Writing $x = \prod_{i=1}^{m} \overset{\overset{\lambda_i}{*}}{(a_i\ b_i)} \prod_{j=1}^{k_e} \overset{\overset{\mu_j}{-}}{(c_j)}$, we define $f(x) = 2\sum_{i=1}^{m} \mid \lambda_i \mid + \sum_{j=1}^{k_e} \mid \mu_j \mid$ ($* = +$ or $-$).

Lemma 3.4.1. *Let x and x' be involutions in $W(\tilde{C}_n)$ with labelled cycle type $(m, k_e, 0, 0)$. Then $x' \in x^{W(\tilde{B}_n)}$ if and only if $f(x') \equiv f(x) (\mathrm{mod}\ 4)$.*

Proof. Let $x = \prod_{i=1}^{m} \overset{\overset{\lambda_i}{*}}{(a_i\ b_i)} \prod_{j=1}^{k_e} \overset{\overset{\mu_j}{-}}{(c_j)} = (\sigma, \mathbf{u})$. Observe that $u_{a_i} = \lambda_i$, $u_{b_i} = \pm\lambda_i$, $u_{c_j} = \mu_j$ for $1 \le i \le m$, $1 \le j \le k_e$. Thus $f(x)$ is just the sum of the modulo of the coordinates of $\mathbf{u}$. The action of $W(B_n)$ permutes coordinates and changes signs, and thus does not affect this sum. Thus $f(x^{(g,\mathbf{0})}) = f(x)$ for any $(g, \mathbf{0}) \in W(\tilde{B}_n)$. Write $x' = \prod_{i=1}^{m} \overset{\overset{\lambda'_i}{*}}{(a'_i\ b'_i)} \prod_{j=1}^{k_e} \overset{\overset{\mu'_j}{-}}{(c'_j)}$. Now set $\bar{x} = \prod_{i=1}^{m} \overset{\overset{\lambda_i}{+}}{(2i-1\ 2i)} \prod_{j=1}^{k_e} \overset{\overset{\mu_j}{-}}{(2m+j)}$ and $\bar{x}' = \prod_{i=1}^{m} \overset{\overset{\lambda'_i}{+}}{(2i-1\ 2i)} \prod_{j=1}^{k_e} \overset{\overset{\mu'_j}{-}}{(2m+j)}$.
Since $f(\bar{x}) = f(x)$, $f(\bar{x}') = f(x')$, and x is conjugate to $\bar{x}$ and x' is conjugate to $\bar{x}'$ in $W(\tilde{B}_n)$ (via elements of $W(B_n)$), it is sufficient to prove that $\bar{x}' \in \bar{x}^{W(\tilde{B}_n)}$ if and only if $f(\bar{x}') \equiv f(\bar{x})$ (mod 4).
Consider the vector $\mathbf{w} = (w_1, ..., w_n)$ where $w_{2i-1} = \lambda'_i$, $w_{2i} = \lambda_i$, $w_{2m+j} = \frac{1}{2}(\mu'_j - \mu_j)$

for $1 \leq i \leq m$, $1 \leq j \leq k_e$. Now, putting $\bar{x} = (\bar{\sigma}, \bar{\mathbf{u}})$ we get

$$\begin{aligned}
\bar{x}^{(1,\mathbf{w})} &= (\bar{\sigma}, \bar{\mathbf{u}} + \mathbf{w} - \mathbf{w}^{\bar{\sigma}}) \\
&= (\bar{\sigma}, \bar{\mathbf{u}} + (w_1 - w_2, w_2 - w_1, ..., w_{2m-1} - w_{2m}, w_{2m} - w_{2m-1}, 2w_{2m+1}, ..., 2w_n)) \\
&= (\bar{\sigma}, (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, ..., \lambda_m, -\lambda_m, \mu_1, ..., \mu_{k_e}) \\
&+ (\lambda'_1 - \lambda_1, \lambda_1 - \lambda'_1, ..., \lambda_m, -\lambda'_m, \mu'_1 - \mu_1, ..., \mu'_{k_e} - \mu_{k_e}) \\
&= (\bar{\sigma}, (\lambda'_1, -\lambda'_1, ..., \lambda'_m, -\lambda'_m, \mu'_1, ..., \mu'_{k_e})) \\
&= \bar{x}'.
\end{aligned}$$

By lemma 3.3.3, we get that $\bar{x}' \in \bar{x}^{W(\tilde{B}_n)}$ if and only if $(1, \mathbf{w}) \in W(\tilde{B}_n)$ which is if and only if the coordinate sum of $\mathbf{w}$ is even. Now

$$\begin{aligned}
\sum_{i=1}^{n} w_i &= \sum_{i=1}^{m} (\lambda_i + \lambda'_i) + \sum_{j=1}^{k_e} \frac{1}{2}(\mu'_j - \mu_j) \\
&\equiv -\left\{ \sum_{i=1}^{m} (\mid \lambda_i \mid + \frac{1}{2} \sum_{j=1}^{k_e} \mid \mu_j \mid)) + (\sum_{i=1}^{m} (\mid \lambda'_i \mid + \frac{1}{2} \sum_{j=1}^{k_e} \mid \mu'_j \mid) \right\} \pmod 2.
\end{aligned}$$

So $\bar{x}' \in \bar{x}^{W(\tilde{B}_n)}$ if and only if

$$\textstyle\sum_{i=1}^{m} (\mid \lambda'_i \mid + \frac{1}{2} \sum_{j=1}^{k_e} \mid \mu'_j \mid) \equiv \sum_{i=1}^{m} \mid \lambda_i \mid + \frac{1}{2} \sum_{j=1}^{k_e} \mid \mu_j \mid \pmod 2,$$

which is if and only if $f(\bar{x}') \equiv f(\bar{x}) \mod 4$. Thus $x' \in x^{W(\tilde{B}_n)}$ if and only if $f(x') \equiv f(x)$ (mod 4). □

We summarise the results for $W(\tilde{B}_n)$ in the following theorem.

Theorem 3.4.2. *Consider an involution x of $W(\tilde{B}_n)$ with labelled cycle type (m, k_e, k_o, l). If either l or k_o (or both) is nonzero, then $x^{W(\tilde{B}_n)} = x^{W(\tilde{C}_n)}$. If $l = 0$ and $k_o = 0$, then $x^{W(\tilde{C}_n)}$ splits as two conjugacy classes in $W(\tilde{B}_n)$. In this case, write $x = \prod_{i=1}^{m} (a_i \overset{\overset{\lambda_i}{\ast}}{\ } b_i) \prod_{j=1}^{n-2m} (\overset{\overset{\mu_j}{-}}{c_j})$ where the μ_j are all even. If $m \geq 1$, then x is conjugate to exactly one of the following:*

$$\begin{aligned}
w_1 &= (1 \overset{\overset{1}{-}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m - \overset{\overset{0}{+}}{1}\ 2m)\ (\overset{\overset{0}{-}}{2m + 1}) \cdots (\overset{\overset{0}{-}}{n}), \\
w_2 &= (1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m - \overset{\overset{0}{+}}{1}\ 2m)\ (\overset{\overset{0}{-}}{2m + 1}) \cdots (\overset{\overset{0}{-}}{n}).
\end{aligned}$$

In particular, x is conjugate to w_2 if and only if $\sum_{i=1}^{m} 2 \mid \lambda_i \mid + \sum_{j=2m+1}^{n} \mid \mu_j \mid \equiv 0 \pmod 4$. If $m = 0$, then x is conjugate to

$(\overset{\overset{0}{-}}{1}) \cdots (\overset{\overset{0}{-}}{n-1})(\overset{\overset{0}{-}}{n})$ *if and only if* $\sum_{j=1}^n \mu_j \equiv 0 \pmod 4$, *and to* $(\overset{\overset{0}{-}}{1}) \cdots (\overset{\overset{0}{-}}{n-1})(\overset{\overset{2}{-}}{n})$ *otherwise.*

We next prove results for $W(\tilde{D}_n)$.

Suppose $x \in W(\tilde{C}_n)$ with labelled cycle type $(m, 0, k, 0)$ where here $k = k_o$.

$$\text{Writing} \qquad x = \prod_{i=1}^m (a_i \overset{\overset{\lambda_i}{*}}{\;} b_i) \prod_{j=1}^k (\overset{\overset{\mu_j}{-}}{c_j}),$$

we define $t(x)$ to be the number of transpositions of x that have minus signs above them, and

$$g(x) = 2\sum_{i=1}^m |\lambda_i| + \sum_{j=1}^k \mu_j.$$

Note that $g(x)$ differs from $f(x)$ in that we consider μ_j rather than $|\mu_j|$.

Lemma 3.4.3. *Let x and x' be involutions of $W(\tilde{D}_n)$ with labelled cycle type $(m, 0, k, 0)$. If $t(x) \equiv t(x') \pmod 2$, then $x' \in x^{W(\tilde{D}_n)}$ if and only if $g(x') \equiv g(x) \pmod 4$. If $t(x) \not\equiv t(x') \pmod 2$, then $x' \in x^{W(\tilde{D}_n)}$ if and only if $g(x') \not\equiv g(x) \pmod 4$.*

Proof. Note that since $x, x' \in W(\tilde{D}_n)$, we must have $k > 0$ and k even. Thus $g(x)$ is congruent to 0 or 2 modulo 4. Observe first that conjugation in the underlying S_n permutes coordinates but does not change signs, so does not affect $|\lambda_i|$ or μ_j, nor does it affect the number of minus signs above transpositions. Thus x is conjugate in $W(\tilde{D}_n)$ to an element

$$\bar{x} = \prod_{i=1}^m (2i - \overset{\overset{\lambda_i}{*}}{1}\; 2i) \prod_{j=1}^k (\overset{\overset{\mu_j}{-}}{2m+j})$$

with the property that $g(x) = g(\bar{x})$ and $t(x) = t(\bar{x})$. Similarly x' is conjugate to some

$$\bar{x}' = \prod_{i=1}^m (2i - \overset{\overset{\lambda'_i}{*}}{1}\; 2i) \prod_{j=1}^k (\overset{\overset{\mu'_j}{-}}{2m+j})$$

with the property that $g(\bar{x}') = g(x')$ and $t(\bar{x}') = t(x')$. So we have that $x' \in x^{W(\tilde{D}_n)}$ if and only if $\bar{x}' \in \bar{x}^{W(\tilde{D}_n)}$.

Now, conjugating by pairs $(\overset{\overset{0}{-}}{2i_1})(\overset{\overset{0}{-}}{2i_2})$ with $1 \leq i_1 < i_2 \leq m$ will again leave $g(\bar{x})$ unaffected, and will alter $t(\bar{x})$ by a multiple of 2.
Thus $\bar{x}$ is conjugate in $W(\tilde{D}_n)$ to an element

$$y = (1 \overset{\overset{\lambda_1}{\pm}}{\;} 2)(3 \overset{\overset{\lambda_2}{+}}{\;} 4) \cdots (2m - \overset{\overset{\lambda_m}{+}}{1}\; 2m) \prod_{j=1}^k (\overset{\overset{\mu_j}{-}}{2m+j})$$

and $\bar{x}'$ is conjugate to

$$y' = (1 \overset{\overset{\lambda'_1}{\pm}}{\;} 2)(3 \overset{\overset{\lambda'_2}{+}}{\;} 4) \cdots (2m - \overset{\overset{\lambda'_m}{+}}{1}\; 2m) \prod_{j=1}^k (\overset{\overset{\mu'_j}{-}}{2m+j})$$

with

$$g(x) = g(\bar{x}) = g(y),\ t(x) = t(\bar{x}) \equiv t(y) \pmod 2$$
$$g(x') = g(\bar{x}') = g(y'),\ t(x') = t(\bar{x}') \equiv t(y') \pmod 2.$$

Again, $x' \in x^{W(\tilde{D}_n)}$ if and only if $y' \in y^{W(\tilde{D}_n)}$. Suppose first that $t(y) \equiv t(y') \pmod 2$. That is, the sign above $(1\ 2)$ in both elements is the same.
Let $\mathbf{w} = (w_1, ..., w_n)$ where we set $w_{2i-1} = \lambda_i'$, $w_{2i} = \lambda_i$, $w_{2m+j} = \frac{1}{2}(\mu_j' - \mu_j)$ for $1 \leq i \leq m$, $1 \leq j \leq k$. Just as in the proof of Lemma 3.4.1 we see that $y^{(1,\mathbf{w})} = y'$.
Now $(1, \mathbf{w}) \in W(\tilde{D_n})$ if and only if the coordinate sum of $\mathbf{w}$ is even. We have

$$\begin{aligned}\sum_{i=1}^{n} w_i &= \sum_{i=1}^{m}(\lambda_i + \lambda_i') + \sum_{j=1}^{k}\frac{1}{2}(\mu_j' - \mu_j)\\ &\equiv \left\{\sum_{i=1}^{m}(\mid \lambda_i' \mid + \frac{1}{2}\sum_{j=1}^{k}\mu_j')) - \sum_{i=1}^{m}(\mid \lambda_i \mid + \frac{1}{2}\sum_{j=1}^{k_e}\mu_j)\right\} \pmod 2.\end{aligned}$$

(Note that as $x, x' \in W(\tilde{D}_n)$ we have that $\sum_{j=1}^{k}\mu_j$ and $\sum_{j=1}^{k}\mu_j'$ are both even, because k is even and all μ are odd.) Thus

$$\textstyle\sum_{i=1}^{n} w_i \equiv \frac{1}{2}g(y') - \frac{1}{2}g(y) \pmod 2.$$

So $(1, \mathbf{w}) \in W(\tilde{D_n})$ if and only if $g(y) \equiv g(y') \pmod 4$. Hence if $g(y) \equiv g(y') \pmod 4$, then $y' \in y^{W(\tilde{D}_n)}$.

Now suppose $g(y) \not\equiv g(y') \pmod 4$, meaning $(1, \mathbf{w}) \notin W(\tilde{D_n})$. If $k > 0$ we see that $\overset{\underline{\mu_k}}{(n)}$ centralises y, meaning

$$y^{\overset{\underline{\mu_k}}{(n)}(1,\mathbf{w})} = y',$$

and hence we have an element of $W(\tilde{B_n})\backslash W(\tilde{D_n})$ conjugating y to y'. By Lemma 3.3.3, this means $y' \notin y^{W(\tilde{D}_n)}$. If $k = 0$, then $f(y) = g(y)$ and $f(y') = g(y')$, so $f(y) \not\equiv f(y') \pmod 4$, meaning these elements are not even conjugate in $W(\tilde{B}_n)$.
Thus we have shown that if $t(x) \equiv t(x') \pmod 2$, then $x' \in x^{W(\tilde{D}_n)}$ if and only if $g(x) \equiv g(x') \pmod 4$.
Now we suppose $t(y) \not\equiv t(y') \pmod 2$. This forces $m \geq 1$. Moreover if $k = 0$ then by Theorem 1.3.3 x is not conjugate to x'. So assume $k > 0$. Let $z = y^{\overset{\underline{0}}{(2)}}$. Then $z \in y^{W(\tilde{B}_n)} \backslash y^{W(\tilde{D}_n)}$. We observe that $t(z) \equiv t(y) + 1 \pmod 2$ and consequently $t(z) \equiv t(y') \pmod 2$. Therefore $y' \in z^{W(\tilde{D}_n)}$ if and only if $g(y') \equiv g(z) \pmod 4$.
Now $y' \in z^{W(\tilde{D}_n)}$ precisely when $y' \notin y^{W(\tilde{D}_n)}$. Thus $y' \in y^{W(\tilde{D}_n)}$ if and only if $g(y') \not\equiv g(y) \pmod 4$. We have therefore shown that if $t(x) \not\equiv t(x') \pmod 2$, then $x' \in x^{W(\tilde{D}_n)}$ if and only if $k > 0$ and $g(y') \not\equiv g(y) \pmod 4$, and this completes the proof. $\square$

Theorem 3.4.4. *Consider an involution x in $W(\tilde{D}_n)$ with labelled cycle type (m, k_e, k_o, l). If l or k_e are nonzero, then $x^{W(\tilde{D}_n)} = x^{W(\tilde{B}_n)}$. If $l = 0$ and $k_e = 0$, then $x^{W(\tilde{B}_n)}$ splits as two conjugacy classes in $W(\tilde{D}_n)$. In this case, write $x = \prod_{i=1}^{m} (a_i \overset{\overset{\lambda_i}{*}}{\ } b_i) \prod_{j=1}^{n-2m} (\overset{\overset{\sigma_j}{-}}{c_j})$ where the σ_j are all odd. If $0 < 2m < n$, then x is conjugate to exactly one of the following:*

$$w_3 = (1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m-1 \overset{\overset{0}{+}}{\ } 2m)\ (\overset{\overset{1}{-}}{2m+1}) \cdots (\overset{\overset{1}{-}}{n});$$
$$w_4 = (1 \overset{\overset{0}{-}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (2m-1 \overset{\overset{0}{+}}{\ } 2m)\ (\overset{\overset{1}{-}}{2m+1}) \cdots (\overset{\overset{1}{-}}{n}).$$

In particular, elements in $w_3^{W(\tilde{D}_n)}$ are precisely those x where if $t(x) \equiv 0 \pmod 2$, then $g(x) \equiv n - 2m \pmod 4$. Otherwise $g(x) \equiv n - 2m + 2 \pmod 4$.

Because of these results, everything from $W(\tilde{C}_n)$ carries across to $W(\tilde{B}_n)$ except possibly the cases where $l = 0$ and $k_o = 0$. If $m = 0$ the graph is completely disconnected by Lemma 1.5.4. It remains to deal with the cases $n = 2m$, and where both m and k_e are greater than zero. Similarly for $W(\tilde{D}_n)$ we inherit results from $W(\tilde{C}_n) \cdots$ and /or $W(\tilde{B}_n)$ except where $l = 0$ and $k_e = 0$. Again if $m = 0$ the graph is completely disconnected. So here we only need to consider the case where $n = 2m$, and the case where both m and k_o are greater than zero. We look first, for both $W(\tilde{B}_n)$ and $W(\tilde{D}_n)$, at the case $n = 2m$.

Proposition 3.4.5. *Suppose $n = 2m$.*

(i) If G is of type $\tilde{B}_n$, then Diam $\mathcal{C}(G, X) \leq 3$.

(ii) If G is of type $\tilde{D}_n$, then Diam $\mathcal{C}(G, X) \leq 4$.

Proof. Let $x \in X$ and assume first that $G = W(\tilde{B}_n)$. We can write $x = \prod_{i=1}^{m} (a_i \overset{\overset{\lambda_i}{+/-}}{\ } b_i)$. By Proposition 3.3.7, x is conjugate to either
$w_I = (1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (n-1 \overset{\overset{0}{+}}{\ } n)$, where $I = \{r_2, r_4, ..., r_{n-2}, r_n\}$, or to
$w_I = (1 \overset{\overset{0}{+}}{\ } 2) \cdots (n-3 \overset{\overset{0}{+}}{\ } n-2)(n-1 \overset{\overset{1}{-}}{\ } n)$, where $I = \{r_2, r_4, ..., r_{n-2}, r_{n+1}\}$.
By Theorem 3.4.2, x is conjugate to $(1 \overset{\overset{0}{+}}{\ } 2)(3 \overset{\overset{0}{+}}{\ } 4) \cdots (n-1 \overset{\overset{0}{+}}{\ } n)$ if and only if $\sum_{i=1}^{m} \lambda_i \equiv 0 \pmod 2$.

There is an automorphism of the Coxeter graph of $W(\tilde{B}_n)$ mapping $\{r_2, r_4, ..., r_{n-2}, r_n\}$ to $\{r_2, r_4, ..., r_{n-2}, r_{n+1}\}$. Therefore the two conjugacy classes have isomorphic commuting involution graphs. Without loss of generality then, we may assume that $\sum_{i=1}^{m} \lambda_i \equiv 0 \pmod 2$. Then $c = \prod_{i=1}^{m} (a_i \overset{\overset{0}{-/+}}{\ } b_i) \in X$ and by Lemma 1.5.5, x commutes with c. Now c is in a subgroup isomorphic to $W(B_n)$ and so by Theorem 1.5.1 (ii), $d(c, w_I) \leq 2$.

Hence, Diam $C(G,X) \leq 3$.
Now suppose $G = W(\tilde{D}_n)$. In this case x is conjugate to exactly one of the following.

$$\begin{aligned}
w_{I_1} =& \overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)} \cdots \overset{\overset{0}{+}}{(n-1\;n)}, \text{ where } I_1 = \{r_1, r_4, r_6, ..., r_{n-2}, r_n\},\\
w_{I_2} =& \overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)} \cdots \overset{\overset{1}{-}}{(n-1\;n)}, \text{ where } I_2 = \{r_1, r_4, r_6..., r_{n-2}, r_{n+1}\},\\
w_{I_3} =& \overset{\overset{0}{-}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)} \cdots \overset{\overset{0}{+}}{(n-1\;n)}, \text{ where } I_3 = \{r_2, r_4, r_6..., r_{n-2}, r_n\},\\
w_{I_4} =& \overset{\overset{0}{-}}{(1\,2)}\overset{\overset{0}{+}}{(3\,4)} \cdots \overset{\overset{1}{-}}{(n-1\;n)}, \text{ where } I_4 = \{r_2, r_4, r_6, ..., r_{n-2}, r_{n+1}\}.
\end{aligned}$$

There are automorphisms of the Coxeter graph of $W(\tilde{D}_n)$ mapping each I_i to any other I_j. Therefore, the four conjugacy classes have isomorphic commuting involution graphs. Without loss of generality then, assume x is conjugate to w_{I_1}. By Lemma 3.4.3, this occurs if and only if $t(x)$ is even and $g(x) \equiv 0 \pmod 4$, which is equivalent to $\sum_{i=1}^m \lambda_i \equiv 0 \pmod 2$.

If m is even, let $c = \prod_{i=1}^m \overset{\overset{0}{-/+}}{(a_i\, b_i)}$. We have $t(c) \equiv m + t(x) \equiv t(x) \pmod 2$ and $g(c) = 0$. Therefore $c \in X$. Now x commutes with c, and since c lies in a subgroup isomorphic to $W(D_n)$ we have $d(c, w_{I_1}) \leq 2$. Thus $d(x, w_{I_1}) \leq 3$. If m is odd, then as $\sum_{i=1}^m \lambda_i$ is even, there must be at least one even λ_i. Without loss of generality then, we can suppose λ_1 is even. Now let $x' = \overset{\overset{\lambda_1}{+/-}}{(a_1\, b_1)}\overset{\overset{0}{-/+}}{(a_2\, b_2)} \prod_{i=3}^m \overset{\overset{0}{-/+}}{(a_i\, b_i)}$ and $c' = \overset{\overset{0}{-/+}}{(a_1\, b_1)}\overset{\overset{0}{+/-}}{(a_2\, b_2)} \prod_{i=3}^m \overset{\overset{0}{-/+}}{(a_i\, b_i)}$. We have that $t(x') \equiv (m-1) + t(x) \equiv t(x) \pmod 2$ and $g(x') = 2|\lambda_1| \equiv 0 \pmod 4$. Also note that $t(c') = t(x')$ and $g(c') = 0$. Thus $x', c' \in X$ and x' commutes with both x and c'. Moreover $d(c, w_{I_1}) \leq 2$. Therefore $d(x, w_{I_1}) \leq 4$. Hence Diam $\mathcal{C}(G,X) \leq 4$. □

Proposition 3.4.6. *Let $x \in W(\tilde{B}_n)$ be an element with labelled cycle type $(1, k_e, 0, 0)$, where $k_e > 0$. If $k_e \leq 2$, then $\mathcal{C}(G,X)$ is disconnected. Otherwise,* Diam $\mathcal{C}(G,X) \leq n+1$.

Proof. Let $x \in X$. By Theorem 3.4.2, x is conjugate either to $w_1 = \overset{\overset{1}{-}}{(1\,2)}\overset{\overset{0}{-}}{(3)} \cdots \overset{\overset{0}{-}}{(n)}$ or to $w_2 = \overset{\overset{0}{+}}{(1\,2)}\overset{\overset{0}{-}}{(3)} \cdots \overset{\overset{0}{-}}{(n)}$. Suppose first that x is conjugate to w_1. Then $f(x) \equiv 2 \pmod 4$.

If $k_e \leq 2$, then the graph is not connected even in the underlying $W(B_n)$, so by Lemma 1.5.7, $\mathcal{C}(G,X)$ must be disconnected. So assume $k_e \geq 3$, meaning $n \geq 5$. We will show that if the transposition of x is $\overset{\overset{\lambda}{*}}{(1\,2)}$ for some λ, then $d(x,a) \leq n+1$. Otherwise $d(x,a) \leq n$.

We proceed by induction on n. The first case to consider is $n = 5$, where we have $x = \overset{\overset{\lambda}{*}}{(a\,b)}\overset{\overset{2p}{-}}{(c)}\overset{\overset{2q}{-}}{(d)}\overset{\overset{2r}{-}}{(e)}$ for some integers λ, p, q, r. Since $f(x) \equiv 2 \pmod 4$, we must have $\lambda + p + q + r \equiv 1 \pmod 2$. Conjugation by elements of the centraliser of w_1 (which includes the subgroup $\langle \overset{0}{\overset{+}{(1\,2)}}, \overset{0}{\overset{+}{(3\,4)}}, \overset{0}{\overset{+}{(4\,5)}}, \overset{0}{\overset{-}{(5)}} \rangle$) does not affect $d(x, w_1)$. Hence we can assume without loss of generality that x contains one of the transpositions $\overset{\lambda}{\overset{+}{(1\,5)}}$, $\overset{\lambda}{\overset{+}{(3\,4)}}$ or $\overset{\lambda}{\overset{*}{(1\,2)}}$. Suppose first that x contains $\overset{\lambda}{\overset{+}{(1\,5)}}$. Then $x = \overset{\lambda}{\overset{+}{(1\,5)}}\overset{2p}{\overset{-}{(2)}}\overset{2q}{\overset{-}{(3)}}\overset{2r}{\overset{-}{(4)}}$ for some integers p, q, r such that $\lambda + p + q + r \equiv 1 \pmod 2$. Define $x_1 = \overset{1-p-q-r}{\overset{-}{(1\,5)}}\ \overset{2p}{\overset{-}{(2)}}\overset{2q}{\overset{-}{(3)}}\overset{2r}{\overset{-}{(4)}}$, $x_2 = \overset{p+q}{\overset{-}{(2\,3)}}\ \overset{2(1-p-q)}{\overset{-}{(1)}}\ \overset{2r}{\overset{-}{(4)}}\overset{-2r}{\overset{-}{(5)}}$ and $x_3 = \overset{0}{\overset{-}{(4\,5)}}\ \overset{2(1-p-q)}{\overset{-}{(1)}}\ \overset{2(p+q)}{\overset{-}{(2)}}\ \overset{0}{\overset{-}{(3)}}$. Note that for each i we have $f(x_i) = f(w_1) = 2$, and hence $x_i \in X$. Moreover, by Lemma 1.5.5, x, x_1, x_2, x_3, w_1 is a path in $\mathcal{C}(G, X)$. Hence $d(x, w_1) \leq 4$.

If x contains $\overset{\lambda}{\overset{+}{(3\,4)}}$, then $x = \overset{\lambda}{\overset{+}{(3\,4)}}\overset{2p}{\overset{-}{(1)}}\overset{2q}{\overset{-}{(2)}}\overset{2r}{\overset{-}{(5)}}$ where again $\lambda+p+q+r \equiv 1 \pmod 2$, and x commutes with $x' = \overset{p-r}{\overset{+}{(1\,5)}}\overset{2q}{\overset{-}{(2)}}\overset{2\lambda}{\overset{-}{(3)}}\overset{0}{\overset{-}{(4)}}$. Now $f(x') = 2(p-r+q+\lambda) \equiv 2(\lambda+p+q+r) = f(x) \pmod 4$. Thus $x' \in X$ and we have seen above that $d(x', w_1) \leq 4$. Consequently $d(x, w_1) \leq 5$.

Finally, if $x = \overset{\lambda}{\overset{+/-}{(1\,2)}}\overset{2p}{\overset{-}{(3)}}\overset{2q}{\overset{-}{(4)}}\overset{2r}{\overset{-}{(5)}}$, then x commutes with $x'' = \overset{p-q}{\overset{+}{(3\,4)}}\overset{2\lambda}{\overset{-}{(1)}}\overset{0}{\overset{-}{(2)}}\overset{2r}{\overset{-}{(5)}}$ and $f(x'') \equiv f(x) \pmod 4$, meaning $x'' \in X$. We have seen above that $d(x'', w_1) \leq 5$. Hence $d(x, w_1) \leq 6$. Therefore the inductive hypothesis holds for $n = 5$.

We now assume $n \geq 6$ and proceed inductively. Suppose the transposition of x contains some a with $a > 2$. Then x contains $\overset{\lambda}{\overset{*}{(a\ b)}}\overset{\mu}{\overset{-}{(c)}}\overset{\mu'}{\overset{-}{(d)}}$ for some integer λ and even integers μ and μ'. Then, by Lemma 1.5.5, x commutes with the element y containing $\overset{(\mu+\mu')/2}{\overset{-}{(c\ d)}}\ \overset{0}{\overset{-}{(a)}}\overset{2\lambda}{\overset{-}{(b)}}$, with all its other cycles the same as x.
Now $f(y) - f(x) = 2|\lambda| + |\mu + \mu'| - (2|\lambda| + |\mu| + |\mu'|)$, which is congruent modulo 4 to $f(x)$ because μ and μ' are both even. Thus $y \in X$. Ignoring the cycle $\overset{0}{\overset{-}{(a)}}$ we can work within $G_{\{1,\ldots,a-1,a+1,\ldots,n\}}$, to see that inductively $d(y, w_1) \leq n - 1$. Hence $d(x, w_1) \leq n$.
If the transposition of x is $\overset{\lambda}{\overset{*}{(1\ 2)}}$ then, again by Lemma 1.5.5, x certainly commutes with an element of X which does not have this transposition. So $d(x, w_1) \leq n + 1$ as required and $\mathrm{Diam}\, \mathcal{C}(G, X) \leq n + 1$.

Finally, the case where x is conjugate to w_2 follows from the w_1 case. This is because the conjugacy class X of w_1 in $W(\tilde{B}_n)$ is mapped by any element g of $W(\tilde{C}_n) \setminus W(\tilde{B}_n)$ to the class Y of w_2 in $W(\tilde{B}_n)$. This map induces an isomorphism of their respective commuting involution graphs. Therefore the graphs have the same diameters. $\square$

Proposition 3.4.7. *Let $G = W(\tilde{B}_n)$. Suppose X is a conjugacy class whose elements have labelled cycle type $(m, k_e, 0, 0)$ where both $m > 1$ and $k_e \geq 2$. Then $\mathcal{C}(G, X)$ is connected with diameter at most $n - 1$.*

Proof. There are two conjugacy classes for each labelled cycle type.

Let $x = \prod_{i=1}^{m} (a_i \overset{\overset{\lambda_i}{*}}{\;} b_i) \prod_{j=1}^{n-2m} (\overset{\overset{\mu_j}{-}}{c_j})$ where the μ_j are all even. Then x is conjugate to exactly one of the following:

$$
\begin{aligned}
w_1 &= (1 \overset{\overset{1}{-}}{\;} 2)(3 \overset{\overset{0}{+}}{\;} 4) \cdots (2m-1 \overset{\overset{0}{+}}{\;} 2m)\ (\overset{\overset{0}{-}}{2m+1}) \cdots (\overset{\overset{0}{-}}{n}),\\
w_2 &= (1 \overset{\overset{0}{+}}{\;} 2)(3 \overset{\overset{0}{+}}{\;} 4) \cdots (2m-1 \overset{\overset{0}{+}}{\;} 2m)\ (\overset{\overset{0}{-}}{2m+1}) \cdots (\overset{\overset{0}{-}}{n})\ .
\end{aligned}
$$

In particular, x is conjugate to w_2 if and only if $f(x) \equiv 0 \pmod 4$ which, as $2\lambda_i$ and μ_j are even, is if and only if $2\sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{m} \mu_j \equiv 0 \pmod 4$.
By assumption, $m \geq 1$ and $k_e \geq 2$. We proceed by induction on k_e to show that $\mathrm{Diam}\, \mathcal{C}(G,X) \leq n-1$. Suppose $k_e = 2$. Then x is distance at most 2 from an element y of X which has the transposition $(n-1 \overset{\overset{0}{+/-}}{\;} n)$. To see this, note that if both $n-1$ and n appear in transpositions of x, or if both appear in 1-cycles of x, then Lemma 1.5.5 or Lemma 1.5.6, as appropriate, imply that x commutes with some x' in X which contains a transposition of the form $(n-1 \overset{\overset{\lambda}{-/+}}{\;} n)$ for some λ. If $n-1$ and n appear in transpositions of x, then we note that for each pair of double transpositions that commute in Lemma 1.5.6, the numbers above the first double transposition are λ_1 and λ_2, the numbers above the second pair are μ_1 and μ_2, and in every case $\lambda_1+\lambda_2 \equiv \mu_1+\mu_2 \pmod 2$, which means x' is guaranteed to be conjugate to x. If $n-1$ and n are in 1-cycles of x, then x contains cycles of the form $(\alpha \overset{\overset{\lambda_1}{+/-}}{\;} \beta)(\gamma \overset{\overset{\lambda_2}{+/-}}{\;} \delta)(\overset{\overset{\mu_1}{-}}{n-1})(\overset{\overset{\mu_2}{-}}{n})$; so, writing $\mu = (\mu_1 - \mu_2)/2$, we may choose x' to be x with those cycles replaced with $(n-1 \overset{\overset{\mu}{+}}{\;} n)\ (\gamma \overset{\overset{\lambda_1+\lambda_2-\mu}{-/+}}{\;} \delta)\ (\overset{\overset{0}{-}}{\alpha})(\overset{\overset{2\lambda_1}{-}}{\beta})$. In either case, x' is an element of X that commutes with a suitable y. The remaining case is when x contains (for example) the 1-cycle $(\overset{\overset{\mu}{-}}{n})$ and $n-1$ appears in a transposition $(\varepsilon \overset{\overset{\sigma}{+/-}}{\;} n-1)$ for some $\varepsilon < n-1$. Then x commutes with x'' in X containing the transpositions $(\varepsilon \overset{\overset{\lambda}{-/+}}{\;} n-1)$ and $(\varepsilon' \overset{\overset{\lambda'}{-/+}}{\;} n)$ for some λ, λ' and ε', where we can choose λ to ensure that $x'' \in X$. Lemma 1.5.6 now implies that x' commutes with an appropriate y, in particular one containing the transpositions $(\varepsilon \overset{\overset{\sigma\pm\lambda}{-}}{\;} \varepsilon')$ and $(n-1 \overset{\overset{0}{-}}{\;} n)$. Now y in turn commutes with some z in X with the 1-cycles $(\overset{\overset{0}{-}}{n-1})$ and $(\overset{\overset{0}{-}}{n})$. If we ignore these cycles and work in G_{n-2}, then a quick check confirms that when $n-2 = 4$ we have $d(z,a) \leq 2$, and when $n-2 > 4$, Proposition 3.4.5 tells us that $d(z,a) \leq 3$. Therefore $\mathrm{Diam}\, \mathcal{C}(G,X) \leq n-1$.

Finally, suppose $m \geq 2$ and $k_e > 2$. Suppose there is some transposition of x containing an element α with $\alpha > 2m$. Then by Lemma 1.5.5(iii) x commutes with some $y \in X$ such that y has the 1-cycle $\overset{\overset{0}{-}}{(\alpha)}$. By induction $d(y,a) \leq n-2$. Hence $d(x,a) \leq n-1$. The final possibility is that the elements of a transpositions of x are $\{1,2,\ldots,2m\}$. Since $m>1$ we can use Lemma 1.5.6 to show that x commutes with some y in X containing a transposition $\overset{\overset{0}{*}}{(1\,2)}$ with ($*=+$ or $-$). Working in $G_{\{3,4,\ldots,n\}}$ (using the case $m=1$ and induction on m) we see that $d(y,a) \leq n-2$. Hence $d(x,a) \leq n-1$, which completes the proof of Proposition 3.4.7. $\square$

We now move on to $W(\tilde{D}_n)$. We proved that if $n=2m$, then the diameter is at most 4 and if $m=0$, then the commuting involution graph is completely disconnected. Hence, the remaining case is when $k_e=0$, $l=0$, $k_o>0$, $m>0$. The following lemma shows that in certain circumstances if two elements that commute are obtained from another by interchanging a transposition with two 1-cycles, then they are guaranteed to be conjugate.

Lemma 3.4.8. *Suppose x and y are elements of $W(\tilde{D}_n)$ with cycle type $(m,0,k_o,0)$, where $x = \overset{\overset{\lambda}{*}}{(a\,b)}\overset{\overset{\mu_c}{-}}{(c)}\overset{\overset{\mu_d}{-}}{(d)}\bar{x}$ [with $\bar{x}$ being comprised of $m-1$ transpositions and k_o-2 negative 1-cycles] and $y = \overset{\overset{\sigma}{*}}{(c\,d)}\overset{\overset{\mu_a}{-}}{(a)}\overset{\overset{\mu_b}{-}}{(b)}\bar{x}$. If x commutes with y, then $y \in X^{W(\tilde{D}_n)}$.*

Proof. Suppose x commutes with y. Let $\gamma = \mu_a+\mu_b+\mu_c+\mu_d+g(\bar{x})$ and $\tau = 1+t(\bar{x})$. Using Lemma 1.5.5, we see that the transposition of x is either $\overset{\overset{\frac{1}{2}(\mu_a+\mu_b)}{-}}{(a\,b)}$ or $\overset{\overset{\frac{1}{2}(\mu_a-\mu_b)}{+}}{(a\,b)}$. In the first case we get $t(x)=\tau$ and $g(x) = |\mu_a+\mu_b|+\mu_c+\mu_d+g(\bar{x}) \equiv \gamma \pmod 4$ (because each μ is odd so the sum of any pair is even); in the second case $t(x) \equiv \tau+1 \pmod 2$ and $g(x) = |\mu_a-\mu_b|+\mu_c+\mu_d+g(\bar{x}) \equiv \gamma+2 \pmod 4$. Similarly, the transposition of y is either $\overset{\overset{\frac{1}{2}(\mu_c+\mu_d)}{-}}{(c\,d)}$ or $\overset{\overset{\frac{1}{2}(\mu_c-\mu_d)}{+}}{(c\,d)}$. So we either get $t(y)=\tau$ and $g(y) \equiv \gamma \pmod 4$, or $t(y) \equiv \tau+1 \pmod 2$ and $g(y) \equiv \gamma+2 \pmod 4$. All four possibilities belong to the same conjugacy class by Theorem 3.4.4. $\square$

Proposition 3.4.9. *Let $x \in W(\tilde{D}_n)$ be an element with labelled cycle type $(1,0,k_o,0)$, where k_o is an even number greater than zero. If $k_o=2$, then $\mathcal{C}(G,X)$ is disconnected. Otherwise,* Diam $\mathcal{C}(G,X) \leq n$.

Proof. Let $x \in X$. Note that k_o must be even, or x would not be contained in $W(\tilde{D}_n)$. If $k_o=2$ then the corresponding graph in the underlying $W(D_n)$ is disconnected, so $\mathcal{C}(G,X)$ is disconnected. Therefore we assume from now on that k_o is even and greater than 2. Hence $n \geq 6$. By Theorem 3.4.4, x is conjugate either to $w_3 = \overset{\overset{0}{+}}{(1\,2)}\overset{\overset{1}{-}}{(3)}\cdots\overset{\overset{1}{-}}{(n)}$ or to $w_4 = \overset{\overset{0}{-}}{(1\,2)}\overset{\overset{1}{-}}{(3)}\cdots\overset{\overset{1}{-}}{(n)}$. Suppose first that x is conjugate to w_3. This occurs if and only if either $t(x)=0$ and $g(x) \equiv n-2 \pmod 4$, or $t(x)=1$ and $g(x) \equiv n \pmod 4$.

That is, $x \in X$ if and only if $g(x) + 2t(x) \equiv n - 2 \pmod 4$.

Let $x \in X$. We will show that in almost all cases x is of distance at most $n-5$ from an element

$$y = \overset{\overset{\sigma}{-}}{(a_1\, a_2)}\overset{\overset{\sigma_3}{-}}{(a_3)}\overset{\overset{\sigma_4}{-}}{(a_4)}\overset{\overset{\sigma_5}{-}}{(a_5)}\overset{\overset{1}{-}}{(a_6)}\cdots\overset{\overset{1}{-}}{(a_n)} \in X$$

such that $\sigma_3 \equiv \sigma_4 \mod 4$ and $\{1,2\} \cap \{a_2, a_5\} = \emptyset$. Consider such a y. We have

$$g(y) + 2t(y) = (2|\sigma| + \sigma_3 + \sigma_4 + \sigma_5 + n - 5) + 2 \equiv 2\sigma - 1 + \sigma_5 + (n-2) \pmod 4.$$

Since $y \in X$, it must be that $2\sigma - 1 + \sigma_5 \equiv 2 \pmod 4$, and so $\frac{1}{2}(2\sigma - 1 + \sigma_5)$ is odd (as is $\frac{1}{2}(\sigma_3 + \sigma_4)$). Now define

$$\begin{aligned}
y_1 &= \overset{\overset{\frac{1}{2}(\sigma_3+\sigma_4)}{-}}{(a_3\, a_4)}\overset{\overset{1}{-}}{(a_1)}\overset{\overset{2\sigma-1}{-}}{(a_2)}\overset{\overset{\sigma_5}{-}}{(a_5)}\overset{\overset{1}{-}}{(a_6)}\cdots\overset{\overset{1}{-}}{(a_n)},\\
y_2 &= \overset{\overset{0}{+}}{(a_3\, a_4)}\overset{\overset{1}{-}}{(a_1)}\overset{\overset{2\sigma-1}{-}}{(a_2)}\overset{\overset{\sigma_5}{-}}{(a_5)}\overset{\overset{1}{-}}{(a_6)}\cdots\overset{\overset{1}{-}}{(a_n)},\\
y_3 &= \overset{\overset{\frac{1}{2}(2\sigma-1+\sigma_5)}{-}}{(a_2\, a_5)}\overset{\overset{1}{-}}{(a_1)}\overset{\overset{1}{-}}{(a_3)}\overset{\overset{1}{-}}{(a_4)}\overset{\overset{1}{-}}{(a_6)}\cdots\overset{\overset{1}{-}}{(a_n)},\\
y_4 &= \overset{\overset{0}{+}}{(a_2\, a_5)}\overset{\overset{1}{-}}{(a_1)}\overset{\overset{1}{-}}{(a_3)}\overset{\overset{1}{-}}{(a_4)}\overset{\overset{1}{-}}{(a_6)}\cdots\overset{\overset{1}{-}}{(a_n)}.
\end{aligned}$$

By repeated use of Lemma 1.5.5 and Lemma 3.4.8, we see that each $y_i \in X$ and furthermore that $y, y_1, y_2, y_3, y_4, w_3$ is a path in $\mathcal{C}(G,X)$ from y to w_3. For example, y_1 commutes with y by Lemma 1.5.5, and thus is an element of X by Lemma 3.4.8. Therefore $d(y, w_3) \leq 5$. We next show that for all $x \in X$, either there is such a y with $d(x,y) \leq n-5$, or $d(x, w_3) \leq 6$ (or both).

Firstly, suppose $n \geq 8$ and write $x = \overset{\overset{\lambda}{*}}{(a\, b)} \prod_{i=3}^{n} \overset{\overset{\mu_i}{-}}{(c_i)}$.

Let $x_1 = \overset{\overset{\frac{1}{2}(\mu_3+\mu_4)}{*}}{(c_3\, c_4)}\overset{\overset{1}{-}}{(a)}\overset{\overset{*(1-2\lambda)}{-}}{(b)} \prod_{i=5}^{n} \overset{\overset{\mu_i}{-}}{(c_i)}$. By Lemma 1.5.5, x_1 commutes with x, and thus $x_1 \in X$ by Lemma 3.4.8. We can repeat this step to produce for example x_2 containing $\overset{\overset{1}{-}}{(a)}\overset{\overset{1}{-}}{(c_3)}$. At each stage we create an additional 1-cycle $\overset{\overset{1}{-}}{(c)}$ for some c. As $n \geq 8$, after at most three steps we can ensure we have an element containing $\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}$. Therefore, x is of distance at most $n-5$ from an element y with $n-5$ cycles $\overset{\overset{1}{-}}{(c)}$, including $\overset{\overset{1}{-}}{(1)}$ and $\overset{\overset{1}{-}}{(2)}$; the remaining three 1-cycles have odd numbers over them, each congruent to either 1 or 3 modulo 4, and so at least two of these odd numbers are congruent to each other. We also note that at each step, including for y, the transposition has a minus sign over it. Hence y is of the required form. Thus $d(x, w_3) \leq (n-5) + n = n$.

The remaining case to consider is $n = 6$. Since $d(x, w_3)$ is unaffected by conjugation by any element in the centraliser of w_3, we may assume without loss of generality that the transposition of x is $\overset{-}{(1\,3)}$, $\overset{-}{(3\,4)}$, $\overset{+}{(1\,2)}$ or $\overset{-}{(1\,2)}$. We are looking to show, where possible, that x commutes with some $y = \overset{\substack{\sigma\\ -}}{(a_1\,a_2)}\overset{\substack{\sigma_3\\ -}}{(a_3)}\overset{\substack{\sigma_4\\ -}}{(a_4)}\overset{\substack{\sigma_5\\ -}}{(a_5)}\overset{\substack{1\\ -}}{(a_6)}$ in X, with $\{1,2\} \notin \{a_2, a_5\}$ and $\sigma_3 \equiv \sigma_4 \pmod 4$.

If $x = \overset{\substack{\lambda\\ -}}{(1\,3)}\overset{\substack{\mu_2\\ -}}{(2)}\overset{\substack{\mu_4\\ -}}{(4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$, then set $y = \overset{\substack{\frac{1}{2}(\mu_2+\mu_4)\\ -}}{(2\,4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}\overset{\substack{-(1-2\lambda)\\ -}}{(3)}\overset{\substack{1\\ -}}{(1)}$. By Lemma 1.5.5 and Lemma 3.4.8, $y \in X$ and x commutes with y. It is clear that y has the required form. Hence $d(x, w_3) \leq 6$.

Suppose $x = \overset{\substack{\lambda\\ -}}{(3\,4)}\overset{\substack{\mu_1\\ -}}{(1)}\overset{\substack{\mu_2\\ -}}{(2)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$. If μ_1 or μ_2 is congruent modulo 4 to either μ_5 or μ_6, then without loss of generality suppose $\mu_2 \equiv \mu_6 \pmod 4$. Then we can set $y = \overset{\substack{\frac{1}{2}(\mu_1+\mu_5)\\ -}}{(1\,5)}\overset{\substack{\mu_2\\ -}}{(2)}\overset{\substack{\mu_6\\ -}}{(6)}\overset{\substack{*(1-2\lambda)\\ -}}{(4)}\overset{\substack{1\\ -}}{(3)}$, and y has the required form. If on the other hand neither μ_1 nor μ_2 is congruent to μ_5 or μ_6, then we must have $\mu_1 \equiv \mu_2 \pmod 4$. Thus we can set $y = \overset{\substack{\frac{1}{2}(\mu_5+\mu_6)\\ -}}{(5\,6)}\overset{\substack{\mu_1\\ -}}{(1)}\overset{\substack{\mu_2\\ -}}{(2)}\overset{\substack{*(1-2\lambda)\\ -}}{(3)}\overset{\substack{1\\ -}}{(4)}$, and again y has the required form, which implies $d(x, w_3) \leq 6$.

Next suppose $x = \overset{\substack{\lambda\\ +}}{(1\,2)}\overset{\substack{\mu_3\\ -}}{(3)}\overset{\substack{\mu_4\\ -}}{(4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$. If $1-2\lambda$ is congruent to any μ_i, then without loss of generality we can assume $1-2\lambda \equiv \mu_5 \pmod 4$, and set $y = \overset{\substack{\frac{1}{2}(\mu_3+\mu_4)\\ -}}{(3\,4)}\overset{\substack{1-2\lambda\\ -}}{(2)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}\overset{\substack{1\\ -}}{(1)}$. So $d(x, w_3) \leq 6$. If this does not happen, then we must have $\mu_3 \equiv \mu_4 \equiv \mu_5 \equiv \mu_6 \pmod 4$. Thus $g(x) + 2t(x) \equiv 2|\lambda| + 4\mu_3 \equiv 2\lambda \pmod 4$. Since x is conjugate to w_3 and $n = 6$, this implies λ is even. The following is a path in $\mathcal{C}(G, X)$ from x to w_3: x, $\overset{\substack{1\\ -}}{(1\,2)}\overset{\substack{\mu_3\\ -}}{(3)}\overset{\substack{\mu_4\\ -}}{(4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$, $\overset{\substack{\frac{1}{2}(\mu_3+\mu_4)\\ -}}{(3\,4)}\overset{\substack{1\\ -}}{(1)}\overset{\substack{1\\ -}}{(2)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$, $\overset{\substack{0\\ +}}{(3\,4)}\overset{\substack{1\\ -}}{(1)}\overset{\substack{1\\ -}}{(2)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$, $\overset{\substack{\frac{1}{2}(\mu_5+\mu_6)\\ -}}{(5\,6)}\overset{\substack{1\\ -}}{(1)}\overset{\substack{1\\ -}}{(2)}\overset{\substack{1\\ -}}{(3)}\overset{\substack{1\\ -}}{(4)}$, $\overset{\substack{0\\ +}}{(5\,6)}\overset{\substack{1\\ -}}{(1)}\overset{\substack{1\\ -}}{(2)}\overset{\substack{1\\ -}}{(3)}\overset{\substack{1\\ -}}{(4)}$, w_3. Thus $d(x, w_3) \leq 6$.

Finally suppose $x = \overset{\substack{\lambda\\ -}}{(1\,2)}\overset{\substack{\mu_3\\ -}}{(3)}\overset{\substack{\mu_4\\ -}}{(4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$. If $2\lambda - 1$ is congruent to any μ_i, then without loss of generality we can assume $2\lambda - 1 \equiv \mu_5 \pmod 4$, and set $y = \overset{\substack{\frac{1}{2}(\mu_3+\mu_4)\\ -}}{(3\,4)}\overset{\substack{2\lambda-1\\ -}}{(2)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}\overset{\substack{1\\ -}}{(1)}$. So $d(x, w_3) \leq 6$. If this does not happen, then again we must have $\mu_3 \equiv \mu_4 \equiv \mu_5 \equiv \mu_6 \pmod 4$. Thus $g(x) + 2t(x) \equiv 2\lambda + 2 \pmod 4$, which forces λ to be odd. Hence x commutes with $\overset{\substack{0\\ +}}{(1\,2)}\overset{\substack{\mu_3\\ -}}{(3)}\overset{\substack{\mu_4\\ -}}{(4)}\overset{\substack{\mu_5\\ -}}{(5)}\overset{\substack{\mu_6\\ -}}{(6)}$, which is distance at most 4 from w_3.
Thus again $d(x, w_3) \leq 6$. In all cases we have shown that $d(x, w_3) \leq n$ and so $\operatorname{Diam} \mathcal{C}(G, X) \leq n$. $\square$

Proposition 3.4.10. *Let $G = W(\tilde{D}_n)$. Suppose X is a conjugacy class whose elements have labelled cycle type $(m, 0, k_o, 0)$, where $m > 1$ and $k_o \geq 2$. Then $\mathcal{C}(G, X)$ is connected with diameter at most n.*

Proof. Let $x = \prod_{i=1}^{m} \overset{\substack{0\\+}}{(a_i\, b_i)} \prod_{j=1}^{k_o} \overset{\substack{\mu_j\\-}}{(c_j)}$, where the μ_j are all odd. By Theorem 3.4.4 there are two conjugacy classes with labelled cycle type $(m, 0, k_o, 0)$, namely w_3^G and w_4^G, and we recall that $w_3 = \overset{\substack{0\\+}}{(1\, 2)} \cdots \overset{\substack{0\\+}}{(2m-1\ 2m)}\overset{\substack{1\\-}}{(2m+1)} \cdots \overset{\substack{1\\-}}{(n)}$. We note that w_3 and w_4 are interchanged by an isomorphism induced by a graph automorphism of the Coxeter graph for $W(\tilde{D}_n)$ and this isomorphism in turn induces an isomorphism between the respective commuting graphs. Hence it is sufficient to prove the result for the case $X = w_3^G$. Here, we have $x \in X$ if and only if either $t(x)$ is even and $g(x) \equiv k_o \pmod 4$, or t is odd and $g(x) \equiv k_o + 2 \pmod 4$. Since $g(x) = \sum_{i=1}^{m} 2|\lambda_i| + \sum_{j=1}^{k_o} \mu_j$, we note that $g(x) \equiv \sum_{i=1}^{m} 2\lambda_i + \sum_{j=1}^{k_o} \mu_j \pmod 4$.

We claim that x is distance at most 2 from an element y of X which has the transposition $\overset{\substack{0\\+}}{(n-1\ n)}$. There are three possibilities. If $n-1$ and n appear in 1-cycles of x, then by Lemmas 1.5.5 and 3.4.8, x commutes with some $x' \in X$ having the transposition $\overset{\substack{\mu\\-}}{(n-1\ n)}$ for some μ, and since $m > 1$ there is at least one other transposition $\overset{\substack{\lambda\\+/-}}{(a\, b)}$ in x'. If μ is odd, then set y to be x' with $\overset{\substack{\mu\\-}}{(n-1\ n)}$ replaced by $\overset{\substack{0\\+}}{(n-1\ n)}$; otherwise set y to be x' with $\overset{\substack{\mu\\-}}{(n-1\ n)}\overset{\substack{\lambda\\+/-}}{(a\, b)}$ replaced by $\overset{\substack{0\\+}}{(n-1\ n)}\overset{\substack{\lambda\\-/+}}{(a\, b)}$. Either way a quick check shows that $y \in X$ and y commutes with x'. So $d(x, y) \leq 2$.

The second possibility is that $n-1$ and n both appear in transpositions of x. If they are in the same transposition $\overset{\substack{\mu\\-}}{(n-1\ n)}$ for some μ, then we proceed as for x' above and so $d(x, y) = 1$. If they are in a transposition $\overset{\substack{\sigma\\+}}{(n-1\ n)}$, then x commutes with an x' containing $\overset{\substack{\mu\\-}}{(n-1\ n)}$ where $\mu = \sigma + 1$, so we have $x' \in X$ and as above x' commutes with a suitable y in X, meaning $d(x, y) \leq 2$. If $n-1$ and n appear in different transpositions $\overset{\substack{\sigma\\ *}}{(n-1\ \varepsilon)}$ and $\overset{\substack{\sigma'\\ *}}{(n\, \varepsilon)}$ of x, then by Lemma 1.5.6 there is an element y which commutes with x and which contains $\overset{\substack{0\\+}}{(n-1\ n)}\overset{\substack{\mu\\+/-}}{(\varepsilon\ \varepsilon')}$. We note further that for each pair of double transpositions that commute in Lemma 1.5.6, the sum of the numbers above the first double transposition is congruent modulo 2 to the sum of the numbers above the second double transposition, meaning $g(x) \equiv g(y) \pmod 4$, and also that the number of minus signs above each double transposition is congruent modulo 2, so that $t(x) \equiv t(y) \pmod 2$. Thus $y \in X$ and $d(x, y) = 1$.

Finally, we consider the case where (for example) x contains $\overset{\substack{\sigma\\ *}}{(n-1\ \varepsilon)}\overset{\substack{\mu\\-}}{(n)}$. Then, by Lemmas 1.5.5 and 3.4.8 (remembering that $m > 1$) we see that x commutes with an x' in X containing $(n \overset{\overset{\sigma}{*}}{-} 1\ \varepsilon)(\overset{\overset{\sigma'}{*}}{n}\,\varepsilon)$ with $(* = +$ or $-)$. As above, x' commutes with an appropriate y in X and so $d(x, y) \leq 2$.

In each case we have shown that x is distance at most 2 from an element y of X containing $(n \overset{\overset{0}{+}}{-} 1\ n)$. By Lemmas 1.5.5 and 2.4.8 therefore, y commutes with some z in X which contains $(n \overset{\overset{1}{-}}{-} 1)(\overset{\overset{1}{-}}{n})$. We now ignore these cycles and proceed in G_{n-2}, using induction on k_o. If $k_o = 2$, then in G_{n-2} we are in the '$2m = n$' case. When $n-2 = 4$, a quick check verifies that $d(z, w_3) \leq 3$. Otherwise $d(z, w_3) \leq 4$ by Proposition 3.4.5. Thus in all cases $d(x, w_3) \leq n$. If $k_o > 2$, then we use induction on k_o. If $k_o > 2$, then in G_{n-2} we have elements with labelled cycle type $(m, 0, k_o - 2, 0)$, and so by induction $d(z, w_3) \leq n - 2$. Hence again $d(x, w_3) \leq n$, and the proof is complete. $\square$

We have now established all the results needed to prove our main results on the connectedness and diameters of commuting involution graphs in $W(\tilde{B}_n)$ and $W(\tilde{D}_n)$. We restate them here for convenience. Theorem 3.1.1 stated that $\mathcal{C}(G, X)$ is disconnected if and only if one of the following holds:

(i) $m = 0$ and $l = 0$;

(ii) $m > 0$, $l = 0$, $k_e = 1$;

(iii) $m > 0$ and $\max\{k_e, k_o, l\} = 1$;

(iv) $n = 4$ and $m = 1$;

(v) $n = 6$, $m = 1$, $k_o = k_e = 2$.

This is identical to Theorem 2.1.2 for $W(\tilde{C}_n)$ except that the case $k_o = 1$ is excluded in (ii) because this is not possible in $W(\tilde{B}_n)$ or $W(\tilde{D}_n)$. Theorem 3.1.2 stated that if $m = 0$ and $l \geq 1$, then $\mathcal{C}(G, X)$ is connected, and gave diameter bounds which are exactly the same as those stated in Theorem 2.3.2 for the corresponding labelled cycle types in $W(\tilde{C}_n)$. Theorem 3.1.3 stated that if $m \geq 1$, then we have four possibilities.

(i) If $2m = n$, then Diam $\mathcal{C}(G, X) \leq 4$.

(ii) If $l = 0$, $0 \in \{k_e, k_o\}$, and either $m > 1$ or $\max\{k_e, k_o\} \geq 3$, then Diam $\mathcal{C}(G, X) \leq n + 1$.

(iii) If $l = 0$ and $\min\{k_e, k_o\} \geq 2$, then Diam $\mathcal{C}(G, X) \leq n + 2$.

(iv) If $l \geq 1$ and $\max\{k_e, k_o, l\} > 1$, then Diam $\mathcal{C}(G, X) \leq n$.

Proof of Theorems 3.1.1, 3.1.2 and 3.1.3 Let x be an involution in G where G is either $W(\tilde{B}_n)$ or $W(\tilde{D}_n)$, with labelled cycle type (m, k_e, k_o, l) and conjugacy class X. Note that k_o must always be even in order for x to be an element of either of these groups. For Theorem 3.1.1, the proof that the given cases (i) - - (v) have disconnected graphs is identical to the proof of Theorem 2.3.1; for example if $m = 0$ and $l = 0$, then $\mathcal{C}(G, X)$ is completely disconnected by Lemma 1.5.4, if $l = 0$ and $k_e = 1$, then by Lemma 1.5.4 and 1.5.5, any element that commutes with an x containing $\overset{\overset{0}{-}}{(n)}$ must also contain $\overset{\overset{0}{-}}{(n)}$ and so on. The reverse implication, that any other labelled cycle type has a connected commuting involution graph, will follow from Theorems 3.1.2 and 3.1.3 which give diameter bounds in these cases.

For Theorem 3.1.2, we are dealing with labelled cycle types where $l \geq 1$. Here, by Proposition 3.3.4, $x^G = x^{W(\tilde{C}_n)}$. Therefore Theorem 3.1.2 holds, by Theorem 2.3.2.

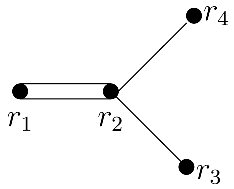
Finally we deal with Theorem 3.1.3. Part (i) is just Proposition 3.4.5. Recall that if $x \in W(\tilde{B}_n)$ and either $k_o > 0$ or $l > 0$, then $X = x^{W(\tilde{C}_n)}$. Thus part (ii) when $k_o > 0$, along with parts (iii) and (iv), follow immediately from the corresponding results for $W(\tilde{C}_n)$ (Proposition 2.3.4, Theorem 2.3.6, Lemmas 2.3.7 and 2.3.9, and Theorem 2.3.8). When $k_o = 0$ part (ii) holds by Propositions 3.4.6 and 3.4.7. If $x \in W(\tilde{D}_n)$ and either $k_e > 0$ or $l > 0$, then $X = x^{W(\tilde{C}_n)}$. So this time part (ii) when $k_e > 0$, along with parts (iii) and (iv), follow again from the corresponding results for $W(\tilde{C}_n)$. When $k_e = 0$ part (ii) holds by Propositions 3.4.9 and 3.4.10. This completes the proof of Theorems 3.1.1, 3.1.2 and 3.1.3, showing that when $\mathcal{C}(G, X)$ is connected, its diameter is at most $n + 2$.

3.5 Involution Conjugacy Classes in G where $|R| \leq 5$

In this section we illustrate our results on conjugacy classes of involutions for groups of small rank, namely $W(\tilde{B}_3)$, $W(\tilde{B}_4)$ and $W(\tilde{D}_4)$. As ever, we use the fact that any involution in one of these groups G is conjugate to the central longest element w_I of some standard parabolic subgroup W_I. For ease of notation we write X_I for w_I^G, the conjugacy class of w_I in G.

3.5.1 G is of type $\tilde{B}_3$

Let G be an affine Coxeter group of type $\tilde{B}_3$. Then the set of generators of G is $R = \{r_1, r_2, r_3, r_4\}$ with $r_1 = \overset{\overset{0}{-}}{(1)}$, $r_2 = \overset{\overset{0}{+}}{(1\ 2)}$, $r_3 = \overset{\overset{0}{+}}{(2\ 3)}$ and $r_4 = \overset{\overset{1}{-}}{(2\ 3)}$. Its Coxeter graph is as follows:



We can see that the connected components of the Coxeter graph for W_I are of types B_i for $i \in \{1,2,3\}$ and A_1. So for $|I| = 1$, we have two conjugacy classes with subgraph A_1. One graph is as follows:

Here we have $I = \{r_1\}$, so we consider this class as type B_1 to differentiate between $\overset{0}{\overset{-}{(1)}}$ and $\overset{0}{\overset{+}{(1\ 2)}}$. In this case $w_I = \overset{0}{\overset{-}{(1)}}$. We can see that $\overset{\lambda}{\overset{+}{(i\ j)}}\overset{0}{\overset{-}{(i)}}\overset{\lambda}{\overset{+}{(i\ j)}} = \overset{-2\lambda}{\overset{-}{(j)}}$ and $\overset{\lambda}{\overset{-}{(i\ j)}}\overset{0}{\overset{-}{(i)}}\overset{\lambda}{\overset{-}{(i\ j)}} = \overset{2\lambda}{\overset{-}{(j)}}$. It is well known that elements of $W(B_3)$ are conjugate if they have the same signed cycle type. For example, a 1-cycle can not be conjugate to a transposition. Thus we have

$$X_1 = \{\overset{\mu}{\overset{-}{(j)}} \colon 1 \leq j \leq 3, \mu \in 2\mathbb{Z}\}.$$

The second class is of type A_1, when $I = \{r_2\}$. Its graph is as follows:

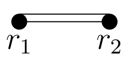
Here, $w_I = \overset{0}{\overset{+}{(1\ 2)}}$. We can see that

$$\begin{aligned} \overset{\mu}{\overset{-}{(i)}}\overset{0}{\overset{+}{(i\ j)}}\overset{\mu}{\overset{-}{(i)}} &= \overset{\mu}{\overset{-}{(i\ j)}}, \\ \overset{\lambda}{\overset{-}{(j\ k)}}\overset{0}{\overset{+}{(i\ j)}}\overset{\lambda}{\overset{-}{(j\ k)}} &= \overset{\lambda}{\overset{-}{(i\ k)}}, \\ \overset{\mu}{\overset{-}{(i)}}\overset{\lambda}{\overset{-}{(i\ j)}}\overset{\mu}{\overset{-}{(i)}} &= \overset{\mu-\lambda}{\overset{+}{(i\ j)}}. \end{aligned}$$

Then, the set of elements conjugate to r_2 is:

$$X_I = \{\overset{\lambda}{\overset{\pm}{(i\ j)}} \colon 1 \leq i < j \leq 3, \lambda \in \mathbb{Z}\}.$$

Now for $|I| = 2$ we have four conjugacy classes. The first class is a subgroup of type B_2 and has subgraph of type B_2, when $I = \{r_1, r_2\}$. Its graph is as follows:



We have $w_I = \overset{0}{\overset{-}{(1)}}\overset{0}{\overset{-}{(2)}}$. So we consider $w^{-1}\ \overset{0}{\overset{-}{(1)}}\overset{0}{\overset{-}{(2)}}\ w$ for $w \in G$, and have

$$\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda}{-}}{(i\ j)}=\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(j)},$$
$$\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda}{+}}{(i\ j)}=\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{-2\lambda}{-}}{(j)},$$
$$\overset{\overset{\lambda}{-}}{(i\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda}{-}}{(i\ k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)},$$
$$\overset{\overset{\lambda}{+}}{(i\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda}{+}}{(i\ k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{-2\lambda}{-}}{(k)}\ .$$

In the calculation above, we see that the labels above i and j are even, and in fact we have

$$X_I=\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\colon 1\le i<j\le 3, \lambda_i,\lambda_j\in 2\mathbb{Z}\}.$$

There are two conjugacy classes which are subgroups correspond to w_I for a parabolic subgroup of type $B_1\times A_1$ and their subgraphs are of type A_1^2. One of these arises when $I=\{r_1,r_3\}$. Its graph is as follows:

$$\underset{r_1}{\bullet}\qquad\underset{r_3}{\bullet}$$

Any involution in this class is conjugate to $w_I=\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{+}}{(2\ 3)}$. We see that

$$\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{+}}{(j\ k)}\overset{\overset{\lambda}{-}}{(i\ j)}=\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{\lambda}{-}}{(i\ k)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{+}}{(j\ k)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}=\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j\ k)}\ (\text{with } \lambda_i+\lambda_j \text{ even}),$$
$$\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{+}}{(j\ k)}\overset{\overset{\lambda}{+}}{(i\ j)}=\overset{\overset{-2\lambda}{-}}{(j)}\overset{\overset{\lambda}{+}}{(i\ k)}\ .$$

Hence, $X_I=\{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\pm\lambda}{\mp}}{(j\ k)},\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j\ k)}\colon 1\le i<j\le 3, \{\lambda,\lambda_i,\lambda_j\}\subset\mathbb{Z}\}$.

When $I=\{r_1,r_4\}$, then its graph is as follows:

$$\underset{r_1}{\bullet}\qquad\underset{r_4}{\bullet}$$

An involution is conjugate to $w_I=\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2\ 3)}$. We have

$$\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j\ k)}\overset{\overset{\lambda}{-}}{(i\ j)}=\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{\lambda-1}{+}}{(i\ k)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j\ k)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}=\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j-1}{+}}{(j\ k)}\ (\text{with } \lambda_i+\lambda_j \text{ even}),$$
$$\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j\ k)}\overset{\overset{\lambda}{+}}{(i\ j)}=\overset{\overset{-2\lambda}{-}}{(j)}\overset{\overset{1+\lambda}{-}}{(i\ k)}\ .$$

Hence, $X_I = \{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)},\ \overset{\overset{-2\lambda}{-}}{(i)}\overset{\overset{\lambda+1}{-}}{(j\ k)},\ \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j-1}{+}}{(j\ k)} \colon 1 \le i < j \le 3, \{\lambda, \lambda_i, \lambda_j\} \subset \mathbb{Z}\}$.

The fourth conjugacy class corresponds to w_I for a parabolic subgroup of type A_1^2. It has a subgraph of type A_1^2 corresponding to $I = \{r_3, r_4\}$. Its graph is as follows:

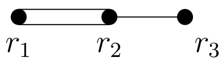
Any involution in this class is conjugate to $w_I = \overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}$. Then we have

$$\begin{aligned}
\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda}{+}}{(i\ j)} &= \overset{\overset{1}{-}}{(i)}\overset{\overset{1+2\lambda}{-}}{(k)},\\
\overset{\overset{\lambda}{+}}{(j\ k)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda}{+}}{(j\ k)} &= \overset{\overset{1+2\lambda}{-}}{(j)}\overset{\overset{1-2\lambda}{-}}{(k)},\\
\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i\ j)} &= \overset{\overset{2\lambda-1}{-}}{(i)}\overset{\overset{1}{-}}{(k)},\\
\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_j}{-}}{(j)} &= \overset{\overset{2\lambda_j-1}{-}}{(j)}\ \overset{\overset{2\lambda_k-1}{-}}{(k)}\ \text{(with } \lambda_j + \lambda_k \text{ even).}
\end{aligned}$$

In the calculation above, in X_I we have that all the labels above i, j and k are odd. We have

$$X_I = \{\overset{\overset{\lambda}{-}}{(i)}\overset{\overset{\lambda'}{-}}{(j)} \colon 1 \le i < j \le 3,\ \lambda, \lambda' \text{ are odd integers}\}.$$

Now, when $|I| = 3$, by Theorem 1.4.2 we have three conjugacy classes where two correspond to w_I for parabolic subgroups of type B_3 and they have subgraphs of type B_3. Thus, when $I = \{r_1, r_2, r_3\}$, the graph is as follows:



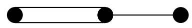
Then an involution is conjugate to $w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}$. We have

$$\begin{aligned}
\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i\ j)} &= \overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{0}{-}}{(k)},\\
\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{\lambda}{+}}{(i\ j)} &= \overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{-2\lambda}{-}}{(j)}\overset{\overset{0}{-}}{(k)},\\
\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)} &= \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(k)},\\
\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)} &= \overset{\overset{4\lambda}{-}}{(i)}\ \overset{\overset{2\lambda-2}{-}}{(j)}\ \overset{\overset{2-2\lambda}{-}}{(k)},\\
\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)} &= \overset{\overset{4\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)}.
\end{aligned}$$

Hence, if we add the labels above i, j and k for any element in this class, then we obtain one of these integers 4λ, 0 or 8λ. Then we can see that

$$X_I = \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)} \colon 1 \leq i < j < k \leq 3, \lambda \in 2\mathbb{Z}, (\lambda_i + \lambda_j + \lambda_k) \in 4\mathbb{Z}\}.$$

The second class arises when $I = \{r_1, r_2, r_4\}$. Its graph is as follows:



Then an involution is conjugate to $w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{2}{-}}{(3)}$. We see that

$$\begin{aligned}
&\overset{\overset{\lambda}{-}}{(i\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i\ k)} = \overset{\overset{2\lambda-2}{-}}{(i)}\ \overset{\overset{0}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)},\\
&\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2}{-}}{(k)}\overset{\overset{\lambda}{+}}{(i\ j)} = \overset{\overset{2+2\lambda}{-}}{(i)}\ \overset{\overset{0}{-}}{(j)}\overset{\overset{-2\lambda}{-}}{(k)},\\
&\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)} = \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\ \overset{\overset{2\lambda_j-2}{-}}{(k)},\\
&\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)} = \overset{\overset{4\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{2-2\lambda}{-}}{(k)},\\
&\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)} = \overset{\overset{4\lambda}{-}}{(i)}\ \overset{\overset{2\lambda-2}{-}}{(j)}\ \overset{\overset{2\lambda}{-}}{(k)}.
\end{aligned}$$

Hence, if we add the labels above i, j and k for any element in this class, then we have the following results ± 2 or $4\lambda - 2$ or $4\lambda + 2$ or $8\lambda - 2$. Then we can see that

$$X_I = \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)} \colon 1 \leq i < j < k \leq 3, \lambda \in 2\mathbb{Z}, (\lambda_i + \lambda_j + \lambda_k) \equiv 2 \bmod 4\ \}.$$

The third class correspond to w_I for a parabolic subgroup of type $B_1 \times A_1^2$ where $I = \{r_1, r_3, r_4\}$. Its subgraph is of type A_1^3. The graph is as follows:

We see that any involution in this class is conjugate to $w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}$. Then we have

$$\begin{aligned}
&\overset{\overset{\lambda}{-}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i\ j)} = \overset{\overset{2\lambda-1}{-}}{(i)}\ \overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{1}{-}}{(k)},\\
&\overset{\overset{\lambda}{+}}{(i\ j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda}{+}}{(i\ j)} = \overset{\overset{1+2\lambda}{-}}{(i)}\ \overset{\overset{-2\lambda}{-}}{(j)}\overset{\overset{1}{-}}{(k)},\\
&\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)} = \overset{\overset{2\lambda_i}{-}}{(i)}\ \overset{\overset{2\lambda_j-1}{-}}{(j)}\ \overset{\overset{1}{-}}{(k)},\\
&\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)} = \overset{\overset{4\lambda}{-}}{(i)}\ \overset{\overset{2\lambda-1}{-}}{(j)}\ \overset{\overset{3-2\lambda}{-}}{(k)},\\
&\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{1}{-}}{(j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)} = \overset{\overset{4\lambda}{-}}{(i)}\ \overset{\overset{2\lambda-1}{-}}{(j)}\ \overset{\overset{2\lambda-1}{-}}{(k)}.
\end{aligned}$$

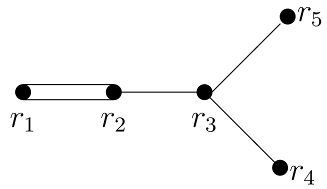
By using the above calculation, we can see that

$$X_I = \{\overset{\overset{\lambda_1}{-}}{(i)}\overset{\overset{\lambda_2}{-}}{(j)}\overset{\overset{\lambda_3}{-}}{(k)}\colon 1 \le i < j < k \le 3, \text{ exactly one } \lambda_i \text{ even}, i \in \{1,2,3\}\}.$$

Hence, we have 9 conjugacy classes of involutions when G is a Coxeter group of type $\tilde{B}_3$.

3.5.2 G is of type $\tilde{B}_4$

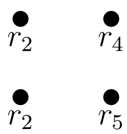
Let G be an affine Coxeter group of type $\tilde{B}_4$, generated by $\{r_1, r_2, r_3, r_4, r_5\}$ with $r_1 = \overset{\overset{0}{-}}{(1)}$, $r_2 = \overset{\overset{0}{+}}{(1\ 2)}$, $r_3 = \overset{\overset{0}{+}}{(2\ 3)}$, $r_4 = \overset{\overset{0}{+}}{(3\ 4)}$ and $r_5 = \overset{\overset{1}{-}}{(3\ 4)}$. The Coxeter graph of G is as follows:



The connected components of the Coxeter graph for W_I are of types B_i for $i \in \{1,2,3,4\}$, A_1 or D_4. Therefore, Theorem 1.4.2 is used in the following proofs. For $|I| = 1$, two conjugacy classes have subgraphs of type A_1. One class is a subgroup of type B_1 and the other is of type A_1. They have the same definition as when G is a Coxeter group of type $\tilde{B}_3$. Now for $|I| = 2$, we have 6 conjugacy classes of involutions of which four classes have the same proof as when G is a Coxeter group of type $\tilde{B}_3$. They are

$$\begin{aligned}
\left(\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\right)^G &= \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\colon 1 \le i < j \le 4, \{\lambda_i, \lambda_j\} \subset 2\mathbb{Z}\},\\
\left(\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(3\ 4)}\right)^G &= \{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)}, \overset{\overset{-2\lambda}{-}}{(i)}\overset{\overset{\lambda+1}{-}}{(j\ k)}, \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j-1}{+}}{(j\ k)}\colon 1 \le i < j \le 4, \{\lambda, \lambda_i, \lambda_j\} \subset \mathbb{Z}\}\\
\left(\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{+}}{(3\ 4)}\right)^G &= \{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\pm\lambda}{\pm}}{(j\ k)}, \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j\ k)}\colon 1 \le i < j \le 4, \{\lambda, \lambda_i, \lambda_j\} \subset \mathbb{Z}\},\\
\left(\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\right)^G &= \{\overset{\overset{\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j)}\colon 1 \le i < j \le 4, \lambda = 2z+1, z \in \mathbb{Z}\}.
\end{aligned}$$

The two remaining classes have subgraphs of type A_1^2 correspond to w_I for parabolic subgroups of type A_1^2. Their graphs are as:



When $I = \{r_2, r_4\}$, then we have

$$\overset{\overset{\lambda}{-}}{(i\ k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{0}{+}}{(k\ l)}\overset{\overset{\lambda}{-}}{(i\ k)} = \overset{\overset{\lambda}{-}}{(i\ l)}\overset{\overset{\lambda}{-}}{(j\ k)},$$

$$
\begin{aligned}
&\overset{\substack{\lambda\\+}}{(i\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{0\\+}}{(k\ l)}\overset{\substack{\lambda\\+}}{(i\ k)}=\overset{\substack{\lambda\\+}}{(i\ l)}\overset{\substack{\lambda\\+}}{(j\ k)},\\
&\overset{\substack{\lambda\\-}}{(i)}\overset{\substack{\lambda\\-}}{(k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{0\\+}}{(k\ l)}\overset{\substack{\lambda\\-}}{(k)}\overset{\substack{\lambda\\-}}{(i)}=\overset{\substack{\lambda\\-}}{(i\ j)}\overset{\substack{\lambda\\-}}{(k\ l)},\\
&\overset{\substack{2\lambda\\-}}{(i)}\overset{\substack{\lambda\\-}}{(j\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{0\\+}}{(k\ l)}\overset{\substack{\lambda\\-}}{(j\ k)}\overset{\substack{2\lambda\\-}}{(i)}=\overset{\substack{\lambda\\+}}{(i\ k)}\overset{\substack{\lambda\\-}}{(j\ l)},\\
&\overset{\substack{-2\lambda\\-}}{(i)}\overset{\substack{\lambda+1\\-}}{(j\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{0\\+}}{(k\ l)}\overset{\substack{\lambda+1\\-}}{(j\ k)}\overset{\substack{-2\lambda\\-}}{(i)}=\overset{\substack{-3\lambda-1\\-}}{(i\ k)}\overset{\substack{\lambda+1\\+}}{(j\ l)}.
\end{aligned}
$$

We see that the conjugacy class of $\overset{\substack{0\\+}}{(1\ 2)}\overset{\substack{0\\+}}{(3\ 4)}$ is $\{\overset{\substack{\lambda_i\\\pm}}{(i\ j)}\overset{\substack{\lambda_j\\\pm}}{(k\ l)}\colon 1 \le i < j \le 4, \{\lambda_1, \lambda_2\} \subset \mathbb{Z}\}$. For $I = \{r_2, r_5\}$, we have

$$
\begin{aligned}
&\overset{\substack{\lambda\\-}}{(i\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{1\\-}}{(k\ l)}\overset{\substack{\delta\\-}}{(i\ k)}=\overset{\substack{\lambda-1\\+}}{(i\ l)}\overset{\substack{\lambda\\-}}{(j\ k)},\\
&\overset{\substack{\lambda\\+}}{(i\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{1\\-}}{(k\ l)}\overset{\substack{\lambda\\+}}{(i\ k)}=\overset{\substack{1+\lambda\\-}}{(i\ l)}\overset{\substack{\lambda\\+}}{(j\ k)},\\
&\overset{\substack{2\lambda_i\\-}}{(i)}\overset{\substack{2\lambda_k\\-}}{(k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{1\\-}}{(k\ l)}\overset{\substack{2\lambda_k\\-}}{(k)}\overset{\substack{2\lambda_i\\-}}{(i)}=\overset{\substack{2\lambda_i\\-}}{(i\ j)}\overset{\substack{2\lambda_k-1\\+}}{(k\ l)},\\
&\overset{\substack{2\lambda\\-}}{(i)}\overset{\substack{\lambda\\-}}{(j\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{1\\-}}{(k\ l)}\overset{\substack{\lambda\\-}}{(j\ k)}\overset{\substack{2\lambda\\-}}{(i)}=\overset{\substack{\lambda\\+}}{(i\ k)}\overset{\substack{\lambda-1\\+}}{(j\ l)},\\
&\overset{\substack{-2\lambda\\-}}{(i)}\overset{\substack{\lambda+1\\-}}{(j\ k)}\overset{\substack{0\\+}}{(i\ j)}\overset{\substack{1\\-}}{(k\ l)}\overset{\substack{\lambda+1\\-}}{(j\ k)}\overset{\substack{-2\lambda\\-}}{(i)}=\overset{\substack{-3\lambda-1\\+}}{(i\ k)}\overset{\substack{\lambda\\+}}{(j\ l)}.
\end{aligned}
$$

By the calculation above we see that the conjugacy class of $\overset{\substack{0\\+}}{(1\ 2)}\overset{\substack{1\\-}}{(3\ 4)}$ is

$$\{\overset{\substack{\lambda_i\\\pm}}{(i\ j)}\overset{\substack{\lambda_j\\\pm}}{(k\ l)}\colon 1 \le i < j \le 4, \{(\lambda_i, \lambda_j)\} \subset \{(2z, 2z+1), (2z+1, 2z) : z \in \mathbb{Z}\}\}.$$

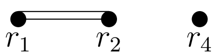
Now let $|I| = 3$. We have five conjugacy classes of involutions. Two classes have the same definition as when G is a Coxeter group of type $\tilde{B}_3$. The first conjugacy class corresponds to $I = \{r_1, r_2, r_3\}$; we have $w_I = \overset{\substack{0\\-}}{(1)}\overset{\substack{0\\-}}{(2)}\overset{\substack{0\\-}}{(3)}$, and

$$X_I = \{\overset{\substack{\lambda_i\\-}}{(i)}\overset{\substack{\lambda_j\\-}}{(j)}\overset{\substack{\lambda_k\\-}}{(k)}\colon 1 \le i < j < k \le 4, \lambda \in 2\mathbb{Z}, (\lambda_i + \lambda_j + \lambda_k) \in 4\mathbb{Z}\}.$$

The second class corresponds to $I = \{r_1, r_4, r_5\}$. We have $w_I = \overset{\substack{0\\-}}{(1)}\overset{\substack{1\\-}}{(2)}\overset{\substack{1\\-}}{(3)}$ and

$$X_I = \{\overset{\substack{\lambda_1\\-}}{(i)}\overset{\substack{\lambda_2\\-}}{(j)}\overset{\substack{\lambda_3\\-}}{(k)}\colon 1 \le i < j < k \le 4, \text{ exactly one } \lambda_i \in 2\mathbb{Z}, i \in \{1, 2, 3\}\}\ .$$

The remaining conjugacy classes have $I = \{r_1, r_2, r_4\}$, $I = \{r_1, r_2, r_5\}$ or $I = \{r_2, r_4, r_5\}$. Hence, the first class has a subgraph and class of type $B_2 \times A_1$, with $w_I = \overset{\substack{0\\-}}{(1)}\overset{\substack{0\\-}}{(2)}\overset{\substack{0\\+}}{(3\ 4)}$. Its graph is as follows:



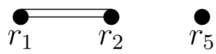
We have to determine the labels above these cycles. We see that

$$\overset{\overset{\lambda}{-}}{(i\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{+}}{(k\,l)}\overset{\overset{\lambda}{-}}{(i\,k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i\,l)},$$
$$\overset{\overset{\lambda}{+}}{(i\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{o}{+}}{(k\,l)}\overset{\overset{\lambda}{+}}{(i\,k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{-2\lambda}{-}}{(k)}\overset{\overset{\lambda}{+}}{(i\,l)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{+}}{(k\,l)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}=\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k\,l)},$$
$$\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{+}}{(k\,l)}\overset{\overset{\lambda}{-}}{(j\,k)}\overset{\overset{2\lambda}{-}}{(i)}=\overset{\overset{4\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(k)}\overset{\overset{\lambda}{-}}{(j\,l)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{+}}{(k\,l)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}=\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k-\lambda_l}{+}}{(k\,l)},$$
$$\overset{\overset{\lambda_i}{-}}{(i\,k)}\overset{\overset{\lambda_j}{-}}{(j\,l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{+}}{(k\,l)}\overset{\overset{\lambda_i}{-}}{(i\,k)}\overset{\overset{\lambda_j}{-}}{(j\,l)}=\overset{\overset{\lambda_i-\lambda_j}{+}}{(i\,j)}\overset{\overset{2\lambda_i}{-}}{(k)}\overset{\overset{2\lambda_l}{-}}{(l)}.$$

Hence, by the calculation above we have

$$X_I=\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{\pm}}{(k\,l)}\colon 1\le i<j\le 4, \{\lambda_i,\lambda_j\}\subset\{(0,2\lambda),(2\lambda,0):\lambda\in\mathbb{Z}\}\cup\{\{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{+}}{(k\,l)}\}.$$

The second class has the same type of subgraph and same type of subgroup as the previous one. Its elements have different labels as we can see in the calculation below. Its graph is as follows:



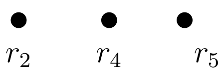
We have

$$\overset{\overset{\lambda}{-}}{(i\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\,l)}\overset{\overset{\lambda}{-}}{(i\,k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)}\overset{\overset{\lambda-1}{+}}{(i\,l)},$$
$$\overset{\overset{\lambda}{+}}{(i\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\,l)}\overset{\overset{\lambda}{+}}{(i\,k)}=\overset{\overset{0}{-}}{(j)}\overset{\overset{-2\lambda}{-}}{(k)}\overset{\overset{1+\lambda}{-}}{(i\,l)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\,l)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}=\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda_k-1}{+}}{(k\,l)},$$
$$\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\,l)}\overset{\overset{\lambda}{-}}{(j\,k)}\overset{\overset{2\lambda}{-}}{(i)}=\overset{\overset{4\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(k)}\overset{\overset{\lambda-1}{+}}{(j\,l)},$$
$$\overset{\overset{-2\lambda}{-}}{(i)}\overset{\overset{\lambda+1}{-}}{(j\,k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\,l)}\overset{\overset{\lambda+1}{-}}{(j\,k)}\overset{\overset{-2\lambda}{-}}{(i)}=\overset{\overset{-4\lambda}{-}}{(i)}\overset{\overset{2\lambda+2}{-}}{(k)}\overset{\overset{\lambda}{+}}{(j\,l)}.$$

Hence, $w_I=\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(3\,4)}$ and

$$X_I=\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda\mp1}{\pm}}{(k\,l)}\colon 1\le i<j\le 4, \{\lambda_i,\lambda_j\}\subset\{(0,2\lambda),(2\lambda,0):\lambda\in\mathbb{Z}\}\cup\{\{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{-}}{(k\,l)}\}.$$

The fifth conjugacy class corresponds to $I=\{r_2,r_4,r_5\}$. This class is of type A_1^3 and has subgraph of type A_1^3. Its graph is as follows, and $w_I=\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}$:



Then we have

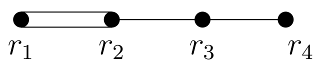
$$\overset{\overset{\lambda}{-}}{(i\,k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{-}}{(i\,k)}=\overset{\overset{2\lambda-1}{-}}{(i)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda-1}{-}}{(j\,k)},$$

$$\overset{\overset{\lambda}{+}}{(i\ k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{+}}{(i\ k)}=\overset{\overset{1+2\lambda}{-}}{(i)}\ \overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{+}}{(j\ k)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)}=\overset{\overset{\lambda_i}{-}}{(i\ j)}\ \overset{\overset{2\lambda_k-1}{-}}{(k)}\ \overset{\overset{1}{-}}{(l)},$$
$$\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{-}}{(j\ k)}\overset{\overset{2\lambda}{-}}{(i)}=\overset{\overset{2\lambda-1}{-}}{(j)}\ \overset{\overset{1}{-}}{(l)}\overset{\overset{2\lambda}{+}}{(i\ k)},$$
$$\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{\lambda-1}{+}}{(k\ l)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda-1}{+}}{(k\ l)}\overset{\overset{2\lambda}{-}}{(i)}=\overset{\overset{2\lambda}{-}}{(i\ j)}\overset{\overset{\lambda+3}{-}}{(k)}\ \overset{\overset{-2\lambda-1}{-}}{(l)}\ .$$

Hence, we see that

$$X_I=\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{-}}{(k\ l)}\colon 1\le i<j\le 4,,\lambda\in\mathbb{Z},\{\lambda_i,\lambda_j\}\subset\{2z+1: z\in\mathbb{Z}\}\}.$$

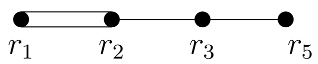
Now let $|I|=4$. Then we have four conjugacy classes, corresponding to $I=\{r_1,r_2,r_3,r_4\}$, $I=\{r_1,r_2,r_3,r_5\}$, $I=\{r_1,r_2,r_4,r_5\}$ or $I=\{r_2,r_3,r_4,r_5\}$. Hence, the first class is of type B_4 and has a subgraph of type B_4. Its graph is as follows:



This conjugacy class of involutions has the same definition as when G is a Coxeter group of type $\tilde{B}_3$. We have

$$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}^{G}=\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}\colon 1\le i<j<k\le 4,\lambda\in 2\mathbb{Z},(\lambda_i+\lambda_j+\lambda_k+\lambda_l)\in 4\mathbb{Z}\}.$$

The second class has subgraph of type B_4 and a class of type B_4 too. Its graph is as follows:



We have

$$\overset{\overset{\lambda}{-}}{(i\ l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}\overset{\overset{\lambda}{-}}{(i\ l)}=\overset{\overset{2\lambda-2}{-}}{(i)}\ \overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(l)},$$
$$\overset{\overset{\lambda}{+}}{(i\ l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}\overset{\overset{\lambda}{+}}{(i\ l)}=\overset{\overset{2\lambda+2}{-}}{(i)}\ \overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{-2\lambda}{-}}{(l)},$$
$$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_l}{-}}{(l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_l}{-}}{(l)}=\overset{\overset{4\lambda_i}{-}}{(i)}\overset{\overset{4\lambda_j}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\ \overset{\overset{4\lambda_l-2}{-}}{(l)}\ ,$$
$$\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(j\ l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}\overset{\overset{\lambda}{-}}{(j\ l)}\overset{\overset{2\lambda}{-}}{(i)}=\overset{\overset{4\lambda}{-}}{(i)}\ \overset{\overset{2\lambda-2}{-}}{(j)}\ \overset{\overset{0}{-}}{(k)}\overset{\overset{2\lambda}{-}}{(l)},$$
$$\overset{\overset{-2\lambda}{-}}{(i)}\overset{\overset{\lambda+1}{-}}{(j\ l)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}\overset{\overset{\lambda+1}{-}}{(j\ l)}\overset{\overset{-2\lambda}{-}}{(i)}=\overset{\overset{-4\lambda}{-}}{(i)}\overset{\overset{2\lambda}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\ \overset{\overset{2\lambda+2}{-}}{(l)}\ .$$

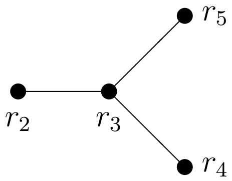
Therefore, by the calculation above, we see that by adding the labels above $(i),(j),(k)$ and (l) for each elements in this class, we have the labels ± 2, $12\lambda-2$ or $8\lambda-2$. Then we have the conjugacy class of $\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{2}{-}}{(l)}$ is

$\{\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}\colon 1 \leq i < j < k < l \leq 4, \lambda_i + \lambda_j + \lambda_k + \lambda_l \equiv 2 \bmod 4, \text{ all } \lambda_i \in 2\mathbb{Z}\}$.

Now when $I = \{r_1, r_2, r_4, r_5\}$, the class is of type $B_2 \times A_1^2$ and has subgraph of the same type. Its graph is as follows:

Hence, the conjugacy class of $\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}$ is $\{\overset{\underline{\lambda_1}}{(i)}\overset{\underline{\lambda_2}}{(j)}\overset{\underline{\lambda_3}}{(k)}\overset{\underline{\lambda_4}}{(l)}\colon 1 \leq i < j < k < l \leq 4, \text{ exactly two of } \lambda_1, \lambda_2, \lambda_3, \lambda_4 \text{ are even}\}$. The proof is the same as that for the conjugacy class of $\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{1}}{(3)}$ when G is a Coxeter group of type $\tilde{B}_3$.

The fourth conjugacy class arises when $I = \{r_2, r_3, r_4, r_5\}$. It corresponds to w_I for a parabolic subgroup of type D_4 and its subgraph is of type D_4 . Its graph is as follows:



We have

$$\overset{\underline{\lambda}}{(i\ k)}\overset{\underline{1}}{(i)}\overset{\underline{1}}{(j)}\overset{\underline{1}}{(k)}\overset{\underline{1}}{(l)}\overset{\underline{\lambda}}{(i\ k)} = \overset{\underline{2\lambda-1}}{(i)}\ \overset{\underline{1}}{(j)}\ \overset{\underline{2\lambda-1}}{(k)}\ \overset{\underline{1}}{(l)},$$

$$\overset{\underset{+}{\lambda}}{(i\ k)}\overset{\underline{1}}{(i)}\overset{\underline{1}}{(j)}\overset{\underline{1}}{(k)}\overset{\underline{1}}{(l)}\overset{\underset{+}{\lambda}}{(i\ k)} = \overset{\underline{2\lambda+1}}{(i)}\ \overset{\underline{1}}{(j)}\ \overset{\underline{-1-2\lambda}}{(k)}\ \overset{\underline{1}}{(l)},$$

$$\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}\overset{\underline{1}}{(i)}\overset{\underline{1}}{(j)}\overset{\underline{1}}{(k)}\overset{\underline{1}}{(l)}\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)} = \overset{\underline{2\lambda_i-1}}{(i)}\ \overset{\underline{2\lambda_j-1}}{(j)}\ \overset{\underline{2\lambda_k-1}}{(k)}\ \overset{\underline{2\lambda_l-1}}{(l)}\ ,$$

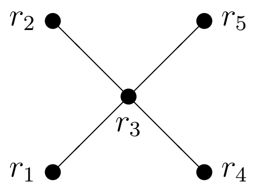
$$\overset{\underline{2\lambda}}{(i)}\overset{\underline{\lambda}}{(j\ k)}\overset{\underline{1}}{(i)}\overset{\underline{1}}{(j)}\overset{\underline{1}}{(k)}\overset{\underline{1}}{(l)}\overset{\underline{\lambda}}{(j\ k)}\overset{\underline{2\lambda}}{(i)} = \overset{\underline{4\lambda-1}}{(i)}\ \overset{\underline{2\lambda-1}}{(j)}\ \overset{\underline{2\lambda-1}}{(k)}\ \overset{\underline{1}}{(l)},$$

$$\overset{\underline{-2\lambda}}{(i)}\overset{\underline{\lambda+1}}{(j\ k)}\overset{\underline{1}}{(i)}\overset{\underline{1}}{(j)}\overset{\underline{1}}{(k)}\overset{\underline{1}}{(l)}\overset{\underline{\lambda+1}}{(j\ k)}\overset{\underline{-2\lambda}}{(i)} = \overset{\underline{4\lambda-1}}{(i)}\ \overset{\underline{2\lambda+1}}{(j)}\ \overset{\underline{2\lambda+1}}{(k)}\ \overset{\underline{1}}{(l)}\ .$$

Therefore, by the calculation above, we see that all the labels above $(i), (j), (k)$ and (l) are odd integers. Then we have the conjugacy class of $\overset{\underline{1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}$ is $\{\overset{\underline{\lambda_1}}{(i)}\overset{\underline{\lambda_2}}{(j)}\overset{\underline{\lambda_3}}{(k)}\overset{\underline{\lambda_4}}{(l)}\colon 1 \leq i < j < k < l \leq 4, \text{ where } \lambda_1, \lambda_2, \lambda_3, \lambda_4 \text{ are odd integers}\}$.
Hence, there are 17 conjugacy classes of involutions when G is a Coxeter group of type $\tilde{B}_4$.

3.5.3 G is of type $\tilde{D}_4$

Let G be a Coxeter group of type $\tilde{D}_4$. Then the set of generators of G is $R = \{r_1, r_2, r_3, r_4, r_5\}$ with $r_1 = \overset{\overset{0}{-}}{(1\ 2)}$, $r_2 = \overset{\overset{0}{+}}{(1\ 2)}$, $r_3 = \overset{\overset{0}{+}}{(2\ 3)}$, $r_4 = \overset{\overset{0}{+}}{(3\ 4)}$ and $r_5 = \overset{\overset{1}{-}}{(3\ 4)}$. The Coxeter graph of G is as follows:



By Theorem 1.4.2, the connected components of the Coxeter graph for W_I are of type A_1 or D_4. As Proposition 1.3 in [7] shows, if the elements of conjugacy classes contain only transpositions, then their numbers of transpositions of type $\overset{-}{(i\ j)}$ are congruent modulo 2. Now for $|I| = 1$, we have one conjugacy class of type A_1 with a subgraph of type A_1. Considering that $W(\tilde{D}_4) \subset W(\tilde{B}_4)$, then this class is the same as when G is of type $\tilde{B}_4$. Hence,

$$r_1{}^G = \{\overset{\overset{\lambda}{\pm}}{(i\ j)}\colon 1 \leq i < j \leq 4, \lambda \in \mathbb{Z}\}.$$

Now for $|I| = 2$, we have 6 conjugacy classes where all are of type A_1^2 and have subgraphs of type A_1^2. By the same calculation as when G is a Coxeter group of type $\tilde{B}_4$, when $I = \{r_1, r_2\}$ and $I = \{r_4, r_5\}$, we see that

$$\text{the conjugacy class of } \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)} \text{ is } \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\colon 1 \leq i < j \leq 4, \{\lambda_i, \lambda_j\} \subset \mathbb{Z}\},$$
$$\text{and the conjugacy class of } \overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)} \text{ is } \{\overset{\overset{2\lambda_i+1}{-}}{(i)}\ \overset{\overset{2\lambda_j+1}{-}}{(j)} : 1 \leq i < j \leq 4, \lambda \in \mathbb{Z}\}.$$

The third class corresponds to $I = \{r_1, r_4\}$. Elements of this signed cycle type are conjugate in G precisely when they have two transpositions of type $\overset{-}{(i\ j)}\overset{-}{(k\ l)}$ or of type $\overset{+}{(i\ j)}\overset{+}{(k\ l)}$.

Then by calculating of $w^{-1}\ \overset{\overset{0}{+}}{(1\ 2)}\overset{\overset{0}{+}}{(3\ 4)}\ w$ for $w \in G$, we see that

$$\overset{\overset{\lambda}{-}}{(i\ k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{0}{-}}{(k\ l)}\overset{\overset{\lambda}{-}}{(i\ k)} = \overset{\overset{\lambda}{-}}{(i\ l)}\overset{\overset{\lambda}{-}}{(j\ k)},$$

$$\overset{\overset{\lambda}{+}}{(i\ k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{0}{+}}{(k\ l)}\overset{\overset{\lambda}{+}}{(i\ k)} = \overset{\overset{\lambda}{+}}{(i\ l)}\overset{\overset{\lambda}{+}}{(j\ k)},$$

$$\overset{\overset{\lambda+i}{-}}{(i)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{0}{+}}{(i\ j)}\overset{\overset{0}{+}}{(k\ l)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_i}{-}}{(i)} = \overset{\overset{\lambda_i}{-}}{(i\ j)}\overset{\overset{\lambda_k}{-}}{(k\ l)}\ .$$

Hence, we have the conjugacy class of $(\overset{\overset{0}{+}}{1}\,2)(\overset{\overset{0}{+}}{3}\,4)$ is $\{(\overset{\overset{\lambda_i}{\pm}}{i}\,j)(\overset{\overset{\lambda_j}{\pm}}{k}\,l)\colon 1 \leq i < j \leq 4, \lambda \in \mathbb{Z}\}$. When $I = \{r_2, r_4\}$, this class has elements of the form $(\overset{+}{i}\,j)(\overset{-}{i}\,j)$ where $\cdots$ $1 \leq i < j < k < l \leq 4$. Hence, we perform the same calculation as when G is of type $\tilde{B}_4$. Then we have

$$\text{the conjugacy class of } (\overset{\overset{0}{-}}{1}\,2)(\overset{\overset{0}{+}}{3}\,4) \text{ is } \{(\overset{\overset{\lambda_i}{\mp}}{i}\,j)(\overset{\overset{\lambda_j}{\pm}}{k}\,l)\colon 1 \leq i < j \leq 4, \lambda \in \mathbb{Z}\}.$$

The fifth conjugacy class corresponds to $I = \{r_1, r_5\}$. We have

$$\begin{aligned}
(\overset{\overset{\lambda}{-}}{i}\,k)(\overset{\overset{0}{+}}{i}\,j)(\overset{\overset{1}{-}}{k}\,l)(\overset{\overset{\lambda}{-}}{i}\,k) &= (\overset{\overset{\lambda-1}{+}}{i}\,l)(\overset{\overset{\lambda}{-}}{j}\,k), \\
(\overset{\overset{\lambda}{+}}{i}\,k)(\overset{\overset{0}{+}}{i}\,j)(\overset{\overset{1}{-}}{k}\,l)(\overset{\overset{\lambda}{+}}{i}\,k) &= (\overset{\overset{1+\lambda}{-}}{i}\,l)(\overset{\overset{\lambda}{+}}{j}\,k), \\
(\overset{\overset{\lambda_i}{-}}{i})(\overset{\overset{\lambda_k}{-}}{k})(\overset{\overset{0}{+}}{i}\,j)(\overset{\overset{1}{-}}{k}\,l)(\overset{\overset{\lambda_k}{-}}{k})(\overset{\overset{\lambda_i}{-}}{i}) &= (\overset{\overset{\lambda_i}{-}}{i}\,j)(\overset{\overset{\lambda_k-1}{+}}{k}\,l)\ .
\end{aligned}$$

Therefore, we see that

$$\text{the conjugacy class of } (\overset{\overset{0}{+}}{1}\,2)(\overset{\overset{1}{-}}{3}\,4) \text{ is } \{(\overset{\overset{\lambda_i}{\pm}}{i}\,j)(\overset{\overset{\lambda_j\pm 1}{\mp}}{k}\,l)\colon \lambda \in \mathbb{Z}\}.$$

The last class is when $I = \{r_2, r_5\}$. By similar methods, to before, and recalling that elements in G of the form $(\overset{+}{i}\,j)(\overset{+}{k}\,l)$ and $(\overset{-}{i}\,j)(\overset{-}{k}\,l)$ are conjugate, we have

$$\text{the conjugacy class of } (\overset{\overset{0}{-}}{1}\,2)(\overset{\overset{1}{-}}{3}\,4) \text{ is } \{(\overset{\overset{\lambda_i}{\pm}}{i}\,j)(\overset{\overset{\lambda_j\pm 1}{\pm}}{k}\,l)\colon \lambda \in \mathbb{Z}\}.$$

Let $I = 3$. Then we have four conjugacy classes of type A_1^3 and their subgraphs are of type A_1^3 where

$$\begin{aligned}
I &= \{r_1, r_4, r_5\}, or \\
I &= \{r_2, r_4, r_5\}, or \\
I &= \{r_1, r_2, r_4\}, or \\
I &= \{r_1, r_2, r_5\}.
\end{aligned}$$

For $I = \{r_1, r_4, r_5\}$, we have

$$\begin{aligned}
\overset{\overset{\lambda}{-}}{(i\,k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{-}}{(i\,k)} &= \overset{\overset{2\lambda-1}{-}}{(i)}\,\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{-}}{(j\,k)},\\
\overset{\overset{\lambda}{+}}{(i\,k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{+}}{(i\,k)} &= \overset{\overset{1+2\lambda}{-}}{(i)}\,\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda}{+}}{(j\,k)},\\
\overset{\overset{\lambda_i}{-}}{(i\,k)}\overset{\overset{\lambda_j}{-}}{(j\,l)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda_i}{-}}{(i\,k)}\overset{\overset{\lambda_j}{-}}{(j\,l)} &= \overset{\overset{\lambda_i-\lambda_j}{+}}{(k\,l)}\,\overset{\overset{2\lambda_i-1}{-}}{(i)}\,\overset{\overset{2\lambda_j-1}{-}}{(j)}\,,\\
\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{1}{-}}{(k)}\overset{\overset{1}{-}}{(l)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)} &= \overset{\overset{\lambda_i-\lambda_j}{+}}{(i\,j)}\,\overset{\overset{2\lambda_k-1}{-}}{(k)}\,\overset{\overset{2\lambda_l-1}{-}}{(l)}\;.
\end{aligned}$$

Hence, for $I = \{r_1, r_4, r_5\}$ and $I = \{r_2, r_4, r_5\}$, we have the conjugacy class of $\overset{\overset{0}{+}}{(1\,2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}$ is $\{\overset{\overset{\lambda}{-}}{(i\,j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)} \colon \{\lambda_k, \lambda_l\} \subset \{1, 2\lambda - 1\}, \lambda \in \mathbb{Z}\} \cup \{\overset{\overset{\lambda}{+}}{(i\,j)}\,\overset{\overset{2\lambda_k+1}{-}}{(k)}\,\overset{\overset{2\lambda+l}{-}}{(l)} \colon \lambda \in \mathbb{Z}\}$.

By the same calculation, when $I = \{r_2, r_4, r_5\}$, the conjugacy class of $\overset{\overset{0}{-}}{(1\,2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}$ is $\{\overset{\overset{\lambda}{+}}{(i\,j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)} \colon \{\lambda_k, \lambda_l\} \subset \{1, 2\lambda - 1\}, \lambda \in \mathbb{Z}\} \cup \{\overset{\overset{\lambda}{-}}{(i\,j)}\,\overset{\overset{2\lambda_k+1}{-}}{(k)}\,\overset{\overset{2\lambda_l+1}{-}}{(l)} : \{\lambda_k, \lambda_l\} \subset \mathbb{Z}\}$.

Now, for $I = \{r_1, r_2, r_4\}$ we have

$$\begin{aligned}
\overset{\overset{\lambda}{-}}{(i\,k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{0}{-}}{(l)}\overset{\overset{\lambda}{-}}{(i\,k)} &= \overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{0}{-}}{(l)}\overset{\overset{\lambda}{-}}{(j\,k)},\\
\overset{\overset{\lambda}{+}}{(i\,k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{0}{-}}{(l)}\overset{\overset{\lambda}{+}}{(i\,k)} &= \overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{0}{-}}{(l)}\overset{\overset{\lambda}{+}}{(j\,k)},\\
\overset{\overset{\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(k)}\overset{\overset{0}{+}}{(i\,j)}\overset{\overset{0}{-}}{(k)}\overset{\overset{0}{-}}{(l)}\overset{\overset{\lambda}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i)} &= \overset{\overset{\lambda}{-}}{(i\,j)}\overset{\overset{2\lambda}{-}}{(k)}\overset{\overset{0}{-}}{(l)}\;.
\end{aligned}$$

Then we see that the conjugacy class of $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{+}}{(3\,4)}$ is
$\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{\pm}}{(k\,l)} \colon 1 \leq i < j \leq 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\} \cup \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{+}}{(k\,l)}\}$.

If $I = \{r_1, r_2, r_5\}$, then we have

$$\overset{\overset{\lambda}{-}}{(i\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{1}{-}}{(k\ l)}\overset{\overset{\lambda}{-}}{(i\ k)} = \overset{\overset{\lambda-1}{+}}{(i\ l)}\overset{\overset{0}{-}}{(j)}\overset{\overset{2\lambda}{-}}{(k)},$$

$$\overset{\overset{\lambda}{+}}{(i\ k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{o}{-}}{(j)}\overset{\overset{1}{-}}{(k\ l)}\overset{\overset{\lambda}{+}}{(i\ k)} = \overset{\overset{\lambda+1}{-}}{(i\ l)}\overset{\overset{0}{-}}{(j)}\overset{\overset{-2\lambda}{-}}{(k)},$$

$$\overset{\overset{\lambda}{-}}{(i)}\overset{\overset{\lambda}{-}}{(k)}\overset{\overset{0}{-}}{(i)}\overset{\overset{0}{-}}{(k)}\overset{\overset{1}{-}}{(k\ l)}\overset{\overset{\lambda}{-}}{(k)}\overset{\overset{\lambda}{-}}{(i)} = \overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{0}{-}}{(j)}\overset{\overset{\lambda-1}{+}}{(k\ l)}\ .$$

Then we see that the conjugacy class of $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(3\ 4)}$ is
$\{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda\mp1}{\pm}}{(k\ l)}\colon 1 \leq i < j \leq 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\} \cup \{\{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{\lambda}{-}}{(k\ l)}\}$.
Hence, if $|I| = 3$, there are four conjugacy classes.

Finally, when $|I| = 4$ we have five conjugacy classes where 4 of them are the same as when G is of type $\tilde{B}_4$. We have

$$I = \{r_1, r_2, r_4, r_5\} \text{ with } w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)},$$

$$I = \{r_1, r_3, r_4, r_5\} \text{ with } w_I = \overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)},$$

$$I = \{r_1, r_2, r_3, r_4\} \text{ with } w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)},$$

$$I = \{r_2, r_3, r_4, r_5\} \text{ with } w_I = \overset{\overset{-1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)} \textit{ or}$$

$$I = \{r_1, r_2, r_3, r_5\} \text{ with } w_I = \overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{2}{-}}{(4)}\ .$$

Therefore, the conjugacy class of $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}$ is
$\{\overset{\overset{\lambda_1}{-}}{(i)}\overset{\overset{\lambda_2}{-}}{(j)}\overset{\overset{\lambda_3}{-}}{(k)}\overset{\overset{\lambda_4}{-}}{(l)}\colon 1 \leq i < j < k < l \leq 4$, where two of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even$\}$.
Now, when $I = \{r_1, r_3, r_4, r_5\}$, then the conjugate of $\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}$ by $w = \overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}$ is
$\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)} = \overset{\overset{2\lambda_i-1}{-}}{(1)}\ \overset{\overset{2\lambda_j-1}{-}}{(2)}\ \overset{\overset{2\lambda_k-1}{-}}{(3)}\ \overset{\overset{2\lambda_l-1}{-}}{(4)}$. Hence, if we add the labels above the cycles $(i), (j), (k)$ and (l), then we obtain $2(\lambda_i + \lambda_j + \lambda_k + \lambda_l) - 4$. We know that every $w \in W(\tilde{D}_n)$ which is of the form $\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}$, has all λ_i even, or all odd, or two even and two odd. Hence in all cases $\lambda_i + \lambda_j + \lambda_k + \lambda_l$ is even, so $2(\lambda_i + \lambda_j + \lambda_k + \lambda_l)$ is a multiple of 4. Therefore, $2(\lambda_i + \lambda_j + \lambda_k + \lambda_l) - 2 \equiv 0 \pmod 4$. Now, if $w = \overset{\overset{\lambda_i}{-}}{(i\ j)}\overset{\overset{\lambda_j}{-}}{(k\ l)}$,

then we have $\overset{\underline{\lambda_i}}{(i\ j)}\overset{\underline{\lambda_j}}{(k\ l)}\overset{\underline{1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}\overset{\underline{\lambda_i}}{(i\ j)}\overset{\underline{\lambda_j}}{(k\ l)} = \overset{\underline{2\lambda_i-1}}{(1)}\ \overset{\underline{2\lambda_i-1}}{(2)}\ \overset{\underline{2\lambda_j-1}}{(3)}\ \overset{\underline{2\lambda_j-1}}{(4)}$. The sum of labels above the negative 1-cycles $(i),(j),(k)$ and (l) is $2(2\lambda_i+2\lambda_j)-4$. Therefore, $4\lambda_i+4\lambda_j \equiv 0 \pmod 4$. Hence, the conjugacy class of

$\overset{\underline{1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}$ is $\{\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}\colon (\lambda_i+\lambda_j+\lambda_k+\lambda_l) \equiv 0 \pmod 4$, all λ_i are odd integers$\}$. We have an involution with the same labelled cycle type as the previous one but it represents a different class in $W(\tilde{D}_4)$.

When $I=\{r_2,r_3,r_4,r_5\}$, then the conjugate of $\overset{\underline{-1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}$ by $w = \overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}$ is $\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}\overset{\underline{-1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)} = \overset{\underline{2\lambda_i+1}}{(1)}\ \overset{\underline{2\lambda_j-1}}{(2)}\ \overset{\underline{2\lambda_k-1}}{(3)}\ \overset{\underline{2\lambda_l-1}}{(4)}$. Hence, if we add the labels obtain the cycles $(i),(j),(k)$ and (l), then we have $2(\lambda_i+\lambda_j+\lambda_k+\lambda_l)-2$. We know that every $w \in W(\tilde{D}_n)$ of the form $\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}$ has all λ_i even or odd, or two even and two odd. Hence in all the cases $\lambda_i+\lambda_j+\lambda_k+\lambda_l$ is even so $2(\lambda_i+\lambda_j+\lambda_k+\lambda_l)$ is a multiple of 4. Therefore, $2(\lambda_i+\lambda_j+\lambda_k+\lambda_l)-2 \equiv \pm 2 \pmod 4$. Now, if $w = \overset{\underline{\lambda_i}}{(i\ j)}\overset{\underline{\lambda_j}}{(k\ l)}$, then we have $\overset{\underline{\lambda_i}}{(i\ j)}\overset{\underline{\lambda_j}}{(k\ l)}\overset{\underline{-1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}\overset{\underline{\lambda_i}}{(i\ j)}\overset{\underline{\lambda_j}}{(k\ l)} = \overset{\underline{2\lambda_i-1}}{(1)}\ \overset{\underline{2\lambda_i+1}}{(2)}\ \overset{\underline{2\lambda_j-1}}{(3)}\ \overset{\underline{2\lambda_j-1}}{(4)}$. The sum of labels above the negative 1-cycles $(i),(j),(k)$ and (l) is $2(2\lambda_i+2\lambda_j)-2 = 4\lambda_i+4\lambda_j-2$. Therefore, $2(2\lambda_i+2\lambda_j) \equiv \pm 2 \pmod 4$. Hence, the conjugacy class of $\overset{\underline{-1}}{(1)}\overset{\underline{1}}{(2)}\overset{\underline{1}}{(3)}\overset{\underline{1}}{(4)}$ is $\{\overset{\underline{\lambda_i}}{(i)}\overset{\underline{\lambda_j}}{(j)}\overset{\underline{\lambda_k}}{(k)}\overset{\underline{\lambda_l}}{(l)}\colon 2(\lambda_i+\lambda_j+\lambda_k+\lambda_l) \equiv \pm 2 \pmod 4$, $\lambda_1,\lambda_2,\lambda_3$ and λ_4 are odd integers$\}$.

For $I=\{r_1,r_2,r_3,r_4\}$ we have the conjugacy class of $\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{0}}{(3)}\overset{\underline{0}}{(4)}$ is $\{\overset{\underline{2\lambda_i}}{(i)}\overset{\underline{2\lambda_j}}{(j)}\overset{\underline{2\lambda_k}}{(k)}\overset{\underline{2\lambda_l}}{(l)}, \lambda \in \mathbb{Z}\}$. Finally, for $I=\{r_1,r_2,r_3,r_5\}$ we have the conjugacy class of $\overset{\underline{0}}{(1)}\overset{\underline{0}}{(2)}\overset{\underline{0}}{(3)}\overset{\underline{2}}{(4)}$ is $\{\overset{\underline{2\lambda_i}}{(i)}\overset{\underline{2\lambda_j}}{(j)}\overset{\underline{2\lambda_k}}{(k)}\overset{\underline{2\lambda_l}}{(l)}\colon 2\lambda_i+2\lambda_j+2\lambda_k+2\lambda_l \equiv \pm 2 \pmod 4\}$.

Hence, there are 16 conjugacy classes of involutions when G is a Coxeter group of type $\tilde{D}_4$. These results are summarised in the following tables, for $\lambda \in \mathbb{Z}$ and $1 \le i < j < k < l \le 4$.

$\tilde{B}_3$	Conjugacy classes
$\lvert I\rvert = 1$	$\overset{\overset{0}{-}}{(1)}^{G} = \{\overset{\overset{2\lambda}{-}}{(i)}\colon \lambda \in \mathbb{Z}\}$, $\overset{\overset{0}{+}}{(1\ 2)}^{G} = \{\overset{\overset{\lambda}{\pm}}{(i\ j)}\colon \lambda \in \mathbb{Z}\}$
$\lvert I\rvert = 2$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}^{G} = \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\colon \lambda_i, \lambda_j \in \mathbb{Z}\}$, $\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}^{G} = \{\ \overset{\overset{2\lambda_i+1}{-}}{(i)}\ \overset{\overset{2\lambda_j+1}{-}}{(j)} : \lambda_i, \lambda_j \in \mathbb{Z}\}$, $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{+}}{(2\ 3)}^{G} = \{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\pm\lambda}{\mp}}{(j\ k)}, \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j\ k)}\colon 1 \leq i < j \leq 3, \{\lambda, \lambda_i, \lambda_j\} \subset \mathbb{Z}\}$ $\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2\ 3)}^{G} = \{\overset{\overset{2\lambda}{-}}{(i)}\overset{\overset{\lambda-1}{+}}{(j\ k)}, \overset{\overset{-2\lambda}{-}}{(i)}\overset{\overset{\lambda+1}{-}}{(j\ k)}, \overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j-1}{+}}{(j\ k)}\colon 1 \leq i < j \leq 3, \{\lambda, \lambda_i, \lambda_j\} \subset \mathbb{Z}\}$
$\lvert I\rvert = 3$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}^{G} = \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{2\lambda_k}{-}}{(k)}\colon (\sum_{i=1}^{3} \lambda_i \equiv 0 \mod 4)\}$, $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{2}{-}}{(3)}^{G} = \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\overset{\overset{2\lambda_k}{-}}{(k)}\colon (\sum_{i=1}^{3} \lambda_i \equiv \pm 2 \mod 4)\}$, $\overset{\overset{0}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}^{G} = \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}\overset{\overset{\lambda_k}{-}}{(k)}\colon \text{exactly one of } \lambda_i \in 2\mathbb{Z})\}$,
Total	9 Conjugacy classes

Table 3.1: G is of type $\tilde{B}_3$

$\tilde{B}_4$	Conjugacy classes
$\|I\| = 1$	$(\overset{\overset{0}{-}}{1})^G = \{(\overset{\overset{2\lambda}{-}}{i})\}$, $(\overset{\overset{0}{+}}{1\,2})^G = \{(\overset{\overset{\lambda}{\pm}}{i\,j})\}$
$\|I\| = 2$	$\left((\overset{\overset{0}{-}}{i})(\overset{\overset{0}{-}}{j})\right)^G = \{(\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{2\lambda_j}{-}}{j})\}$, $\left((\overset{\overset{1}{-}}{i})(\overset{\overset{1}{-}}{j})\right)^G = \{ (\overset{\overset{2\lambda_i+1}{-}}{i})\ (\overset{\overset{2\lambda_j+1}{-}}{j}) \}$, $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{1}{-}}{3\,4})\right)^G = \{(\overset{\overset{2\lambda}{-}}{i})(\overset{\overset{\lambda-1}{+}}{j\,k}), (\overset{\overset{-2\lambda}{-}}{i})(\overset{\overset{\lambda+1}{-}}{j\,k}), (\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{\lambda_j-1}{+}}{j\,k})\}$, $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{+}}{3\,4})\right)^G = \{(\overset{\overset{2\lambda}{-}}{i})(\overset{\overset{\pm\lambda}{\pm}}{j\,k}), (\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{\lambda_j}{-}}{j\,k})\}$, $\left((\overset{\overset{0}{+}}{1\,2})(\overset{\overset{0}{+}}{3\,4})\right)^G = \{(\overset{\overset{\lambda_1}{\pm}}{i\,j})(\overset{\overset{\lambda_2}{\pm}}{k\,l})\}$, $\left((\overset{\overset{0}{+}}{1\,2})(\overset{\overset{1}{-}}{3\,4})\right)^G = \{(\overset{\overset{\lambda_1}{\pm}}{i\,j})(\overset{\overset{\lambda_2}{\pm}}{k\,l})\colon \{\lambda_1, \lambda_2\} \subset \{(2z, 2z+1), (2z+1, 2z), z \in \mathbb{Z}\}$
$\|I\| = 3$	$\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{-}}{3})\right)^G = \{(\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{2\lambda_j}{-}}{j})(\overset{\overset{2\lambda_k}{-}}{k})\colon \sum_{i=1}^{3} \lambda_i \in 4\mathbb{Z}\}$, $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{1}{-}}{2})(\overset{\overset{1}{-}}{3})\right)^G = \{(\overset{\overset{\lambda_i}{-}}{i})(\overset{\overset{\lambda_j}{-}}{j})(\overset{\overset{\lambda_k}{-}}{k})\colon \text{exactly one of } \lambda_i \in 2\mathbb{Z}\}$, $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{+}}{3\,4})\right)^G = \{(\overset{\overset{\lambda_i}{-}}{i})(\overset{\overset{\lambda_j}{-}}{j})(\overset{\overset{\lambda}{\pm}}{k\,l})\colon 1 \leq i < j \leq 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\}$ $\cup\{\{(\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{2\lambda_j}{-}}{j})(\overset{\overset{\lambda}{+}}{k\,l})\}$ $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{1}{-}}{3\,4})\right)^G = \{(\overset{\overset{\lambda_i}{-}}{i})(\overset{\overset{\lambda_j}{-}}{j})(\overset{\overset{\lambda\mp1}{\pm}}{k\,l})\colon 1 \leq i < j \leq 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\}$ $\cup\{(\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{2\lambda_j}{-}}{j})(\overset{\overset{\lambda}{-}}{k\,l})\}.$, $\left((\overset{\overset{1}{-}}{1})(\overset{\overset{1}{-}}{2})(\overset{\overset{0}{+}}{3\,4})\right)^G = \{ (\overset{\overset{2\lambda_i+1}{-}}{i})\ (\overset{\overset{2\lambda_j+1}{-}}{j})\ (\overset{\overset{\lambda}{\pm}}{k\,l})\colon \lambda_i, \lambda_j \in \mathbb{Z}\}$
$\|I\| = 4$	$\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{-}}{3})(\overset{\overset{0}{-}}{4})\right)^G = \{(\overset{\overset{2\lambda_i}{-}}{i})(\overset{\overset{2\lambda_j}{-}}{j})(\overset{\overset{2\lambda_k}{-}}{k})(\overset{\overset{2\lambda_l}{-}}{l})\colon \sum_{i=1}^{3} \lambda_i \equiv 0 \mod 4\}$, $\left((\overset{\overset{1}{-}}{1})(\overset{\overset{1}{-}}{2})(\overset{\overset{1}{-}}{3})(\overset{\overset{1}{-}}{4})\right)^G = \{(\overset{\overset{\lambda_1}{-}}{1})(\overset{\overset{\lambda_2}{-}}{2})(\overset{\overset{\lambda_3}{-}}{3})(\overset{\overset{\lambda_4}{-}}{4})\colon \text{all } \lambda_i \text{ odd}\}$ $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{0}{-}}{3})(\overset{\overset{2}{-}}{4})\right)^G = \{(\overset{\overset{\lambda_1}{-}}{1})(\overset{\overset{\lambda_2}{-}}{2})(\overset{\overset{\lambda_3}{-}}{3})(\overset{\overset{\lambda_4}{-}}{4})\colon \sum_{i=1}^{4} \lambda_i \equiv 2 \mod 4, \text{all } \lambda_i \text{ even}\}$ $\left((\overset{\overset{0}{-}}{1})(\overset{\overset{0}{-}}{2})(\overset{\overset{1}{-}}{3})(\overset{\overset{1}{-}}{1})\right)^G = \{(\overset{\overset{\lambda_1}{-}}{1})(\overset{\overset{\lambda_2}{-}}{2})(\overset{\overset{\lambda_3}{-}}{3})(\overset{\overset{\lambda_4}{-}}{4}) \text{ where exactly two } \lambda_i \text{ are even}\}$
Total	17 Conjugacy classes

Table 3.2: G is of type $\tilde{B}_4$

$\tilde{D}_4$	Conjugacy classes
$\lvert I\rvert = 1$	$(1\,\overset{\overset{0}{+}}{2})^G = (i\,\overset{\overset{\lambda}{\pm}}{j})$
$\lvert I\rvert = 2$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}^G = \{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}\}$, $\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}^G = \{\overset{\overset{2\lambda_i+1}{-}}{(i)}\overset{\overset{2\lambda_j+1}{-}}{(j)}\}$, $(1\,\overset{\overset{0}{-}}{2})(3\,\overset{\overset{0}{+}}{4})^G = \{(i\,\overset{\overset{\lambda_1}{\pm}}{j})(k\,\overset{\overset{\lambda_2}{\mp}}{l})\}$, $(1\,\overset{\overset{0}{+}}{2})(3\,\overset{\overset{0}{+}}{4})^G\ \{(i\,\overset{\overset{\lambda_1}{\pm}}{j})(k\,\overset{\overset{\lambda_2}{\pm}}{l})\}$, $(1\,\overset{\overset{0}{+}}{2})(3\,\overset{\overset{1}{-}}{4})^G = \{(i\,\overset{\overset{\lambda_1}{\pm}}{j})(k\,\overset{\overset{\lambda_2}{\mp}}{l})$ where $\{\lambda_1, \lambda_2\} = \{2z, 2z+1\}$ with $z \in \mathbb{Z}\}$, $(1\,\overset{\overset{0}{-}}{2})(3\,\overset{\overset{1}{-}}{4})^G = \{(i\,\overset{\overset{\lambda_1}{\pm}}{j})(k\,\overset{\overset{\lambda_2}{\pm}}{l})$ where $\{\lambda_1, \lambda_2\} = \{2z, 2z+1\}$ with $z \in \mathbb{Z}\}$,
$\lvert I\rvert = 3$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}(3\,\overset{\overset{0}{+}}{4})^G = \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}(k\,\overset{\overset{\lambda}{\pm}}{l})$: $1 \le i < j \le 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\}$ $\cup\{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}(k\,\overset{\overset{\lambda}{+}}{l})\}$ $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}(3\,\overset{\overset{1}{-}}{4})^G = \{\overset{\overset{\lambda_i}{-}}{(i)}\overset{\overset{\lambda_j}{-}}{(j)}(k\,\overset{\overset{\lambda\mp1}{\pm}}{l})$: $1 \le i < j \le 4, \{\lambda_i, \lambda_j\} \subset \{(0, 2\lambda), (2\lambda, 0) : \lambda \in \mathbb{Z}\}$ $\cup\{\{\overset{\overset{2\lambda_i}{-}}{(i)}\overset{\overset{2\lambda_j}{-}}{(j)}(k\,\overset{\overset{\lambda}{-}}{l})\}$, $(1\,\overset{\overset{0}{+}}{2})\overset{\overset{1}{-}}{(3)}\overset{\overset{l}{-}}{(4)}^G = \{(i\,\overset{\overset{\lambda}{-}}{j})\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}$: $\{\lambda_k, \lambda_l\} \subset \{1, 2\lambda - 1\}$ $\cup\{(i\,\overset{\overset{\lambda}{+}}{j})\overset{\overset{2\lambda_k+1}{-}}{(k)}\overset{\overset{2\lambda_l+1}{-}}{(l)} : \lambda, \lambda_k, \lambda_l \in \mathbb{Z}\}$, $(1\,\overset{\overset{0}{-}}{2})\overset{\overset{1}{-}}{(3)}\overset{\overset{l}{-}}{(4)}^G = \{(i\,\overset{\overset{\lambda}{+}}{j})\overset{\overset{\lambda_k}{-}}{(k)}\overset{\overset{\lambda_l}{-}}{(l)}$: $\{\lambda_k, \lambda_l\} \subset \{1, 2\lambda - 1\}$ $\cup\{(i\,\overset{\overset{\lambda}{-}}{j})\overset{\overset{2\lambda_k+1}{-}}{(k)}\overset{\overset{2\lambda_l+1}{-}}{(l)} : \lambda, \lambda_k, \lambda_l \in \mathbb{Z}\}$,
$\lvert I\rvert = 4$	$\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{0}{-}}{(4)}^G = \{\overset{\overset{2\lambda_1}{-}}{(1)}\overset{\overset{2\lambda_2}{-}}{(2)}\overset{\overset{2\lambda_3}{-}}{(3)}\overset{\overset{2\lambda_4}{-}}{(4)}$: $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{Z}\}$. $\overset{\overset{1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}^G = \{\overset{\overset{\lambda_1}{-}}{(1)}\overset{\overset{\lambda_2}{-}}{(2)}\overset{\overset{\lambda_3}{-}}{(3)}\overset{\overset{\lambda_4}{-}}{(4)}$: $\sum_{i=1}^4 \lambda_i \equiv 0 \mod 4$, all λ_i odd$\}$. $\overset{\overset{-1}{-}}{(1)}\overset{\overset{1}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}^G = \{\overset{\overset{\lambda_1}{-}}{(1)}\overset{\overset{\lambda_2}{-}}{(2)}\overset{\overset{\lambda_3}{-}}{(3)}\overset{\overset{\lambda_4}{-}}{(4)}$: $\sum_{i=1}^4 \lambda_i \equiv 2 \mod 4$, all λ_i odd $\}$. $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{1}{-}}{(3)}\overset{\overset{1}{-}}{(4)}^G = \{\overset{\overset{\lambda_1}{-}}{(1)}\overset{\overset{\lambda_2}{-}}{(2)}\overset{\overset{\lambda_3}{-}}{(3)}\overset{\overset{\lambda_4}{-}}{(4)}$: exactly two λ_i even$\}$. $\overset{\overset{0}{-}}{(1)}\overset{\overset{0}{-}}{(2)}\overset{\overset{0}{-}}{(3)}\overset{\overset{2}{-}}{(4)}^G = \{\overset{\overset{\lambda_1}{-}}{(1)}\overset{\overset{\lambda_2}{-}}{(2)}\overset{\overset{\lambda_3}{-}}{(3)}\overset{\overset{\lambda_4}{-}}{(4)}$: $\sum_{i=1}^4 \lambda_i \equiv 2 \mod 4$, all λ_i even$\}$.
Total	16 Conjugacy classes

Table 3.3: G is of type $\tilde{D}_4$

3.6 Summary Tables

In this final section we summarise our results for $W(\tilde{B}_n)$ and $W(\tilde{D}_n)$ when $n \leq 6$, as illustrative examples. When $n = 2m$, Diam $\mathcal{C}(G, X) \leq 4$, so we assume $n > 2m$. We therefore consider $m = 0$ for $n \in \{2, 3\}$, $m \in \{0, 1\}$ for $n \in \{4, 5\}$ and $m \in \{0, 1, 2\}$ for $n = 6$. Let G be of type $\tilde{B}_2$. The table below shows that when $\mathcal{C}(G, X)$ is connected, Diam $\mathcal{C}(G, X) \leq 2$.

Labelled Cycle Type		Connected	Diameter
$m = 0$	$l = 1$	yes	2
$m = 0$	$l = 0$	No	-

Table 3.4: G is of type $\tilde{B}_2$

Let G be of type $\tilde{B}_3$. If $m = 0$ and $l = 0$ or if $m = 1$ then $C(G, X)$ is disconnected. If $l > 0$, then we state the results in the table below. In each case Diam $\mathcal{C}(G, X) \leq 3$.

Labelled Cycle Type		Connected	Diameter
$m = 0$	$l = 2$	yes	2
$m = 0$	$l = 1$	yes	3

Table 3.5: G is of type $\tilde{B}_3$

Let G be of type $\tilde{B}_4$. Then either $\mathcal{C}(G, X)$ is disconnected or Diam $\mathcal{C}(G, X) \leq 5$.

Labelled Cycle Type		Connected	Diameter
$m = 1$	$l = 0$	No	-
$m = 0$	$l = 0$	No	-
	$l = 1$ and $k_e = 3$	Yes	4
	$l = 1$ and $k_o = 2$	Yes	5
	$l \geq 2$ and $k_e \leq 2$ or $k_o \leq 2$	Yes	2

Table 3.6: G is of type $\tilde{B}_4$

Let G be of type $\tilde{B}_5$. Then either $\mathcal{C}(G, X)$ is disconnected or Diam $\mathcal{C}(G, X) \leq 6$.

Labelled Cycle Type		Connected	Diameter
$m = 0$	$l = 0$	No	-
	$l = 1$ and $k_e = k_o = 2$	Yes	6
	$l = 1$ and $k_e = 4$ or $k_o = 4$	Yes	5
	$l = 2$ and $k_e = 3$	Yes	3
	$l = 2$, $k_o = 2$ and $k_e = 1$	Yes	4
	$3 \leq l \leq 4$	Yes	2
$m = 1$	$l = 0$ and $k_o = 0$	Yes	5
	$l = 1$	yes	5
	$l = 2$	Yes	4
$m > 1$		No	-

Table 3.7: G is of type $\tilde{B}_5$

Finally if G is of type $\tilde{B}_6$, then either $\mathcal{C}(G, X)$ is disconnected or Diam $\mathcal{C}(G, X) \leq 7$, with results summarised in the next table.

Labelled Cycle Type		Connected	Diameter
$m = 1$	$l = 0$ and $k_e = 2$	No	-
	$l = 0$ and $k_e = 4$ or $k_o = 4$	Yes	6
	$l = 1$ and $k_e = 1$	Yes	5
	$l = 1$ and $k_e = 3$	Yes	4
	$l = 2$	Yes	4
	$l = 3$	Yes	3
$m = 0$	$l = 0$	No	-
	$l = 1$ and $k_e, k_o > 0$	Yes	7
	$l = 1$ and $k_o = 0$	Yes	6
$m = 0$	$l = 2$ and $k_e, k_o > 0$	Yes	4
	$l = 2$ and k_e or $k_o = 0$	Yes	3
	$l = 3$ and $k_o, k_e > 0$	Yes	2
	$l = 3$ and $k_o = 0$	Yes	2
$m > 1$	$l = 0$ and k_e or $k_o = 0$	Yes	5
	$l = 1$	No	-

Table 3.8: G is of type $\tilde{B}_6$

Now, let G be of type $\tilde{D}_n$. If $n = 2m$ then Diam $C(G, X) \leq 4$. Assume that $n > 2m$. We present the results in the tables below. Let G be of type $\tilde{D}_4$. Then we have Diam $C(G, X) \leq 2$.

Labelled Cycle Type		Connected	Diameter
$m = 1$		No	-
$m = 0$	$l = 0$	No	-
	$l = 2$	Yes	2

Table 3.9: G is of type $\tilde{D}_4$

Let G be of type $\tilde{D}_5$. Then we have we have Diam $C(G,X) \leq 6$.

Labelled Cycle Type		Connected	Diameter
$m=1$	$k_e = 2$ or $k_o = 2$	Yes	5
$m=0$	$k_e = k_o = 2$	Yes	6
	k_e or $k_o = 4$	Yes	5

Table 3.10: G is of type $\tilde{D}_5$

Finally, let G be of type $\tilde{D}_6$. We state the results in the table below. In each connected case, Diam $C(G,X) \leq 6$.

Labelled Cycle Type		Connected	Diameter
$m=2$	$k_e = 2$ or $k_o = 2$	yes	5
$m=1$	$l=0$ and $k_e, k_o = 2$	No	-
	$l=0$ and k_e or $k_o = 0$	yes	6
	$l=2$	yes	5
$m=0$	$l=0$	No	-
	$l = k_e = k_o = 2$	Yes	4
	$l=2$ and k_e or $k_o = 4$	Yes	3
	$l=4$	Yes	2

Table 3.11: G is of type $\tilde{D}_6$

Chapter 4

Results for Exceptional Affine Weyl Groups

4.1 Introduction

In Chapters 2 and 3, we showed that if X is an involution conjugacy class in the affine Weyl group G of type $\tilde{C}_n$, $\tilde{B}_n$ or $\tilde{D}_n$, then $C(G, X)$ is either disconnected or has diameter at most $n + 2$.

In this chapter we will look at commuting involution graphs for two of the exceptional affine Weyl groups:(these of types $\tilde{F}_4$ and $\tilde{G}_2$). The general set-up will be that we have a finite Weyl group W with associated root system Φ, and a corresponding affine Weyl group $G = \tilde{W}$. As we have seen in earlier chapters, the group $\tilde{W}$ can be viewed as the semidirect product of W with the group of translations of the coroot lattice $Z = L(\Phi^\vee)$. For any element w of $\tilde{W}$, we can write $w = (g, \mathbf{v})$ where $g \in W$ and $\mathbf{v} \in Z$, and we seek to derive information about conjugacy classes and connectivity of the commuting involution graphs from information about these graphs in the underlying finite Weyl group W. In the current chapter, we will use this interpretation for the majority of our work, and exclusively for Sections 4.3–4.4. However, as an introductory example, we look first, in Section 4.2, at type $\tilde{G}_2$. Because affine Weyl groups are Coxeter groups in their own right, there is an alternative geometric interpretation using a more general definition of a (not necessarily finite) root system. This interpretation will be what we use in our work on commuting reflection graphs for arbitrary Coxeter groups in the next chapter, and so we present both interpretations in depth for $\tilde{G}_2$ here, as an illustrative example. We show in Section 4.2, by an analysis of the elements of each conjugacy class of involutions in groups of type $\tilde{G}_2$, that none of the commuting involution graphs are connected. In fact, the disconnectedness of these graphs will turn out to follow from more general results proved in Section 4.3, but we felt the concrete example verifying these results in one instance would still be useful to include.

4.1.1 Root Systems for Arbitrary Coxeter Groups

It is possible to define a nation of root systems for arbitrary Coxeter groups. These root systems are not usually finite.
Let (W, R) be a Coxeter system, so that

$$W = \langle R \mid (rs)^{m_{rs}} = 1 \text{ for } r, s \in R \rangle.$$

Let V be a real vector space with basis

$$\Pi = \{\alpha_r : r \in R\}.$$

We can define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V by setting

$$\langle \alpha_r, \alpha_s \rangle = -\cos \tfrac{\pi}{m_{rs}}$$

for all $r, s \in R$. When W is finite, this form is an inner product, but not otherwise because it fails to be positive definite.
Note that since $m_{rr} = 1$ for all $r \in R$, all basis vectors are unit vectors.
We can then define 'reflection' for a nonzero vector α, via

$$s_\alpha(\mathbf{v}) = \mathbf{v} - 2\tfrac{\langle \alpha, \mathbf{v} \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

as with Euclidean vectors. Note that if $||\alpha|| = 1$, we just get $s_\alpha(\mathbf{v}) = \mathbf{v} - 2\langle \alpha, \mathbf{v} \rangle \alpha$. Now we can define the group action on V by setting $r = s_{\alpha_r}$ for $r \in R$. We define the root system Φ to be

$$\Phi = \{w(\alpha_r) : w \in W, r \in R\}.$$

Note that Φ consists of unit vectors.
It can be shown that Π is a simple system for Φ, with positive and negative roots defined as for finite reflection groups.

4.2 Type $\tilde{G}_2$

This section concentrates on the commuting involution graph $\mathcal{C}(G, X)$ when G is the affine Weyl group of type $\tilde{G}_2$. Our main result is the following.

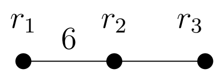
Theorem 4.2.1. *Let X be a conjugacy class of involutions in an affine Weyl group of type $\tilde{G}_2$. Then the commuting involution graph $\mathcal{C}(G, X)$ is disconnected.*

The proof of Theorem 4.2.1 is in Section 4.2.3. In order to state our results about disconnectedness, we need to describe the elements of conjugacy classes of involutions. This is done in Sections 4.2.1 and 4.2.2.

4.2.1 Reflection Groups of Type $\tilde{G}_2$

For a Coxeter group G of type $\tilde{G}_2$ the Coxeter graph is as follows.

$\tilde{G}_2$



We have $m_{11} = m_{22} = m_{33} = 1$, $m_{12} = m_{21} = 6$, $m_{23} = m_{32} = 3$ and $m_{13} = m_{31} = 2$.

There are two useful geometric representations of Coxeter groups G of type $\tilde{G}_2$, one non-Euclidean and one Euclidean. We make use of both.

The non Euclidean representation. Here we use the broader definition of root system given in Section 4.1.1. We have $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{m_{ij}}$ and $r_i = s_{\alpha_i}$ for $i \in \{1, 2, 3\}$.

So, for example

$$
\begin{aligned}
r_1(\alpha_2) &= \alpha_2 - 2\frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle}\alpha_1 \\
&= \alpha_2 - 2\langle \alpha_1, \alpha_2 \rangle \alpha_1 \\
&= \alpha_2 + (2\cos\frac{\pi}{6})\alpha_1 \\
&= \alpha_2 + \sqrt{3}\alpha_1.
\end{aligned}
$$

The Euclidean representation. This interpretation relies on the underlying finite Weyl group and its root system, which exists in Euclidean space. We consider the Euclidean vector space $\mathbb{R}^3$ with orthonormal basis $\{e_1, e_2, e_3\}$, and let V be the subspace of vectors the sum of whose coordinates, with respect to this basis, is zero. Reflection s_α in the hyperplane through the origin orthogonal to a non-zero vector α, and affine reflection in the hyperplane $H_{\alpha,k} = \{\mathbf{u} \in V : \langle \mathbf{u}, \alpha \rangle = k\}$, are defined as follows, for $\mathbf{v} \in V$:

$$
\begin{aligned}
s_\alpha(\mathbf{v}) &= \mathbf{v} - \frac{2\langle \mathbf{v}, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \\
s_{\alpha,k}(\mathbf{v}) &= \mathbf{v} - \frac{2(\langle \mathbf{v}, \alpha \rangle - k)}{\langle \alpha, \alpha \rangle}\alpha.
\end{aligned}
$$

We look first at the root system Φ whose associated finite reflection group W is of type G_2 (which is just a dihedral group of order 12).
We may take $\Phi = V \cap \{\mathbf{v} \in \mathbb{Z}^3 : ||\mathbf{v}|| = \sqrt{2} \text{ or } \sqrt{6}\}$. Here we are writing $\mathbb{Z}^3$ for the

standard lattice $\{ae_1 + be_2 + ce_3 : a, b, c \in \mathbb{Z}\}$. Hence

$$\Phi = \{\pm(e_i - e_j) : 1 \leq i < j \leq 3\} \cup \{\pm(2e_i - e_j - e_k) : \{i, j, k\} = \{1, 2, 3\}\}.$$

Set $\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3$ and $\alpha_3 = 2e_3 - e_1 - e_2$. Further, let $r_1 = s_{\alpha_1}, r_2 = s_{\alpha_2}$ and $r_3 = s_{\alpha_3,1}$. Then we may take $\{r_1, r_2\}$ as the set of fundamental reflections for W, and $R = \{r_1, r_2, r_3\}$ as the set of fundamental reflections for G. Then G acts as an affine reflection group on the real inner product space $\mathbb{R}^3$.

Recall that for a root α, the coroot $\alpha^\vee$ is given by $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$. We write $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$. Observe that $s_{\alpha,k}(\mathbf{v}) = \mathbf{v} - \frac{2(\langle \mathbf{v}, \alpha \rangle - k)}{\langle \alpha, \alpha \rangle}\alpha = s_\alpha(\mathbf{v}) + k\alpha^\vee$. In fact the affine Weyl group G is the semidirect product of W with the translation group Z of the coroot lattice $L = L(\Phi^\vee)$ of W. Every element of G can be written as an ordered pair $(g, \mathbf{u})$ where $g \in W$ and $\mathbf{u} \in L$. For $\mathbf{v} \in V$ we have $(g, \mathbf{u})(\mathbf{v}) = g(\mathbf{v}) + \mathbf{u}$. Hence, for example $s_{\alpha_3,1}$ can be written as $(s_{\alpha_3}, \frac{1}{3}(\alpha_3))$ or equivalently $(s_{\alpha_3}, \frac{1}{3}(-1, -1, 2))$ where the coordinates are with respect to standard basis. It can be shown that for elements w, w' of G, given by $(g, \mathbf{u})$ and $(h, \mathbf{v})$ respectively, multiplication and conjugation are given as follows (note we adopt the convention that $a^b = b^{-1}ab$):

$$ww' = (g, \mathbf{u})(h, \mathbf{v}) = (gh, \mathbf{u}^h + \mathbf{v}), \tag{4.1}$$

$$w^{w'} = (g^h, \mathbf{u}^h + \mathbf{v} - g^h(\mathbf{v})). \tag{4.2}$$

Finally we note that in W, which is dihedral of order 12, there are three conjugacy classes of involutions; two classes of reflections (r_1^W and r_2^W), each with three elements, and one class consisting of the unique central involution $(r_1 r_2)^3$, which we will write as $\tilde{w}$. This element $\tilde{w}$ acts as -1 on the root system Φ, and hence also as -1 on every element of the coroot lattice L, which will be useful later.

4.2.2 Involution Conjugacy Classes

For the rest of this section let G be a Coxeter group of type $\tilde{G}_2$, generated by $\{r_1, r_2, r_3\}$ with relations as given in the Coxeter graph for $\tilde{G}_2$. Let a be an involution in G. By Theorem 1.4.2, a is conjugate to w_I for some finite standard parabolic subgroup W_I of G in which w_I is central.

Then, the components of the graph corresponding to parabolic subgroups W_I for which w_I is central are of type A_1, $A_1 \times A_1$ or $I_2(6)$. We first consider conjugacy classes of reflections (elements conjugate to fundamental reflections). We do this by looking at the non-Euclidean interpretation, where G has a root system in its own right. Here, we take the root system Φ of G to be the set $\{\alpha_1, \alpha_2, \alpha_3\}^G$. For $\alpha \in V$ and $w \in G$, it is easy to show that $s_{w(\alpha)} = ws_\alpha w^{-1}$. Hence every reflection of G is given by s_β for some $\beta \in \Phi$ and from $s_\alpha(\mathbf{v}) = \mathbf{v} - 2\langle \alpha, \mathbf{v} \rangle \alpha$ we can see that $\langle \alpha_i, \alpha_i \rangle = 1$,

$\langle \alpha_1, \alpha_2 \rangle = -\frac{\sqrt{3}}{2}$ and $\langle \alpha_2, \alpha_3 \rangle = -\frac{1}{2}$, meaning that for example $r_1(\alpha_1) = -\alpha_1$ and $r_2(\alpha_1) = \alpha_1 - 2\langle \alpha_1, \alpha_2 \rangle \alpha_2 = \alpha_1 + \sqrt{3}\alpha_2$. Now

$$
\begin{aligned}
r_3(\alpha_1 + \sqrt{3}\alpha_2) &= \alpha_1 + \sqrt{3}\alpha_2 - 2\langle \alpha_1 + \sqrt{3}\alpha_2, \alpha_3 \rangle \alpha_3 = \alpha_1 + \sqrt{3}\alpha_2 + \sqrt{3}\alpha_3, \\
r_1(\alpha_1 + \sqrt{3}\alpha_2) &= \alpha_1 + \sqrt{3}\alpha_2 - 2\langle \alpha_1 + \sqrt{3}\alpha_2, \alpha_1 \rangle \alpha_1 = 2\alpha_1 + \sqrt{3}\alpha_2.
\end{aligned}
$$

We use these tools to draw Figure 4.1 and Figure 4.2, which show the two connected components of the root system.

$$
\begin{tikzcd}
& -\alpha_1 \arrow[d, "r_1"] & \\
& \alpha_1 \arrow[loop right, "r_3"] \arrow[d, "r_2"] & \\
& \alpha_1 + \sqrt{3}\alpha_2 \arrow[dl, "r_3"'] \arrow[dr, "r_1"] & \\
\alpha_1 + \sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[loop left, "r_2"] \arrow[dr, "r_1"] & & 2\alpha_1 + \sqrt{3}\alpha_2 \arrow[loop right, "r_2"] \arrow[dl, "r_3"'] \\
& 2\alpha_1 + \sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[d, "r_2"] & \\
& 2\alpha_1 + 2\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[loop right, "r_3"] \arrow[d, "r_1"] & \\
& 4\alpha_1 + 2\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[loop right, "r_3"] \arrow[d, "r_2"] & \\
& 4\alpha_1 + 3\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[dl, "r_1"'] \arrow[dr, "r_3"] & \\
5\alpha_1 + 3\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 \arrow[loop left, "r_2"] \arrow[dr, "r_3"'] & & 4\alpha_1 + 3\sqrt{3}\alpha_2 + 2\sqrt{3}\alpha_3 \arrow[loop right, "r_2"] \arrow[dl, "r_1"] \\
& 5\alpha_1 + 3\sqrt{3}\alpha_2 + 2\sqrt{3}\alpha_3 & \\
& \vdots &
\end{tikzcd}
$$

Figure 4.1: The connected component ${\alpha_1}^G$ of Φ (non-Euclidean).

Let $\mathbf{u} = 3\alpha_1 + 2\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3$. We can see that $\langle \mathbf{u}, \alpha_1 \rangle = \langle \mathbf{u}, \alpha_2 \rangle = \langle \mathbf{u}, \alpha_3 \rangle = 0$. Observe in Figure 4.1 that $4\alpha_1 + 2\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3 = \mathbf{u} + \alpha_1$. Thus subsequent roots are of the form $\mathbf{u} + \beta$ where β is a root already seen. In fact we can go slightly further to obtain

$$
r_1^G = \{ s_{\lambda\mathbf{u}\pm\alpha_1}, s_{\lambda\mathbf{u}\pm(\alpha_1+\sqrt{3}\alpha_2)}, s_{\lambda\mathbf{u}\pm(2\alpha_1+\sqrt{3}\alpha_2)} : \lambda \in \mathbb{Z} \}.
$$

Now, let $\mathbf{u}' = \sqrt{3}\alpha_1 + 2\alpha_2 + \alpha_3$. We can see that $\mathbf{u} = \sqrt{3}\mathbf{u}'$ and $\langle \mathbf{u}', \alpha_1 \rangle = \langle \mathbf{u}', \alpha_2 \rangle = \langle \mathbf{u}', \alpha_3 \rangle = 0$. By using Figure 4.2, we are able to determine the elements of the conjugacy class of r_2 and r_3:

$$r_2^G = r_3^G = \{s_{\lambda\mathbf{u}'\pm\alpha_2},\, s_{\lambda\mathbf{u}'\pm\alpha_3}, s_{\lambda\mathbf{u}'\pm(\alpha_2+\alpha_3)} \colon \lambda \in \mathbb{Z}\}.$$

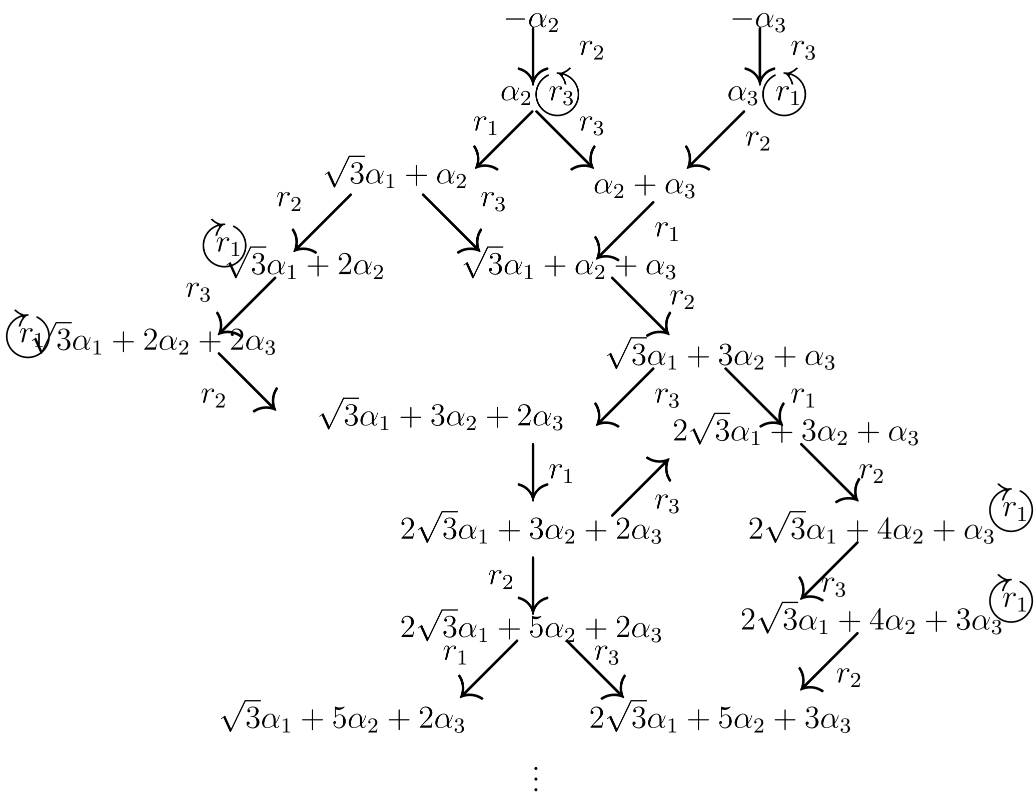


Figure 4.2: The connected component α_2^G of Φ (non-Euclidean).

For the remaining conjugacy classes $[(r_1r_2)^3]^G$ and $(r_1r_3)^G$, we use the Euclidean interpretation to define their elements. So from now on Φ will be the root system for W. There are six positive roots: $e_1 - e_2, e_2 - e_3, e_1 - e_3, -2e_1 + e_2 + e_3, -2e_2 + e_1 + e_3$ and $-2e_3 + e_1 + e_2$. The group G is the affine reflection group generated by $\{r_1, r_2, r_3\}$ where $r_1 = (s_{\alpha_1}, \mathbf{0})$, $r_2 = (s_{\alpha_2}, \mathbf{0})$, $r_3 = (s_{\alpha_3}, \alpha_3^\vee)$, and $G = \{(g, \mathbf{v}) : g \in W, \mathbf{v} \in L(\Phi^\vee)\}$. Note that G contains a subgroup $\langle r_1, r_2 \rangle$ of type G_2, which we identify with W. So we will write things like $r_1 = s_{\alpha_1}$, as long as the meaning is clear.

Lemma 4.2.2. *Let W be the subgroup of type G_2 in G given by $W = \langle r_1, r_2 \rangle$. Then there are two conjugacy classes of reflections in W, each with three elements: $r_1^W = \{s_{e_i - e_j} : 1 \leq i < j \leq 3\}$ and $r_2^W = \{s_{2e_i - e_j - e_k} : \{i, j, k\} = \{1, 2, 3\}\}$.*

Proof. Observe that

$$\begin{aligned}
s_{-2e_i+e_j+e_k}(-2e_i+e_j+e_k) &= -(-2e_i+e_j+e_k),\\
s_{-2e_i+e_j+e_k}(-2e_j+e_i+e_k) &= -(-2e_k+e_i+e_j),\\
s_{-2e_i+e_j+e_k}(e_i-e_j) &= e_k-e_i,\\
s_{-2e_i+e_j+e_k}(e_j-e_k) &= e_j-e_k.
\end{aligned}$$

For all $\mathbf{v} = ae_i + be_j + ce_k$ we have $s_{\pm(e_i-e_j)}(\mathbf{v}) = \pm(be_i + ae_j + ce_k)$ and $s_{\pm(e_i-e_k)}(\mathbf{v}) = \pm(ce_i + be_j + ae_k)$ where $\{i,j,k\} = \{1,2,3\}$. This implies that every element w of W maps short roots $e_i - e_j$ to short roots and long roots $-2e_i + e_j + e_k$ to long roots. Hence ${r_1}^W = \{s_{e_1-e_2}, s_{e_1-e_3}, s_{e_2-e_3}\}$ and ${r_2}^W = \{s_{(-2e_1+e_2+e_3)}, s_{(-2e_2+e_1+e_3)}, s_{(-2e_3+e_1+e_2)}\}$. □

Lemma 4.2.3. *The conjugacy class of* r_1 *in* G *is* $r_1^G = \{s_{e_i-e_j,\delta}\colon 1 \le i < j \le 3,\ \delta \in \mathbb{Z}\}$.

We give some example calculations before the proof (r_1 is underlined for emphasis).

$$\begin{aligned}
r_1\underline{r_1}r_1 &= r_1,\\
r_2\underline{r_1}r_2 &= s_{e_1-e_3},\\
r_3\underline{r_1}r_3 &= r_3r_3r_1 = r_1,\\
r_1r_2\underline{r_1}r_2r_1 &= s_{e_2-e_3},\\
r_3r_2\underline{r_1}r_2r_3 &= s_{e_2-e_3,-1}.
\end{aligned}$$

Proof. Let $w = (g, \mathbf{u}) \in G$ with $g = r_1'r_2'\cdots r_t'$ for some $t \in \mathbb{N}$ and $r_i' \in \{r_1, r_2, r_3\}$ for all $i \in \{1, 2.., t\}$. Then the conjugate to r_1 via w is

$$w^{-1}r_1w = (g^{-1}s_{\alpha_1}g, (0,0,0)^g + \mathbf{u} - \mathbf{u}^{g^{-1}s_{\alpha_1}g}) = (g^{-1}s_{\alpha_1}g, \mathbf{u} - \mathbf{u}^{g^{-1}s_{\alpha_1}g}).$$

Since $\mathbf{u}$ is an element of the coroot lattice of Φ, we can write $\mathbf{u} = \lambda\alpha_1^\vee + \mu\alpha_2^\vee$ for some integers λ and μ.
That is, $\mathbf{u} = \lambda(e_1 - e_2) + \frac{1}{3}\mu(-2e_1 + e_2 + e_3) = \frac{1}{3}((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3)$. By the calculations above and Lemma 4.2.2 we have proved that ${s_{\alpha_1}}^W \in \{s_{e_i-e_j} : 1 \le i < j \le 3\}$. If $g^{-1}s_{\alpha_1}g = s_{e_1-e_2}$, then

$$\begin{aligned}
\mathbf{u} - \mathbf{u}^{g^{-1}s_{\alpha_1}g} &= \mathbf{u} - \mathbf{u}^{s_{e_1-e_2}}\\
&= \tfrac{1}{3}\left((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3\right) - \tfrac{1}{3}\left((\mu - 3\lambda)e_1 + (3\lambda - 2\mu)e_2 + \mu e_3\right)\\
&= \tfrac{1}{3}((6\lambda - 3\mu)e_1 + (3\mu - 6\lambda)e_3\\
&= (2\lambda - \mu)(e_1 - e_2).
\end{aligned}$$

Thus $r_1^w = (s_{e_1-e_2}, (2\lambda-\mu)(e_1-e_2))$ in this case. Setting $\delta = 2\lambda - \mu$ we get $r_1^w = s_{e_1-e_2,\delta}$, and δ is clearly an integer. Also note that since $\mathbf{u}$ is arbitrary we can obtain any integer

δ. If $g^{-1}s_{\alpha_1}g = s_{\pm e_1-e_3}$, then

$$\begin{aligned}
\mathbf{u}-\mathbf{u}^{g^{-1}s_{\alpha_1}g} &= \mathbf{u}-\mathbf{u}^{s_{e_1-e_3}} \\
&= \tfrac{1}{3}\left((3\lambda-2\mu)e_1+(\mu-3\lambda)e_2+\mu e_3\right)-\tfrac{1}{3}\left(\mu e_1+(\mu-3\lambda)e_2+(3\lambda-2\mu)e_3\right) \\
&= (\lambda-\mu)(e_1-e_3).
\end{aligned}$$

Setting $\delta=\lambda-\mu$ we get $r_1^w = s_{e_1-e_3,\delta}$ here, and again δ is an integer. Finally if $g^{-1}s_{\alpha_1}g = s_{\pm e_2-e_3}$, then

$$\begin{aligned}
\mathbf{u}-\mathbf{u}^{g^{-1}s_{\alpha_1}g} &= \mathbf{u}-\mathbf{u}^{s_{e_2-e_3}} \\
&= \tfrac{1}{3}\left((3\lambda-2\mu)e_1+(\mu-3\lambda)e_2+\mu e_3\right)-\tfrac{1}{3}\left((3\lambda-2\mu)e_1+\mu e_2+(\mu-3\lambda)e_3\right) \\
&= -\lambda(e_2-e_3).
\end{aligned}$$

Setting $\delta=-\lambda$ gives $r_1^w = s_{e_2-e_3,\delta}$.
Therefore in all the cases ${r_1}^G \in \{s_{\pm e_i-e_j,\delta}\colon 1\le i<j\le 3, \delta\in\mathbb{Z}\}$, which completes the proof. □

We next consider the conjugacy class of r_2. Here are some examples of the action of conjugates of r_2 on a vector $\mathbf{v}=ae_1+be_2+ce_3$:

$$r_2\underline{r_2}r_2 = \underline{r_2},$$

$$r_1\underline{r_2}r_1(\mathbf{v}) = \frac{1}{3}[(2b+2a-c)e_1+(2a-b+2c)e_2+(-a+2b+2c)e_3].$$

Then,

$$r_1\underline{r_2}r_1 = \underline{s_{-2e_2+e_1+e_3}}$$

$$r_3\underline{r_2}r_3(\mathbf{v}) = \frac{1}{3}[(2b+2a-c+1)e_1+(2a-b+2c-2)e_2+(-a+2b+2c+1)e_3].$$

Then,

$$r_3\underline{r_2}r_3 = \underline{s_{-2e_2+e_1+e_3,1}}$$

$$r_3r_1\underline{r_2}r_1r_3(\mathbf{v}) = \frac{1}{3}[(-a+2b+2c-2)e_1+2a+2b-c+1)e_2+(2a-b+2c+1)e_3].$$

Then,

$$r_3r_1\underline{r_2}r_1r_3 = \underline{s_{-2e_1+e_2+e_3,1}}$$

$$r_1\underline{r_3}r_1 = \underline{r_3}$$

$$r_3\underline{r_3}r_3 = \underline{r_3}$$

$$r_2\underline{r_3}r_2(\mathbf{v}) = \frac{1}{3}[(2a+2b-c+1)e_1+(2a-b+2c-2)e_2+(-a+2b+2c+1)e_3].$$

Then,

$$r_2\underline{r_3}r_2 = \underline{s_{2e_2-e_1-e_3,-1}}.$$

Lemma 4.2.4. *We have* $r_2^G = \{s_{2e_k-e_i-e_j,\gamma} : \{i,j,k\} = \{1,2,3\}, \gamma \in \mathbb{Z}\}$.

Proof. Let $w = (g, \mathbf{u}) \in G$. By Lemma 4.2.2 we have

$$r_2{}^W = \{s_{-2e_1+e_2+e_3}, s_{e_1-2e_2+e_3}, s_{e_1+e_2-2e_3}\},$$

and in the semidirect product notation $r_2 = (s_{-2e_1+e_2+e_3}, 0)$. Then, similarly to Lemma 4.2.3, we use

$$w^{-1}r_2w = (g^{-1}s_{\alpha_2}g, (0,0,0)^g + \mathbf{u} - \mathbf{u}^{g^{-1}s_{\alpha_2}g}) = (g^{-1}s_{\alpha_2}g, \mathbf{u} - \mathbf{u}^{g^{-1}s_{\alpha_2}g})$$

and $\mathbf{u} = \lambda(e_1 - e_2) + \frac{1}{3}\mu(-2e_1 + e_2 + e_3) = \frac{1}{3}((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3)$ for some integers λ, μ. Now $g^{-1}s_{\alpha_2}g = s_\alpha$ for some α in α_2^W, meaning that in particular $\langle \alpha, \alpha \rangle = 6$ and $\alpha^\vee = \frac{1}{3}\alpha$. Note that $\mathbf{u} - \mathbf{u}^{s_\alpha} = \mathbf{u} - (\mathbf{u} - 2\frac{\langle \mathbf{u}, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha) = \frac{1}{3}\langle \mathbf{u}, \alpha \rangle \alpha = \langle \mathbf{u}, \alpha \rangle \alpha^\vee$. We check the three possibilities for α in turn. First suppose $\alpha = -2e_1 + e_2 + e_3$:

$$\begin{aligned}
\langle \mathbf{u}, -2e_1 + e_2 + e_3 \rangle &= \langle \tfrac{1}{3}((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3), (-2e_1 + e_2 + e_3) \rangle \\
&= \tfrac{1}{3}(-6\lambda + 4\mu + \mu - 3\lambda + \mu) = 2\mu - 3\lambda, \\
\mathbf{u} - \mathbf{u}^{s_{-2e_1+e_2+e_3}} &= \langle \mathbf{u}, \alpha \rangle \alpha^\vee = (2\mu - 3\lambda)\alpha^\vee.
\end{aligned}$$

So in this case $w^{-1}r_2w = (s_\alpha, (2\mu - 3\lambda)\alpha^\vee)$, or equivalently, $s_{\alpha,(2\mu-3\lambda)}$. Next let $\alpha = e_1 - 2e_2 + e_3$. Then

$$\begin{aligned}
\langle \mathbf{u}, e_1 - 2e_2 + e_3 \rangle &= \langle \tfrac{1}{3}((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3), (e_1 - 2e_2 + e_3) \rangle \\
&= \tfrac{1}{3}(3\lambda - 2\mu - 2\mu + 6\lambda + \mu) = 3\lambda - \mu, \\
\mathbf{u} - \mathbf{u}^{s_{e_1-2e_2+e_3}} &= \langle \mathbf{u}, \alpha \rangle \alpha^\vee = (3\lambda - \mu)\alpha^\vee,
\end{aligned}$$

and $w^{-1}r_2w = s_{\alpha,3\lambda-\mu}$. Finally suppose $\alpha = e_1 + e_2 - 2e_3$. Then

$$\begin{aligned}
\langle \mathbf{u}, e_1 + e_2 - 2e_3 \rangle &= \langle \tfrac{1}{3}((3\lambda - 2\mu)e_1 + (\mu - 3\lambda)e_2 + \mu e_3), (e_1 + e_2 - 2e_3) \rangle \\
&= \tfrac{1}{3}(3\lambda - 2\mu + \mu - 3\lambda - 2\mu) = -\mu, \\
\mathbf{u} - \mathbf{u}^{s_{e_1+e_2-2e_3}} &= \langle \mathbf{u}, \alpha \rangle \alpha^\vee = (-\mu)\alpha^\vee,
\end{aligned}$$

and $w^{-1}r_2w = s_{\alpha,-\mu}$. Hence in all cases there is an integer γ and an $\alpha \in \alpha_2^W$ such that $r_2^w = s_{\alpha,\gamma}$, and the result follows: □

Observe that the conjugacy classes of r_1 and r_2 exhaust all the reflections. It remains to describe the elements of $(r_1r_3)^G$ and $((r_1r_2)^3)^G$ which are presented in the following lemma. Writing $\tilde{w}$ for the unique central involution $(s_{\alpha_1}s_{\alpha_2})^3$ of W, we note that $r_1r_3 = (s_{\alpha_1}s_{\alpha_3}, \alpha_3^\vee) = (\tilde{w}, \alpha_3^\vee)$ and $(r_1r_2)^3 = (\tilde{w}, \mathbf{0})$.

Experimenting with conjugation of $(r_1r_2)^3$ and acting on a vector $\mathbf{v} = ae_1 + be_2 + ce_3$ we get examples like the following:

$$
\begin{aligned}
r_2\underline{(r_1r_2)^3}r_2 &= (r_2r_1)^3 = \underline{(r_1r_2)^3} = r_1\underline{(r_1r_2)^3}r_1, \\
r_3\underline{(r_1r_2)^3}r_3(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c-2)e_1+(2a-b+2c-2)e_2+(2a+2b-c+4)e_3]. \\
&\text{Then,} \\
r_3\underline{(r_1r_2)^3}r_3 &= \underline{s_{e_1-e_2}s_{2e_3-e_1-e_2,2}}, \\
r_3r_2(r_1r_2)^3r_2r_3(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c-2)e_1+(2a-b+2c-2)e_2+(2a+2b-c+4)e_3]. \\
&\text{Then,} \\
r_3r_2r\underline{(r_1r_2)^3}r_2r_3 &= \underline{s_{e_1-e_2}s_{2e_3-e_1-e_2,2}}, \\
r_1r_3\underline{(r_1r_2)^3}r_3r_1(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c-2)e_1+(2a-b+2c-2)e_2+(2a+2b-c+4)e_3]. \\
&\text{Then,} \\
r_1r_3\underline{(r_1r_2)^3}r_3r_1 &= \underline{s_{e_1-e_2}s_{2e_3-e_1-e_2,2}}, \\
r_3r_2r_1\underline{(r_1r_2)^3}r_1r_2r_3(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c-4)e_1+(2a-b+2c+2)e_2+(2a+2b-c+2)e_3]. \\
&\text{Then,} \\
r_3r_2r_1\underline{(r_1r_2)^3}r_1r_2r_3 &= \underline{s_{e_2-e_3}s_{2e_1-e_2-e_3,-2}}.
\end{aligned}
$$

In each case the resulting vector differs from $\mathbf{v}$ by a vector with coefficients of the form $\frac{2m}{3}$ where $m \in \mathbb{Z}$. On the other hand, for conjugates of r_1r_3 the difference is not an even vector:

$$
\begin{aligned}
r_1\underline{r_1r_3}r_1 = r_3r_1 = r_3\underline{r_1r_3}r_3 &= \underline{r_1r_3}, \\
r_2\underline{r_1r_3}r_2(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c+1)e_1+(2a-b+2c-2)e_2+(2a+2b-c+1)e_3], \\
r_2\underline{r_1r_3}r_2 &= \underline{s_{e_1-e_3}s_{-2e_2+e_1+e_3,1}}, \\
r_1r_2\underline{r_1r_3}r_2r_1(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c)e_1+(2a-b+2c-3)e_2+(2a+2b-c+3)e_3], \\
r_1r_2\underline{r_1r_3}r_2r_1 &= \underline{s_{e_2-e_3,-1}s_{-2e_1+e_2+e_3}}, \\
(r_1r_2)^3\underline{r_1r_3}(r_2r_1)^3(\mathbf{v}) &= \frac{1}{3}[(-a+2b+2c+1)e_1+(2a-b+2c+1)e_2+(2a+2b-c-2)e_3], \\
(r_1r_2)^3\underline{r_1r_3}(r_2r_1)^3 &= \underline{s_{e_1-e_2}s_{2e_3-e_1-e_2,-1}}.
\end{aligned}
$$

The underlying pattern is explained in the next lemma.

Lemma 4.2.5. *We have* $((r_1r_2)^3)^G = \{(\tilde{w}, 2\mathbf{u}) : \mathbf{u} \in L(\Phi^\vee)\}$ *and* $(r_1r_3)^G = \{(\tilde{w}, 2\mathbf{u}+\alpha^\vee) : \mathbf{u} \in L(\Phi^\vee), \alpha \in \alpha_3^W\}$.

Proof. Let $w = (g, \mathbf{u}) \in G$ and $x = (\tilde{w}, \mathbf{v})$ where $\mathbf{u}, \mathbf{v} \in L(\Phi^\vee)$. Recall that $\tilde{w}$ acts as -1 on $L(\Phi^\vee)$. Then by equation (4.2) we have $w^{-1}xw = (g^{-1}\tilde{w}g, \mathbf{v}^g + \mathbf{u} - \mathbf{u}^{g^{-1}\tilde{w}g}) =$

$(\tilde{w}, \mathbf{v}^g + \mathbf{u} - \mathbf{u}^{\tilde{w}}) = (\tilde{w}, \mathbf{v}^g + 2\mathbf{u})$. The result now follows immediately from the fact that $(r_1 r_2)^3 = (\tilde{w}, \mathbf{0})$ and $r_1 r_3 = (\tilde{w}, \alpha_3^\vee)$. $\square$

4.2.3 Commuting Involution Graphs for $\tilde{G}_2$

Now, we are able to prove our main result.

Proposition 4.2.6. *Let X be a conjugacy class of involutions in the affine Coxeter group G of type $\tilde{G}_2$. Then $C(G, X)$ is disconnected.*

Proof. Let $X = r_1^G$ and consider the non-Euclidean root system. In Section 4.2.2 we proved that for all $w \in X$ we have $w = s_{\lambda\mathbf{u}\pm\sigma}$ where $\sigma \in \{\alpha_1, \alpha_1 + \sqrt{3}\alpha_2, 2\alpha_1 + \sqrt{3}\alpha_2\}$ with $\mathbf{u} = 3\alpha_1 + 2\sqrt{3}\alpha_2 + \sqrt{3}\alpha_3$ and $\langle \mathbf{u}, \alpha_i \rangle = 0$ for all $i \in \{1, 2, 3\}$. Let $\sigma, \sigma' \in \{\alpha_1, \alpha_1 + \sqrt{3}\alpha_2, 2\alpha_1 + \sqrt{3}\alpha_2\}$. Then $\langle \sigma, \sigma' \rangle \in \{\pm 1, \pm\frac{1}{2}\}$ and

$$\langle \lambda\mathbf{u} + \sigma, \lambda'\mathbf{u} + \sigma' \rangle = \lambda\lambda' \langle \mathbf{u}, \mathbf{u} \rangle + (\pm 1 \text{ or } \pm\tfrac{1}{2}) = (\pm 1 \text{ or } \pm\tfrac{1}{2}) \neq 0$$

because $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. Hence, we have $\langle \lambda\mathbf{u} \pm \sigma, \lambda'\mathbf{u} \pm \sigma' \rangle \neq 0$. Therefore, by Lemma 1.4.3, $w = s_{\lambda\mathbf{u}\pm\sigma}$ does not commute with any other $w = s_{\lambda'\mathbf{u}\pm\sigma'}$ in G. Hence, $C(G, X)$ is disconnected.

Now, let $X = \{s_{\lambda'\mathbf{u}'\pm\alpha_2}, s_{\lambda'\mathbf{u}'\pm\alpha_3}, s_{\lambda'\mathbf{u}'\pm(\alpha_2+\alpha_3)}\} = r_2^{\tilde{G}_2}$ with $\mathbf{u}' = \sqrt{3}\alpha_1 + 2\alpha_2 + \alpha_3$ and λ an integer. Then, we have

$$\begin{aligned}
\langle \mathbf{u}', \mathbf{u}' \rangle &= 0, \\
\langle \pm\alpha_2, \pm\alpha_2 \rangle &= \pm 1, \\
\langle \pm\alpha_3, \pm\alpha_3 \rangle &= \pm 1, \\
\langle \pm\alpha_2, \pm\alpha_3 \rangle &= \pm\tfrac{1}{2}, \\
\langle \pm\alpha_2, \pm(\alpha_2 + \alpha_3) \rangle &\in \{\tfrac{1}{2}, \tfrac{3}{2}\}, \\
\langle \pm\alpha_3, \pm(\alpha_2 + \alpha_3) \rangle &\in \{\tfrac{1}{2}, \tfrac{3}{2}\}.
\end{aligned}$$

Therefore, for all $s_\alpha, s_\beta \in r_2^G$ we can see that if $\alpha \neq \pm\beta$, then $\langle \pm\alpha, \pm\beta \rangle \neq 0$, so by Lemma 1.4.3, $C(G, X)$ is a disconnected graph. (In fact we stated Lemma 1.4.3 only for finite groups but the proof for arbitrary root systems is identical.)

Now, moving to the Euclidean interpretation, let $(\tilde{w}, \mathbf{u}) \in (r_1 r_2)^{3^G}$ or $(\tilde{w}, \mathbf{u}) \in (r_1 r_3)^G$ where $\tilde{w}$, given by $(s_{\alpha_1} s_{\alpha_2})^3$, is the central longest element of the underlying finite group W. We have noted that $\tilde{w}(\mathbf{v}) = -\mathbf{v}$ for any $\mathbf{v} \in L(\Phi^\vee)$. So if we have two elements, $(\tilde{w}, \mathbf{u})$ and $(\tilde{w}, \mathbf{v})$, we see that $(\tilde{w}, \mathbf{u})(\tilde{w}, \mathbf{v}) = (1, \mathbf{u}^{\tilde{w}} + \mathbf{v}) = (1, \mathbf{v} - \mathbf{u})$ and $(\tilde{w}, \mathbf{v})(\tilde{w}, \mathbf{u}) = (1, \mathbf{v}^{\tilde{w}} + \mathbf{u}) = (1, \mathbf{u} - \mathbf{v})$. These are equal if and only if $\mathbf{u} = \mathbf{v}$. Therefore no element of these two classes commutes with any other. Hence, the two classes are completely disconnected. $\square$

4.3 General Results for Affine Weyl Groups

For the rest of this chapter, we will let $\tilde{W}$ be an affine Weyl group whose underlying finite Weyl group is W, with root system Φ. The reflections of W are s_α, $\alpha \in \Phi^+$, and the reflections of $\tilde{W}$ are $s_{\alpha,k}$, for $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. We will be thinking of $\tilde{W}$ as a semidirect product of W with the translation group Z of the coroot lattice $L(\Phi^\vee)$. This means that strictly speaking W is not a subgroup of $\tilde{W}$, but is of course isomorphic to the subgroup $\{(g, \mathbf{0}) : g \in W\}$. If $\{\alpha_1, \ldots, \alpha_n\}$ are the simple roots of Φ, then $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ is the set of fundamental reflections of W, and (writing $\tilde{\alpha}$ for the highest root of Φ) we have that $\{s_{\alpha_1}, \ldots, s_{\alpha_n}, s_{\tilde{\alpha},1}\}$ is the set of fundamental roots of $\tilde{W}$. Using the semidirect product description, we set $r_i = (s_{\alpha_i}, \mathbf{0})$, for $i \in \{1, \ldots, n\}$, and $r_{n+1} = (s_{\tilde{\alpha}}, \tilde{\alpha}^\vee)$. Then we can take $R = \{r_1, \ldots, r_{n+1}\}$ as the set of fundamental reflections of $\tilde{W}$.

Our aim is to use knowledge about the connectedness or otherwise of commuting involution graphs in the underlying finite Weyl groups to determine connectedness in the corresponding affine case.

There is one case we can deal with that covers several groups, and which generalises what we saw in type $\tilde{G}_2$.

Lemma 4.3.1. *Suppose W is a finite Weyl group that has a central involution $\tilde{w}$. Now suppose a is an involution in the corresponding affine Weyl group $\tilde{W}$ such that $a = (\tilde{w}, \mathbf{v})$ for some $\mathbf{v} \in Z$, and write $X = a^{\tilde{W}}$, the conjugacy class in $\tilde{W}$ of a. Then $\mathcal{C}(\tilde{W}, X)$ is disconnected.*

Proof. Since $\tilde{w}$ is central in W, every conjugate of a in $\tilde{W}$ has the form $b = (\tilde{w}, \mathbf{u})$ for some $\mathbf{u}$ in L. Moreover $\tilde{w}$ acts as -1 on the root system, and hence on L. Now

$$ab = (\tilde{w}, \mathbf{v})(\tilde{w}, \mathbf{u}) = (\tilde{w}^2, \mathbf{v}^{\tilde{w}} + \mathbf{u}) = (1, \mathbf{u} - \mathbf{v}).$$

Thus ab is translation through $\mathbf{u} - \mathbf{v}$. Now a commutes with b precisely when $(ab)^2 = 1$, and hence a commutes with b if and only if $\mathbf{u} = \mathbf{v}$, meaning a does not commute with any other member of its conjugacy class. Therefore $\mathcal{C}(\tilde{W}, X)$ is completely disconnected. $\square$

We recall a lemma we already observed for $\tilde{C}_n$.

Lemma 4.3.2. *Let $a = (g, \mathbf{v})$ be an element of $\tilde{W}$. If $\mathcal{C}(W, g^W)$ is disconnected then $\mathcal{C}(\tilde{W}, a^{\tilde{W}})$ is disconnected.*

Next we have a result which proves connectedness under certain circumstances. Note that the existence of the w_I in the statement of the proposition is guaranteed by applying Richardson's Theorem (Theorem 1.4.2) to W, rather than to $\tilde{W}$.

We use the following notation. We write $g \leftrightarrow h$ to mean $gh = hg$. We write $g \sim h$ to mean that g and h lie in the same connected component of $\mathcal{C}(W, X)$.

Proposition 4.3.3. *Let X be a conjugacy class of involutions in $\tilde{W}$ that contains $(w_I, \mathbf{0})$ where w_I is the central longest element of some standard parabolic subgroup of W. Suppose $(w_I, \mathbf{v}) \sim (w_I, \mathbf{0})$ for all $\mathbf{v} \in Z$ such that $(w_I, \mathbf{v}) \in X$, and that $\mathcal{C}(W, w_I^W)$ is connected. Then $\mathcal{C}(\tilde{W}, X)$ is connected.*

Proof. Let $w = (a, \mathbf{u}) \in X$. Then a is conjugate to w_I in W. That is, there is some $g \in W$ with $a = w_I^g$. Write $w_0 = (y_0, \mathbf{v}_0) = w^{g^{-1}}$. Then $y_0 = w_I$. By hypothesis then, $w_0 \sim (w_I, \mathbf{0})$. That is, there is a sequence $w_0 \leftrightarrow w_1 \cdots \leftrightarrow w_m = (w_I, \mathbf{0})$ of elements of X giving a path in $\mathcal{C}(\tilde{W}, X)$ from w_0 to $(w_I, \mathbf{0})$. But this implies that there is a sequence $w_0^g \leftrightarrow w_1^g \leftrightarrow \cdots \leftrightarrow w_m^g = (w_I, \mathbf{0})^g$ in the commuting involution graph. That is, $w = w_0^g \sim (w_I, \mathbf{0})^g = (a, \mathbf{0})$. Hence $w \sim (a, \mathbf{0})$. Now $\mathcal{C}(W, w_I^W)$ is connected. Thus $a \sim w_I$ in $\mathcal{C}(W, w_I^W)$. Hence $(a, \mathbf{0}) \sim (w_I, \mathbf{0})$ in $\mathcal{C}(\tilde{W}, X)$. Therefore $w \sim (w_I, \mathbf{0})$ and since this holds for all w in X, we deduce that $\mathcal{C}(\tilde{W}, X)$ is connected. $\square$

Lemma 4.3.4. *Suppose $a \in W_I$ for some $I \subseteq R$. If $(a, \mathbf{v})$ is an involution, with $\mathbf{v} = \sum_{r \in R} v_r \alpha_r$, then a is an involution and $v_r = 0$ whenever $r \notin I$. Moreover, let J be the set of reflections s of R that commute with all r in I. If $\mathbf{u} = \sum_{s \in J} u_s \alpha_s$, then $\mathbf{u}^b = \mathbf{u}$ for all $b \in W_I$.*

Proof. We have $(a, \mathbf{v})(a, \mathbf{v}) = (a^2, \mathbf{v}^a + \mathbf{v})$. If $(a, \mathbf{v})$ is an involution, then a is an involution and $\mathbf{v}^a + \mathbf{v} = \mathbf{0}$. Now a is a product of elements of I. We have $\mathbf{u}^r = \mathbf{u} - \langle \mathbf{u}, \alpha_r \rangle \alpha_r^\vee$ for all $\mathbf{u} \in Z$ and $r \in R$. Hence, inductively, $\mathbf{v}^a = \mathbf{v} - \mathbf{x}$ for some $\mathbf{x} \in \langle \alpha_r : r \in I \rangle$. So $\mathbf{v}^a + \mathbf{v} = 2\mathbf{v} - \mathbf{x}$. For this to equal zero, clearly $\mathbf{v} \in \langle \alpha_r : r \in I \rangle$. That is, $v_r = 0$ whenever $r \notin I$. For the second part, observe that for any $r \in I$ and $s \in J$ we have $\langle \alpha_r, \alpha_s \rangle = 0$. Therefore $\mathbf{u}^r = \mathbf{u}$ for all $r \in I$. Hence $\mathbf{u}^b = \mathbf{u}$ for all $b \in W_I$. $\square$

Lemma 4.3.5. *Let $(a, \mathbf{u})$ be an involution in $\tilde{W}$, with $X = (a, \mathbf{u})^{\tilde{W}}$. Suppose b commutes with a in W, where $b \in a^W$. Then there is some $(b, \mathbf{v}) \in X$ such that $(a, \mathbf{u}) \leftrightarrow (b, \mathbf{v})$.*

Proof. Since b is conjugate in W to a, there is $g \in W$ with $b = g^{-1} a g$. Let $\mathbf{w} = \frac{1}{2}(\mathbf{u} - \mathbf{u}^g)$. From equation 1.3 we have that

$$\begin{aligned} (a, \mathbf{u})^{(g, \mathbf{w})} &= (g^{-1} a g, \mathbf{u}^g + \mathbf{w} - \mathbf{w}^{g^{-1} a g}) \\ &= (b, \mathbf{u}^g + \mathbf{w} - \mathbf{w}^b). \end{aligned}$$

Set $\mathbf{v} = \mathbf{u}^g + \mathbf{w} - \mathbf{w}^b$. We claim that $(a, \mathbf{u}) \leftrightarrow (b, \mathbf{v})$. Certainly $(b, \mathbf{v}) \in X$.

We have

$$\begin{aligned}\mathbf{v} &= \mathbf{u}^g + \mathbf{w} - \mathbf{w}^b \\ &= \mathbf{u}^g + \frac{1}{2}(\mathbf{u} - \mathbf{u}^g) - \frac{1}{2}(\mathbf{u}^b - \mathbf{u}^{g^b}).\end{aligned}$$

Note that since $(a, \mathbf{u})$ is an involution, $\mathbf{u}^a = -\mathbf{u}$; also since $g^{-1}ag = b$ we have $gb = ag$ and $\mathbf{u}^{g^b} = \mathbf{u}^{a^g} = -\mathbf{u}^g$.
Thus

$$\begin{aligned}\mathbf{v} &= \mathbf{u}^g + \frac{1}{2}\mathbf{u} - \frac{1}{2}\mathbf{u}^g - \frac{1}{2}\mathbf{u}^b - \frac{1}{2}\mathbf{u}^g \\ &= \frac{1}{2}(\mathbf{u} - \mathbf{u}^b).\end{aligned}$$

Now

$$\begin{aligned}(a, \mathbf{u})(b, \mathbf{v}) &= (ab, \mathbf{u}^b + \mathbf{v}) \\ &= (ab, \mathbf{u}^b + \frac{1}{2}(\mathbf{u} - \mathbf{u}^b)) \\ &= (ab, \frac{1}{2}(\mathbf{u} + \mathbf{u}^b)), \\ (b, \mathbf{v})(a, \mathbf{u}) &= (ba, \mathbf{v}^a + \mathbf{u}) \\ &= (ab, \frac{1}{2}\mathbf{u}^a - \frac{1}{2}\mathbf{u}^{ba} + \mathbf{u}) \\ &= (ab, \frac{1}{2}(\mathbf{u} - \mathbf{u}^{ab})) \\ &= (ab, \frac{1}{2}(\mathbf{u} + \mathbf{u}^b)).\end{aligned}$$

Thus $(a, \mathbf{u}) \leftrightarrow (b, \mathbf{v})$, completing the proof. □

Proposition 4.3.6. *Let X be a conjugacy class of involutions in $\tilde{W}$ that contains $(w_I, \mathbf{u})$ where w_I is the central longest element of some standard parabolic subgroup of W and $\mathbf{u} = \tilde{\sigma}^\vee$. Suppose $(w_I, \mathbf{v}) \sim (w_I, \mathbf{u})$ for all $\mathbf{v} \in Z$ such that $(w_I, \mathbf{v}) \in X$, and that $\mathcal{C}(W, w_I^W)$ is connected. Then $\mathcal{C}(\tilde{W}, X)$ is connected.*

Proof. Let $w = (a, \mathbf{v}) \in X$. Then a is conjugate to w_I in W. That is, there is some $(g, \mathbf{w}) \in \tilde{W}$ with $a = w_I^g$ where $g \in W$. By hypothesis $\mathcal{C}(W, w_I^W)$ is connected, so there is a path $a \leftrightarrow \cdots \leftrightarrow w_I$. Hence by Lemma 4.3.5, there is a path $(a, \mathbf{v}) \leftrightarrow \cdots \leftrightarrow (w_I, \mathbf{v}')$ in the commuting involution graph. By hypothesis $(w_I, \mathbf{v}') \sim (w_I, \mathbf{u})$. Therefore, $w \sim (w_I, \mathbf{u})$.
Write $w_0 = (y_0, \mathbf{v}_0) = w^{(g,0)}$. Then $y_0 = w_I$. By hypothesis then, $w_0 \sim (w_I, \mathbf{u})$. That is, there is a path $w_0 \leftrightarrow w_1 \cdots \leftrightarrow w_m = (w_I, \mathbf{u})$ of elements of X giving a path in $\mathcal{C}(\tilde{W}, X)$ from w_0 to $(w_I, \mathbf{u})$. But this implies that there is a sequence $w_0^{(g,0)} \leftrightarrow w_1^{(g,0)} \leftrightarrow \cdots \leftrightarrow$

$w_m^{(g,0)} = (w_I, \mathbf{u})^{(g,0)}$ in the commuting involution graph. That is, $w_0^{(g,0)} \sim (w_I, \mathbf{u})^{(g,0)}$. Hence $w \sim (a, \mathbf{u}^g)$. Now $\mathcal{C}(W, w_I^W)$ is connected. Thus $a \sim w_I$ in $\mathcal{C}(W, w_I^W)$. Hence, by Lemma 4.3.5, we have $(a, \mathbf{u}^g) \sim (w_I, \mathbf{u}')$ in $\mathcal{C}(\tilde{W}, X)$. Therefore $w \sim (w_I, \mathbf{u})$ and since this holds for all w in X, we deduce that $\mathcal{C}(\tilde{W}, X)$ is connected. $\square$

4.4 Type $\tilde{F}_4$

Let W be of type F_4, with associated root system Φ and simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then $W = \langle s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4} \rangle$ and if the highest root is denoted $\tilde{\alpha}$, then we can take $r_i = (s_{\alpha_i}, 0)$ for $i \in \{1, 2, 3, 4\}$ and $r_5 = (s_{\tilde{\alpha}}, \tilde{\alpha}^\vee)$. The Coxeter graph for $\tilde{W}$ is as shown in Figure 4.3. We have that $\tilde{W} = \langle r_1, r_2, r_3, r_4, r_5 \rangle$. (The subgroup $\langle r_1, r_2, r_3, r_4 \rangle$ is of type F_4). The root system for F_4 has $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \alpha_2 = e_4, \alpha_3 = e_3 - e_4$ and $\alpha_4 = e_2 - e_3$.

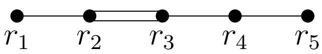


Figure 4.3: Coxeter graph for type $\tilde{F}_4$

It was shown in [7] that there are seven conjugacy classes of involutions in W. Let $a = (s_{\alpha_2} s_{\alpha_3})^2$. Then $\mathcal{C}(W, X)$ is connected, with nine elements in the first disc and eight in the second. Apart from this, and the graph consisting of just the central involution, all the other commuting involution graphs are disconnected. See [7] for more details.

Now let us consider the affine group $\tilde{W}$. The table below gives the involution conjugacy classes, a representative I for which w_I is in the class, and the name of the graph of the underlying class in W if $I \subseteq W$ then, the underlying class has the same graph . If $x = (w, \mathbf{v})$ is a product of k reflections in $\tilde{W}$, then w is a product of k reflections in W, so, for example, the conjugacy class of $\tilde{W}$ corresponding to the B_4 graph must have the class corresponding to F_4 as its underlying class in W. The only instance where this does not immediately tell us the type of the underlying class is the case of A_1^2 when $I = \{r_3, r_5\}$. Here, the underlying class might be type B_2 or type A_1^2. Note that $r_3 = s_{e_3 - e_4}$ and $r_5 = s_{e_1 + e_2, 1}$. So the underlying class is the conjugacy class of $s_{e_3 - e_4} s_{e_1 + e_2}$ in W. One can check that $r_1 r_4 r_3 r_2 r_3 r_4 (e_1 + e_2) = e_3 + e_4$ and $r_1 r_4 r_3 r_2 r_3 r_4 (e_3 - e_4) = e_3 - e_4$. Hence $(s_{e_3 - e_4} s_{e_1 + e_2})^{r_4 r_3 r_2 r_3 r_4 r_1} = s_{e_3 - e_4} s_{e_3 + e_4} = s_{e_3} s_{e_4} = r_3 r_2 r_3 r_2 = (r_3 r_2)^2$. Therefore, the underlying conjugacy class in this case is of type B_2.

Graph	Representative I	Underlying class in W
A_1	$\{r_1\}$	A_1
A_1	$\{r_3\}$	A_1
A_1^2	$\{r_1, r_3\}$	A_1^2
A_1^2	$\{r_3, r_5\}$	B_2
B_2	$\{r_2, r_3\}$	B_2
B_3	$\{r_1, r_2, r_3\}$	B_3
B_3	$\{r_2, r_3, r_4\}$	B_3
A_1^3	$\{r_1, r_3, r_5\}$	B_3
$B_2 \times A_1$	$\{r_2, r_3, r_5\}$	B_3
F_4	$\{r_1, r_2, r_3, r_4\}$	F_4
B_4	$\{r_2, r_3, r_4, r_5\}$	F_4
$B_3 \times A_1$	$\{r_1, r_2, r_3, r_5\}$	F_4

Theorem 4.4.1. *Let $\tilde{W}$ be of type $\tilde{F}_4$, with Coxeter graph shown in Figure 4.3. If X is the conjugacy class of $(r_2r_3)^2$ or r_3r_5 in $\tilde{F}_4$, then $\mathcal{C}(\tilde{W}, X)$ is connected. Otherwise, $\mathcal{C}(\tilde{W}, X)$ is disconnected.*

Proof. Let X be a conjugacy class in $\tilde{W}$. If the underlying class in W is anything other than type B_2 or F_4, then $\mathcal{C}(\tilde{W}, X)$ is disconnected, by Lemma 4.3.2. If the underlying class is type F_4 then $\mathcal{C}(\tilde{W}, X)$ is disconnected by Lemma 4.3.1. So we are reduced to the case where the underlying class in W is type B_2. There are two classes in $\tilde{W}$ where this happens, one containing $(r_2r_3)^2 = ((s_{\alpha_2}s_{\alpha_3})^2, \mathbf{0})$, and one containing $r_3r_5 = ((s_{\alpha_2}s_{\alpha_3})^2, \tilde{\alpha}^\vee)$.

Let X be the conjugacy class of $(r_2r_3)^2$ in $\tilde{W}$; its underlying class in W is of type B_2. Then in Proposition 4.3.3 we can take $w_I = ((s_{\alpha_2}s_{\alpha_3})^2 = s_{e_3}s_{e_4}$, and write $a = (w_I, \mathbf{0})$. Let g be an element of W that conjugates $s_{e_3}s_{e_4}$ to $s_{e_1}s_{e_2}$ (for example we could take $g = s_{e_1-e_3}s_{e_2-e_4}$). Then let $b = a^{(g,0)} = (s_{e_1}s_{e_2},$. Note that $b \in X$. Finally, let $c = (s_{e_3}s_{e_4}, \mathbf{v})$ be an element of X. Now $bc = (s_{e_1}s_{e_2}s_{e_3}s_{e_4}, \mathbf{v}')$. This is clearly an involution because $s_{e_1}s_{e_2}s_{e_3}s_{e_4}$ acts as -1 on Z. Thus b commutes with c for all c of the form $(w_I, \mathbf{v})$ in X. Hence, in particular, $a \leftrightarrow b \leftrightarrow c$. That is, $(w_I, \mathbf{v}) \sim (w_I,\ )$ for all $\mathbf{v} \in Z$ such that $(w_I, \mathbf{v}) \in X$. Now, since $\mathcal{C}(W, w_I^W)$ is connected, Proposition 4.3.3 shows that $\mathcal{C}(\tilde{W}, X)$ is connected.

Let X be the conjugacy class of r_3r_5 in $\tilde{W}$; its underlying class in W is of type B_2. Then in Proposition 4.3.6 we can take $w_I = s_{e_3}s_{e_4}$ and write $a = (s_{e_3}s_{e_4}, (0, 0, 1, 1))$ where $\mathbf{u} = (0, 0, 1, 1)$. Let $(g, 0) \in \tilde{W}$, that conjugate $(s_{e_3}s_{e_4}, \mathbf{u})$ to $(s_{e_1}s_{e_2}, \mathbf{u}')$ (for instance we could take $g = s_{e_1-e_3}s_{e_2-e_4}$). Now, let $b = (s_{e_1}s_{e_2}, \mathbf{u}')$ which is an element of X. Finally, let $c = (s_{e_3}s_{e_4}, \mathbf{v})$. Then $bc = (s_{e_1}s_{e_2}s_{e_3}s_{e_4}, \mathbf{v}')$ is an involution. Thus b

commutes with c for all c of the form $(w_I, \mathbf{v})$ be an element of X. Hence, in particular, $a \leftrightarrow b \leftrightarrow c$. That is, $(w_I, \mathbf{v}) \sim (w_I, \mathbf{u})$ for all $\mathbf{v} \in Z$ such that $(w_I, \mathbf{v}) \in X$. Now, since $\mathcal{C}(W, w_I^W)$ is connected, Proposition 4.3.6 shows that $\mathcal{C}(\tilde{W}, X)$ is connected. $\square$

It should be possible to carry out similar calculations for the remaining exceptional affine Weyl groups, namely those of types $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$. This could be a topic of further study.

Chapter 5

Commuting Reflection Graphs

5.1 Introduction

In this chapter we consider the class of commuting involution graphs $\mathcal{C}(W, X)$ where W is a Coxeter group and X is the set of all reflections in W. We call these *commuting reflection graphs.*

Recall that a Coxeter system (W, R) consists of a group W, with distinguished generating set R (the simple reflections), subject only to relations of the form $(rs)^{m_{rs}} = 1$, where $m_{rr} = 1$ and $m_{rs} = m_{sr} \geq 2$ for all $r, s \in R$. An element of W is called a *reflection* if it is conjugate to a simple reflection. The set X of all reflections is thus either a single conjugacy class or a union of conjugacy classes.

In [6], Bates et al looked into the commuting involution graph $\mathcal{C}(G, X)$ where X is a conjugacy class of involutions in G and G is $\mathrm{Sym}(n)$. Commuting involution graphs for the remaining finite Coxeter groups were analysed in [7]. Commuting involution graphs in affine Coxeter groups of type $\tilde{A}_n$ have been considered in [23]. The cases of $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{F}_4$ and $\tilde{G}_2$ have been considered in this thesis. However in many of these cases the set of all reflections does not form a single conjugacy class, so more work is usually required to deal with the commuting reflection graphs considered here.

For the rest of this chapter we set (W, R) to be a Coxeter system and $X = R^W$ its set of reflections. Our main objective will be to establish, wherever possible, for which Coxeter groups W the commuting reflection graph $\mathcal{C}(W, X)$ is connected. At the end of this chapter we present Coxeter graphs of diameter at most 2 and diameter greater than 2 and show examples of disconnected commuting reflection graphs. We will use the following notation.

Notation 5.1.1. We write $g \leftrightarrow h$ to mean $gh = hg$. We write $g \sim h$ to mean that g and h lie in the same connected component of $\mathcal{C}(W, X)$.

5.2 Preliminary Results

Lemma 5.2.1. *Let W be a Coxeter group. If W is reducible, then $\mathcal{C}(W,X)$ is connected with diameter at most 2.*

Proof. If W is reducible, then by definition we can divide R into two disjoint sets R_1 and R_2 where if $r \in R_1$ and $s \in R_2$, then $r \leftrightarrow s$, and $W \cong \langle R_1 \rangle \times \langle R_2 \rangle$. In particular $X = R_1^{\langle R_1 \rangle} \cup R_2^{\langle R_2 \rangle}$.
Let $x, y \in X$ and suppose without loss of generality that $x \in R_1^{\langle R_1 \rangle}$. If $y \in R_2^{\langle R_2 \rangle}$, then $x \leftrightarrow y$ and $d(x,y) = 1$. Otherwise, $y \in R_1^{\langle R_1 \rangle}$. Let $z \in R_2$. Then $x \leftrightarrow z$ and $z \leftrightarrow y$, so $d(x,y) \leq 2$. Thus $\mathcal{C}(W,X)$ is connected with diameter at most 2. □

The next lemma deals with the dihedral case.

Lemma 5.2.2. *Let W be an irreducible Coxeter group of rank 2. Then $\mathcal{C}(W,X)$ is disconnected.*

Proof. Suppose $|R| = 2$. Then we can set $R = \{r, s\}$, and write $m = m_{rs}$. Let the simple roots corresponding to r and s be α_r and α_s, respectively. Let $w \in X$. Then $w = s_\alpha$ for some $\alpha \in \Phi^+$. So there are non-negative real numbers λ_r and λ_s with $\alpha = \lambda_r \alpha_r + \lambda_s \alpha_s$. By Lemma 1.4.4, which as remarked in Chapter 4 holds for arbitrary root systems, not just finite ones, we have $w \leftrightarrow r$ if and only if either $w = r$ or $\langle \alpha, \alpha_r \rangle = 0$. Assuming $w \neq r$ we get

$$\begin{aligned} \langle \alpha, \alpha_r \rangle &= 0, \\ \langle \lambda_r \alpha_r + \lambda_s \alpha_s, \alpha_r \rangle &= 0, \\ \lambda_r &= \lambda_s \cos \tfrac{\pi}{m}. \end{aligned}$$

That is, $\alpha = \lambda_s \cos \frac{\pi}{m} \alpha_r + \lambda_s \alpha_s$. From $\langle \alpha, \alpha \rangle = 1$, we get $1 = {\lambda_s}^2 (1 - \cos^2 \frac{\pi}{m}) = {\lambda_s}^2 \sin^2 \frac{\pi}{m}$. Then, $1 = {\lambda_s}^2 \sin^2 \frac{\pi}{m}$ which implies that λ_s and $\sin \frac{\pi}{m}$ are both nonzero. Now, suppose $w' = s_\beta$, where $\beta = \mu_r \alpha_r + \mu_s \alpha_s$. If $w \leftrightarrow w'$, then $\langle \alpha, \beta \rangle = 0$.

$$\begin{aligned} 0 &= \langle \alpha, \beta \rangle \\ &= \langle \lambda_s \cos \tfrac{\pi}{m} \alpha_r + \lambda_s \alpha_s, \mu_r \alpha_r + \mu_s \alpha_s \rangle, \\ 0 &= \mu_r \cos \tfrac{\pi}{m} - \mu_s \cos^2 \tfrac{\pi}{m} - \mu_r \cos \tfrac{\pi}{m} + \mu_s \\ &= \mu_s \sin^2 \tfrac{\pi}{m}. \end{aligned}$$

Therefore $\mu_s = 0$, which implies $w' = r$. We can further observe that since $\langle \alpha, \alpha \rangle = 1$ for all roots α, at most one positive root $\alpha = \lambda_s \left(\cos \frac{\pi}{m} \alpha_r + \alpha_s \right)$ can exist. Therefore the connected component of $\mathcal{C}(W,X)$ containing r is a clique with either one element or two elements. In particular, $\mathcal{C}(W,X)$ is disconnected. □

Note that Coxeter groups of rank 2 are dihedral, so we could prove Lemma 5.2.2 without using root system properties, but we have included the proof here because we use more arguments involving roots later and this is an example of the techniques we will use.

Proposition 5.2.3. *Suppose there exists r in R such that for all s in R we have $r \sim s$ and $r \sim srs$. Then $\mathcal{C}(W, X)$ is connected.*

Proof. Since $\sim$ is an equivalence relation, it suffices to show that $r \sim w$ for all $w \in X$. Let $w \in X$. Then $w = gsg^{-1}$ for some $s \in R$ and $g \in W$. We proceed by induction on $\ell(g)$. If $\ell(g) = 0$, then $w = s$ and so by hypothesis $r \sim w$ and we are done. Suppose $\ell(g) > 0$. Then $g = th$ for some $t \in R$ and $h \in W$ with $\ell(h) < \ell(g)$. Inductively $r \sim hsh^{-1}$. This implies $trt \sim thsh^{-1}t = gsg^{-1} = w$. By hypothesis $r \sim trt$. So we have $r \sim tst \sim w$. Hence $r \sim w$, which completes the proof. $\square$

Proposition 5.2.3 allows us quickly to show the connectedness of several special cases.

Lemma 5.2.4. *Let W be an irreducible Coxeter group whose associated Coxeter graph Γ has diameter at least 3. Then $\mathcal{C}(W, X)$ is connected.*

Proof. Since Γ has diameter at least 3, there exist elements r and r' of R, where $d(r, r') \geq 3$. Let $s \in R$. If $s \leftrightarrow r$ then trivially $r \sim s$; moreover $srs = r$, so $r \sim srs$ as well. Now suppose s does not commute with r. Then r and s are joined by an edge in Γ. If s is also joined by an edge to r', then this would contradict the fact that $d(r, r') \geq 3$. Therefore $r' \leftrightarrow s$. Since $r \leftrightarrow r'$, we obtain $r \sim s$. Finally, since both r and s commute with r', we see that $r' \leftrightarrow srs$. Hence $r \sim srs$. Therefore, for all $s \in R$ we have $r \sim s$ and $r \sim srs$. Therefore, by Proposition 5.2.3, $\mathcal{C}(W, X)$ is connected. $\square$

Proposition 5.2.5. *Let W be a finite Coxeter group. Then $\mathcal{C}(W, X)$ is disconnected if W is of type A_2, A_3, B_2, D_4, H_3 or dihedral. For all other finite Coxeter groups $\mathcal{C}(W, X)$ is connected.*

Proof. Suppose $\mathcal{C}(W, X)$ is disconnected. By Lemmas 5.2.1 and 5.2.4, W must be irreducible with a Coxeter graph Γ of diameter at most 2. From the classification of finite Coxeter groups, this implies W is of type $A_1, A_2, A_3, B_2, B_3, D_4, H_3$ or is dihedral. The reflections in $W(A_{n-1})$ correspond to transpositions in $\mathrm{Sym}(n)$. A quick check shows that $\mathcal{C}(W, X)$ is connected for type A_1, and disconnected for types A_2 and A_3. If W is dihedral, then $\mathcal{C}(W, X)$ is either completely disconnected (for dihedral of twice odd order) or consists of cliques of size 2 (for dihedral of twice even order). Either way, $\mathcal{C}(W, X)$ is disconnected (see also Lemma 5.2.2). This deals with type B_2 as well, as then W is dihedral of order 8. For W of type B_3, $\mathcal{C}(W, X)$ is connected. To see this, note that the reflections are s_{e_i}, $s_{e_i \pm e_j}$ for $1 \leq i \leq 3$ and $1 \leq i < j \leq 3$. Now s_{e_1}

commutes with s_{e_2} and s_{e_3}; moreover, every other reflection commutes with at least one s_{e_i}. Hence $\mathcal{C}(W, X)$ is connected with diameter 2. For W of type D_4, $\mathcal{C}(W, X)$ is disconnected with each connected component being a clique with four members. Finally $W(H_3)$ is isomorphic to the direct product of $\mathrm{Alt}(5)$ with $\mathbb{Z}_2$ and hence there are three conjugacy classes of involutions, one of which is central. The commuting graphs of the other two (one of which is the class consisting of the fifteen reflections) are disconnected with each connected component being a clique with three members. Therefore the only finite Coxeter groups for which $\mathcal{C}(W, X)$ is disconnected are dihedral or of type A_2, A_3, B_2, D_4 or H_3. □

Proposition 5.2.6. *Let W be an affine Weyl group. Then $\mathcal{C}(W, X)$ is disconnected if W is of type $\tilde{A}_1$, $\tilde{A}_2, \tilde{A}_3, \tilde{B}_2, \tilde{D}_4$ or $\tilde{G}_2$. For all other affine Weyl groups $\mathcal{C}(W, X)$ is connected.*

Proof. Given a Weyl group G with root system Φ, and set of reflections $Y = \{s_\alpha | \alpha \in \Phi^+\}$, the set X of (affine) reflections of the corresponding affine Weyl group W is $X = \{s_{\alpha,k} | \alpha \in \Phi^+, k \in \mathbb{Z}\}$. The affine reflection $s_{\alpha,k}$ is given by

$$s_{\alpha,k}(\mathbf{v}) = s_\alpha(\mathbf{v}) + k\alpha^\vee.$$

Here, as before $\alpha^\vee$ denotes the coroot $\frac{2\alpha}{\langle \alpha, \alpha \rangle}$. Thus for positive roots α, β and integers k, l, by Lemma 1.4.4, we have $s_{\beta,l} \leftrightarrow s_{\alpha,k}$ if and only if either $\alpha = \beta$ and $k = l$, or $\alpha \neq \beta$ and $s_\alpha \leftrightarrow s_\beta$. In particular, if $\mathcal{C}(G, Y)$ is disconnected, then $\mathcal{C}(G, X)$ is disconnected. Conversely, suppose that $\mathcal{C}(G, Y)$ is connected. Let $s_{\alpha,k}$ be a reflection in W. Suppose G contains more than one reflection. Let s_γ be a reflection in G with $s_\alpha \leftrightarrow s_\gamma$. Then for all integers l we have $s_{\alpha,k} \leftrightarrow s_{\gamma,0} \leftrightarrow s_{\alpha,l}$. Let $s_{\beta,m}$ be a reflection in W with $\beta \neq \alpha$. Then, since the commuting reflection graph for G is connected, there is a path $s_\alpha \leftrightarrow s_{\gamma_1} \leftrightarrow \ldots \leftrightarrow s_{\gamma_j} \leftrightarrow s_\beta$ in $\mathcal{C}(G, Y)$. This induces a path $s_{\alpha,k} \leftrightarrow s_{\gamma_1,0} \leftrightarrow \ldots \leftrightarrow s_{\gamma_j,0} \leftrightarrow s_{\beta,l}$ in $\mathcal{C}(G, X)$. Therefore, unless W is of type $\tilde{A}_1$, we can say that $\mathcal{C}(W, X)$ is connected if and only if the commuting reflection graph of the underlying finite Coxeter group is connected. Since $W(\tilde{A}_1)$ is infinite dihedral, its commuting reflection graph is completely disconnected.

For the remaining cases, using Proposition 5.2.5, we see that $\mathcal{C}(W, X)$ is disconnected when W is of type $\tilde{A}_2, \tilde{A}_3, \tilde{B}_2, \tilde{D}_4$ or $\tilde{G}_2$ (because $\tilde{G}_2$ is the affine group corresponding to $I_2(6)$). In all other cases, apart from $\tilde{A}_1$, we have that $\mathcal{C}(W, X)$ is connected. □

The results so far resolve all cases except where W is an infinite, irreducible, non-affine Coxeter group whose Coxeter graph Γ has diameter 1 or 2. For these remaining cases we have been able to obtain some partial results by looking at reflection centralizers. These are described in the next section.

5.3 Reflection Centralizers

In order to gain more information about the commuting reflection graph, centralizers of reflections are clearly relevant. Recall that for a given reflection a of W, and a non-negative integer i, we write $\Delta_i(a) = \{b \in X : d(a,b) = i\}$, where d is the standard distance metric on $\mathcal{C}(W,X)$. We see that $\Delta_1(a) = X \cap C_W(a)$. Fortunately, a paper by Allcock [4], describes how to construct the reflection subgroups of centralizers of reflections in arbitrary Coxeter groups. We briefly outline his results here.

Let (W,R) be a Coxeter system with associated Coxeter graph Γ. An edge of Γ is *even* when its label is even, and odd otherwise. We write Γ^{odd} for the graph Γ with all even edges removed. For $r \in R$, we denote by $\Gamma^{\text{odd}}(r)$ the connected component of Γ^{odd} containing r.

Definition 5.3.1. Suppose $\gamma = (t_0, t_1, \ldots, t_n)$ is an edge-path in Γ^{odd}, with $2l_i + 1$ being the label on the edge joining t_{i-1} and t_i. Then we set

$$p_\gamma := (t_1t_0)^{l_1}(t_2t_1)^{l_2}\cdots(t_nt_{n-1})^{l_n}$$

with $p_\gamma = 1$ if $n = 0$. If there exists a vertex u of Γ which is not joined to t_n in Γ^{odd} (that is, t_nu has even order 2λ for some integer λ), then we define

$$r_{\gamma,u} := p_\gamma(ut_n)^{\lambda-1}up_\gamma^{-1}.$$

Note that in particular if $m_{t_nu} = 2$ then $r_{\gamma,u} = p_\gamma u p_\gamma^{-1}$. If in addition $\gamma = (r)$ then $r_{\gamma,u} = u$.

Theorem 5.3.2 (Allcock [4])**.** *Let $r \in R$ and write H_r for the subgroup of $C_W(r)$ generated by all the reflections contained in $C_W(r)$ apart from r. Then H_r is a Coxeter group and the set of elements of W of the form $r_{\gamma,u}$, where γ is an edge path starting at r, forms a Coxeter system for H_r.*

Observe that we specify the elements rather than the expressions. It may well be the case that $r_{\gamma,u} = r_{\gamma',u'}$ even if $(\gamma,u) \neq (\gamma',u')$. In our examples it will be clear when this happens.
To illustrate these ideas we give a series of examples; in each case W is an irreducible Coxeter group of rank 3.

Example 5.3.3. Let W be of type A_3. The Coxeter graph is $\overset{r_1}{\bullet}\!-\!\!-\!\!-\overset{r_2}{\bullet}\!-\!\!-\!\!-\overset{r_3}{\bullet}$, and we can take $W = S_4$, $r_1 = (12)$, $r_2 = (23)$ and $r_3 = (34)$. Here $\Gamma^{\text{odd}} = \Gamma$ because all edge labels are odd (every edge label is 3). Let $r = r_1$. Then $\Gamma^{\text{odd}}_r = \Gamma$. Possible edge paths are (r_1), (r_1, r_2), and (r_1, r_2, r_3). Note that we don't need to consider paths which go back on themselves, as for example $\gamma' = (r_1, r_2, r_3, r_2, r_1)$, as in these cases p'_γ is equal to p_γ for a shorter γ. In the case $\gamma' = (r_1, r_2, r_3, r_2, r_1)$, we have $p_{\gamma'} = (r_2 r_1)(r_3 r_2)(r_2 r_3)(r_1 r_2) = 1 = p_{(r_1)}$. In fact whenever Γ^{odd} is a tree, it is sufficient to consider only shortest edge paths between vertices.

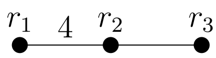
So, with $r = r_1$ we have three possible (shortest) edge paths: (r_1), (r_1, r_2) and (r_1, r_2, r_3). If $\gamma = (r_1)$, then $p_\gamma = 1$. To create $r_{\gamma,u}$ we must have a $u \in R$ with u evenly joined to the endpoint of γ, in this case r_1. The only possibility for u is therefore r_3. We get $r_{(r_1),r_3} = r_3$. If $\gamma = (r_1, r_2)$, then $p_\gamma = r_2 r_1$. However there is no u evenly joined to the endpoint r_2. So no $r_{\gamma,u}$ is possible. Finally suppose $\gamma = (r_1, r_2, r_3)$. Then $p_\gamma = (r_2 r_1)(r_3 r_2)$ and the only possible choice of u is r_1. So we get

$$r_{\gamma,r_1} = p_\gamma r_1 p_\gamma^{-1} = r_2 r_1 r_3 r_2 r_1 r_2 r_3 r_1 r_2 = r_2 r_3 r_1 r_2 r_1 r_2 r_1 r_3 r_2 = r_2 r_3 r_2 r_3 r_2 = r_3.$$

So we find that $H_{r_1} = \langle r_3 \rangle$. That is, there is exactly one reflection (other than r_1 itself) which commutes with r_1, namely r_3. This agrees with the observation that the only transposition that commutes with (12) in S_4 (other than (12) itself) is (34).
If we now take $r = r_2$, then we need to consider three (shortest) edge paths: (r_2), $(r_2 r_1)$ and $(r_2 r_3)$. We can eliminate $\gamma = (r_2)$ because there is no u evenly joined to r_2. For $\gamma = (r_2 r_1)$ the only choice of u is r_3 so we get $p_\gamma = (r_1 r_2)$ and $r_{\gamma,u} = r_1 r_2 r_3 r_2 r_1$. For $\gamma = (r_2 r_3)$ the only choice of u is r_1, we get $p_\gamma = r_3 r_2$ and

$$r_{\gamma,u} = r_3 r_2 r_1 r_2 r_3 = r_3 r_1 r_2 r_1 r_3 = r_1 r_3 r_2 r_3 r_1 = r_1 r_2 r_3 r_2 r_1.$$

So again there is just one reflection that commutes with r_2, and this is $r_1 r_2 r_3 r_2 r_1$. In S_4 this agrees with the fact that (23) commutes only with (14) (and itself). Also note that all reflections are conjugate in this group so we should get isomorphic groups H_r, and this is indeed the case.

Example 5.3.4. Our second example is W of type B_3, with graph 

.
Here there are two conjugacy classes of reflections, but we perform calculations for r_1, r_2 and r_3 as examples; H_{r_2} and H_{r_3} are conjugate and hence isomorphic. Recall that we can represent elements of W as signed permutations; the reflections are $(\overset{-}{1}) = r_1$, $(\overset{-}{2})$, $(\overset{-}{3})$, $(\overset{++}{12}) = r_2$, $(\overset{++}{23}) = r_3$, $(\overset{++}{13})$, $(\overset{--}{12})$, $(\overset{--}{23})$, and $(\overset{--}{13})$. The calculations for $r_{\gamma,u}$ are as follows.

Table 5.1: W is of type B_3

r	Γ_r^{odd}	γ	u	p_γ	$r_{\gamma,u}$
$r_1 = (\overset{-}{1})$	r_1 ●	(r_1)	r_2	1	$r_2r_1r_2 = (\overset{-}{2})$
		(r_1)	r_3	1	$r_3 = (\overset{++}{23})$
$r_2 = (\overset{++}{12})$	r_2 ●—● r_3	(r_2)	r_1	1	$r_1r_2r_1 = (\overset{--}{12})$
		(r_2, r_3)	r_1	r_3r_2	$r_3r_2r_1r_2r_3 = (\overset{-}{3})$
$r_3 = (\overset{++}{23})$	r_2 ●—● r_3	(r_3)	r_1	1	$r_1 = (\overset{-}{1})$
		(r_3, r_2)	r_1	r_2r_3	$r_2r_3r_1r_2r_1r_3r_2 = (\overset{--}{23})$

Hence $H_{r_1} = \langle(\overset{-}{2}), (\overset{++}{23})\rangle$, which is dihedral of order 8. The reflections that commute with r_1 are precisely the reflections in H_{r_1}, namely $(\overset{-}{2}), (\overset{++}{23}), (\overset{-}{3}), (\overset{--}{23})$. Meanwhile H_{r_2} and H_{r_3} are both Klein 4-groups containing exactly two reflections. The reflections commuting with r_2 are $(\overset{--}{12})$ and $(\overset{-}{3})$; the reflections commuting with r_3 are $(\overset{--}{23})$ and $(\overset{-}{1})$.

Example 5.3.5. Our next example considers the case when the graph of W is a tree with three vertices and two odd edges. So $\Gamma = \underset{r_1}{\bullet}\overset{m}{\text{———}}\underset{r_2}{\bullet}\overset{k}{\text{———}}\underset{r_3}{\bullet}$. Let $m = 2l_1 + 1$ and $l = 2l_2 + 1$. The calculations for $r_{\gamma,u}$ are as follows. Note that for any path γ ending in r_2, there is no valid choice of u, because no vertex is evenly joined to r_2. Therefore we need only consider shortest paths γ ending in either r_1 or r_3. In such cases we have $r_{\gamma,u} = p_\gamma u p_\gamma^{-1}$.

Table 5.2: Γ has two odd edges

r	Γ_r^{odd}	γ	u	p_γ	$r_{\gamma,u}$
r_1	Γ	(r_1)	r_3	1	r_3
		(r_1, r_2, r_3)	r_1	$(r_2r_1)^{l_1}(r_3r_2)^{l_2}$	$(r_2r_1)^{l_1}(r_3r_2)^{l_2}r_1(r_2r_3)^{l_2}(r_1r_2)^{l_1}$
r_2	Γ	(r_2, r_1)	r_3	$(r_1r_2)^{l_1}$	$(r_1r_2)^{l_1}r_3(r_2r_1)^{l_1}$
		(r_2, r_3)	r_1	$(r_3r_2)^{l_2}$	$(r_3r_2)^{l_2}r_1(r_2r_3)^{l_2}$
r_3	Γ	(r_3)	r_1	1	r_1
		(r_3, r_2, r_1)	r_3	$(r_2r_3)^{l_2}(r_1r_2)^{l_1}$	$(r_2r_3)^{l_2}(r_1r_2)^{l_1}r_3(r_2r_1)^{l_1}(r_3r_2)^{l_2}$

Example 5.3.6. Now we suppose the graph of W is a tree with three vertices, one odd edge and one even edge. Here there are 2 conjugacy classes of reflections. Let $m = 2l_1$ and $k = 2l_2 + 1$.

Table 5.3: Γ has one even and one odd edge.

r	Γ_r^{odd}	γ	u	p_γ	$r_{\gamma,u}$
r_1	Γ	(r_1)	r_2	1	$(r_2r_1)^{l_1-1}r_2$
		(r_1)	r_3	1	r_3
r_2	Γ	(r_2)	r_1	1	$(r_1r_2)^{l_1-1}r_1$
		(r_2, r_3)	r_1	$(r_3r_2)^{l_2}$	$(r_3r_2)^{l_2}r_1(r_2r_3)^{l_2}$
r_3	Γ	(r_3)	r_1	1	r_1
		(r_3, r_2)	r_1	$(r_2r_3)^{l_2}$	$(r_2r_3)^{l_2}(r_1r_2)^{l_1-1}r_1(r_3r_2)^{l_2}$

Example 5.3.7. Next we look at the case when the graph of W is a tree with three vertices and two even edges. Here we have three conjugacy classes of reflections. Let $m = 2l_1$ and $k = 2l_2$.

Table 5.4: Γ has two even edges

r	Γ_r^{odd}	γ	u	p_γ	$r_{\gamma,u}$
r_1	Γ	(r_1)	r_2	1	$(r_2r_1)^{l_1-1}r_2$
		(r_1)	r_3	1	r_3
r_2	Γ	(r_2)	r_1	1	$(r_1r_2)^{l_1-1}r_1$
		(r_2)	r_3	1	$(r_3r_2)^{l_2-1}r_3$
r_3	Γ	(r_3)	r_1	1	r_1
		(r_3)	r_2	1	$(r_2r_3)^{l_2-1}r_2$

Let W be of type $\tilde{B}_2$. Then $l_1 = l_2 = 2$ and $H_{r_1} = \langle (\overset{0}{\bar{2}}), (\overset{1}{\bar{2}}) \rangle$, $H_{r_2} = \langle ((\overset{0}{\bar{1}}\ 2)), ((\overset{1}{\bar{1}}\ 2)) \rangle$ and $H_{r_3} = \langle (\overset{0}{\bar{1}}), (\overset{1}{\bar{1}}) \rangle$.

Example 5.3.8. We now deal with the possibility that the graph of W is a cycle with three vertices. If all the edges are odd, then there are no possibilities for u in any $r_{\gamma,u}$. Hence the commuting reflection graph is completely disconnected. So we can assume there is at least one even edge. That means Γ^{odd} is a tree, so we only need to look at shortest edge paths γ.
Let $p = 2l_1$, $m = 2l_2 + 1$ and $k = 2l_3 + 1$. The Coxeter graph of W is as follows.

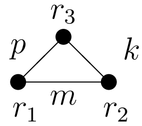


Figure 5.1: A Coxeter group of rank 3

Table 5.5: Γ has one even and two odd edges.

r	Γ_r^{odd}	γ	u	p_γ	$r_{\gamma,u}$
r_1	Γ	(r_1)	r_3	1	$(r_3r_1)^{l_1-1}r_3$
		(r_1, r_2)			
		(r_1, r_2, r_3)	r_1	$(r_2r_1)^{l_2}(r_3r_2)^{l_3}$	$(r_2r_1)^{l_2}(r_3r_2)^{l_3}(r_1r_3)^{l_1-1}r_1(r_2r_3)^{l_3}(r_1r_2)^{l_2}$
r_2	Γ	(r_2)		1	
		(r_2, r_1)	r_3	$(r_1r_2)^{l_2}$	$(r_1r_2)^{l_2}(r_3r_1)^{l_1-1}r_3(r_2r_1)^{l_2}$
		(r_2, r_3)	r_1	$(r_3r_2)^{l_3}$	$(r_3r_2)^{l_3}(r_1r_3)^{l_1-1}r_1(r_2r_3)^{l_3}$
r_3	Γ	(r_3)	r_1	1	$(r_1r_3)^{l_1-1}r_1$
		(r_3, r_2)		1	
		(r_3, r_2, r_1)	r_3	$(r_2r_3)^{l_3}(r_1r_2)^{l_2}$	$(r_2r_3)^{l_3}(r_1r_2)^{l_2}(r_3r_1)^{l_1-1}r_3(r_2r_1)^{l_2}(r_3r_2)^{l_3}$

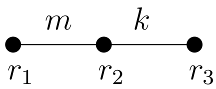
The observation from this last example allows us to deal with a special case.

Lemma 5.3.9. *Suppose the Coxeter graph of W has diameter 1 (that is, it is the complete graph K_n for some n), and suppose that all edge labels are odd. Then $\mathcal{C}(W, X)$ is disconnected.*

Proof. As all edges are odd, there are no possibilities for u or any $r_{\gamma,u}$. So the commuting reflection graph is completely disconnected. $\square$

5.4 Coxeter Groups of Rank 3

In this section we look at case where W has rank 3 and its Coxeter graph is as follows;



If $R = \{r_1, \ldots, r_n\}$, let α_i be the simple root corresponding to r_i, and let $m_{ij} = m_{r_ir_j}$. In the special case we consider in this section, we have $m_{13} = 2$, $m_{12} = m$, $m_{23} = k$. We assume further that m and k are odd.

Conjecture 5.4.1. *If the Coxeter graph of W is as in Figure 5.1, where m and k are odd, then $\mathcal{C}(W, X)$ is disconnected.*

Recall that, for a reflection s we write H_s for the subgroup of $C_W(s)$ generated by the reflections in $C_W(s)$ other than s.

Proposition 5.4.2. *Suppose $m \leq 7$ and $k = 3$. Then $\mathcal{C}(W, X)$ is connected when $m = 4$ and disconnected otherwise.*

Proof. If m is 3, 4, 5 or 6, then W is of type A_3, B_3, H_3 or $\tilde{G}_2$ respectively and these cases are covered by Proposition 5.2.5 and Proposition 5.2.6. The only connected graph is for B_3. So suppose $m = 7$. By Example 5.3.5, $C_W(r_1) = \langle gr_1g^{-1}, g^{-1}r_1g\rangle$, where $g = (r_2r_1)^3(r_3r_2)$; this means that $\Delta_1(r_1) = X \cap H_{r_1}$. Now $C_W(r_1)$ is an infinite dihedral group. If we temporarily write $x = g^{-1}r_1g$ and $y = gr_1g^{-1}$, then $X \cap H_{r_1} = \{x, y, xyx, yxy, xyxyx, yxyxy, \ldots\}$. Let $H = \langle r_1, g\rangle$. Observe that every element of $\Delta_1(r_1)$ is r_1 conjugated by an element of H. That is, $X \cap H_{r_1} \subseteq r_1^H$. Inductively, assume every element of $\Delta_i(r_1)$ is of the form hr_1h^{-1} for some $h \in H$. Then every element t of $\Delta_{i+1}(r_1)$ is a reflection contained in H_z for some $z \in \Delta_i(r_1)$. Thus t is contained in $X \cap H_{hr_1h^{-1}}$ for some $h \in H$. But $X \cap H_{hr_1h^{-1}} = h(X \cap H_{r_1})h^{-1} \subseteq h(r_1^H)h^{-1} = r_1^H$. Hence $\Delta_i(r_1) \subseteq r_1^H$, which means, since $r_1 \in H$, that $\Delta(r_1) \subseteq H$.
In the case $m = 7$ we will see that $r_2 \notin H$, which means $\mathcal{C}(W, X)$ is disconnected. In GAP [18], the technique known as coset enumeration can be carried out for finitely presented groups. The following commands set up W in GAP as a quotient of the free group on 3 generators.

```
F3:= FreeGroup("r","s","t");
W:=F3/[F3.1^2,F3.2^2,F3.3^2,(F3.1*F3.2)^7,(F3.2*F3.3)^3,(F3.1*F3.3)^2];
```

We can now define the subgroup $H = H_{r_1}$.

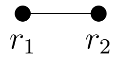
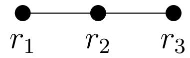
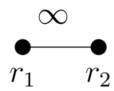
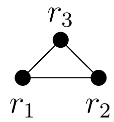
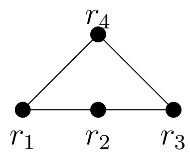
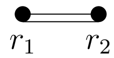
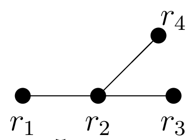
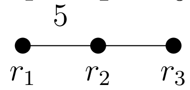
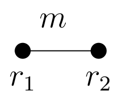
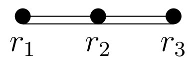
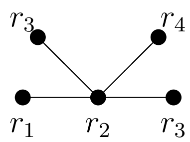
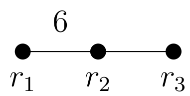
```
H:=Subgroup(W,[W.1,(W.2*W.1)^3*(W.3*W.2)]);
```

The command `Index(W,H)` can now be deployed, and it turns out that H has index 9 in W. (A list of coset representatives can be obtained using the command `Elements(RightTransversal(W,H))`.) In particular, since H contains r_3 and r_1, it cannot also contain r_2, otherwise we would have $H = W$. Hence $r_2 \notin H$, which means $r_2 \not\sim r_1$. Therefore $\mathcal{C}(W, X)$ is disconnected. □

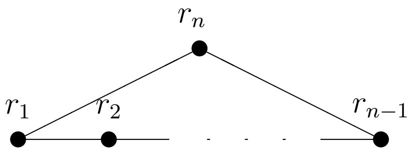
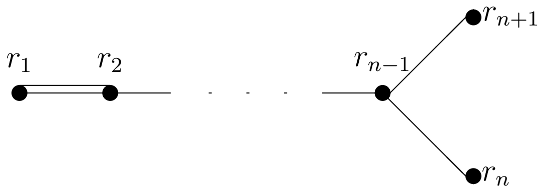
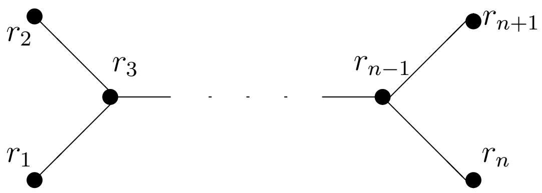
Remark We can try to get a more general result by looking at the group H for other odd values of m and k. If $m = 2l + 1$ and $k = 2q + 1$, then $g = (r_2r_1)^l(r_3r_2)^q$ and $H = \langle r_1, g\rangle$. For example when $m = k = 3$ and the group is type A_3, isomorphic to S_4, we have that $H = \langle(12), (23)(12)(34)(23)\rangle$, which is dihedral of order 8 and has index 3 in W. So $H \neq W$. When $m = 5$ and $k = 3$ then W is of type H_3 and $H = \langle r_1, (r_2r_1)^2(r_3r_2)\rangle$. Here, it turns out that H has index 5 in W. GAP does not seem able to calculate the index of H for cases when $m > 7$.
However, there are some instances we have been able to find where there is a subgroup A containing H, and A has index 3 in W, which means of course in particular that $H \neq W$ and so $r_2 \notin H$. This occurs when m is a multiple of 3, and $A = \langle r_1, r_3, r_2r_1r_3r_2\rangle$.
In the rest of this chapter we gather together some examples. Section 5.5 lists the graphs for finite and affine Coxeter groups that have diameter at most 2. Section 5.6

lists the graphs for finite and affine Coxeter groups with diameter at least 3. Finally in Section 5.7 we give worked examples for two groups ($W(A_3)$ and $W(\tilde{A}_2)$) of rank 3.

5.5 Coxeter graphs with diameter at most 2

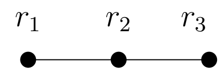
Type	Coxeter graph
A_1	
A_2	
A_3	
$\tilde{A}_1$	
$\tilde{A}_2$	
$\tilde{A}_3$	
B_2	
D_4	
H_3	
$I_2(m)$	
$\tilde{B}_2$	
$\tilde{D}_4$	
$\tilde{G}_2$	

5.6 Coxeter graphs with diameter at least 3

Type	Coxeter graph
$A_{n-1}, n > 4$	
$\tilde{A}_{n-1}, n > 4$	
$B_n,\ n > 3$	
$\tilde{B}_n,\ n > 3$	
$\tilde{C}_n,\ n > 3$	
$\tilde{D}_n,\ n > 4$	

5.7 Examples of Coxeter Groups of Rank 3

Let G be a Weyl group of type A_3. The Coxeter graph is as follows:



We can see that $m_{11} = m_{22} = m_{33} = 1$, $m_{12} = m_{21} = m_{23} = m_{32} = 3$ and $m_{13} = m_{31} = 2$.
We know that $s_{w(\alpha)} = w s_\alpha w^{-1}$. Then, the conjugacy class of reflections in A_3 is represented in the graph below.

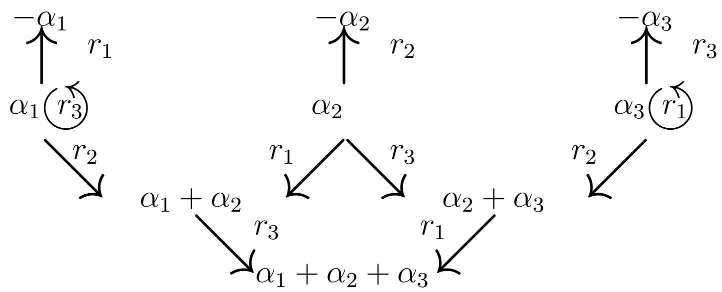


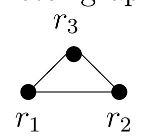
Figure 5.2: $\alpha_1^{A_3}$

Hence, we can deduce from the figure above that there is one conjugacy class of reflections, namely

$$r_1{}^G = \{\pm S_{\alpha_1}, \pm S_{\alpha_2}, \pm S_{\alpha_3}, \pm S_{\alpha_1+\alpha_2}, \pm S_{\alpha_2+\alpha_3}, \pm S_{\alpha_1+\alpha_2+\alpha_3}\}.$$

The commuting involution graph is disconnected by Lemma 1.4.4, as we have $\langle \alpha_2, \alpha_1 \rangle = \langle \alpha_2, \alpha_3 \rangle = \frac{-1}{2}$, $\langle \alpha_2, \alpha_1 + \alpha_2 \rangle = \langle \alpha_2, \alpha_2 + \alpha_3 \rangle = \frac{1}{2}$ and $\langle \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 \rangle = 0$. Then, we can see that s_{α_2} commutes only with $s_{\alpha_1+\alpha_2+\alpha_3}$. On the other hand, $s_{\alpha_1+\alpha_2+\alpha_3}$ does not commute with any $w \in X$ apart from s_{α_2}, as we have $\langle \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 \rangle = \langle \alpha_1 + \alpha_2 + \alpha_3, \alpha_3 \rangle = \frac{1}{2}$ and $\langle \alpha_1 + \alpha_2 + \alpha_3, \alpha_3 \rangle = \langle \alpha_1 + \alpha_2 + \alpha_3, \alpha_2\alpha_3 \rangle = \frac{1}{2}$.

Now, if G is of type $\tilde{A}_2$, the Coxeter graph is as follows:



We have one conjugacy class of reflections, $r_1{}^G = \{S_{\lambda'' k'' \pm \alpha_i} \colon i \in \{1, 2, 3\}, \lambda'' \in \mathbb{Z}\}$ with $k'' = \alpha_1 + \alpha_2 + \alpha_3$. The representation of this conjugacy class is in the figure below.

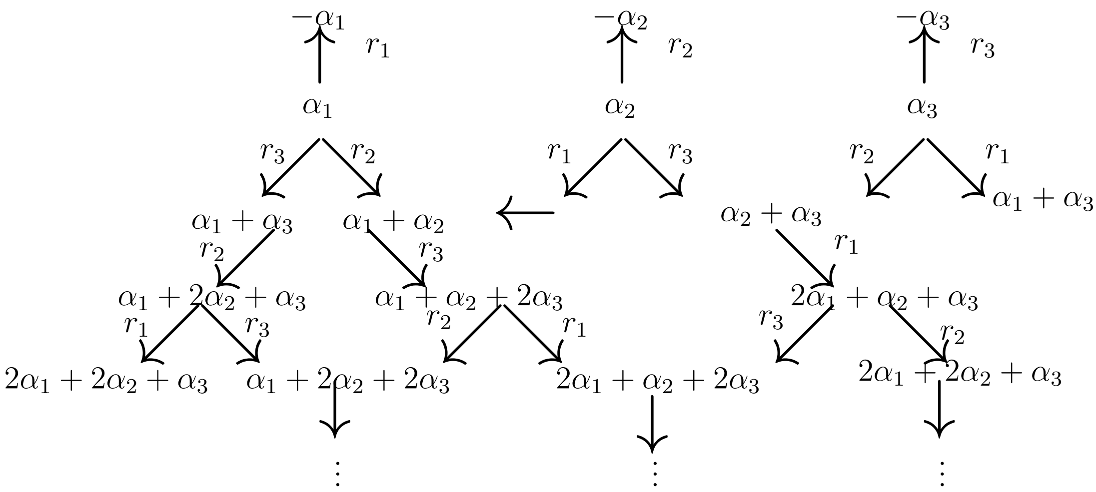


Figure 5.3: $\alpha_1^{\tilde{A}_2}$

Hence, the commuting involution graph $\mathcal{C}(G, X)$ where $X = r_1^G = r_2^G = r_3^G$, is disconnected by Lemma 1.4.4. To see this, $\langle \alpha_i, \alpha_j \rangle = \frac{-1}{2} \neq 0$ and $\langle k'', \alpha_i \rangle = 0$ for all $\{i, j\} \in \{1, 2, 3\}$, so $\langle \pm\lambda'' k'' \pm \alpha_i, \alpha_j \rangle \neq 0$ where $\{i, j\} \subset \{1, 2, 3\}$.

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