The infinite in early modern philosophy

THE INFINITE IN EARLY MODERN PHILOSOPHY

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2018

The work presented in this thesis is my own

ABSTRACT

The Infinite in Early Modern Philosophy: Berkeley's Objections to the Calculus and the Implications for Realism in the Philosophy of Mathematics

The calculus was developed in the seventeenth and eighteenth centuries in order to solve certain mathematical problems. In 'The Analyst' Berkeley gave his objections to the calculus. In particular, he objected to its dependence upon infinitesimals. He argued that these infinitely small distances were contradictory. Further he objected that the calculus, despite leading to true conclusions was not genuine science. So, although the calculus worked, it raised philosophical problems.

I will look at Berkeley's objections, both logical and metaphysical and explore how philosophically problematic they are. I will then consider what consequences arise from these objections. Specifically, I will look at the consequences for mathematical realism, and aim to answer the following questions:

If Berkeley's philosophical arguments are valid how can we explain that the calculus enables us to derive useful results? If we rely on non-entities such as infinitesimals in our mathematical proofs, does this mean that platonism cannot be the true metaphysical framework? If we reject platonism, does the dependence upon non-entities in the proofs of the calculus mean that all forms of mathematical realism are off the table? Or, is this methodology consistent with non-platonic forms of mathematical realism such as truth-value realism. Or, must we look to an alternative picture, such as fictionalism?

Contents

The Infinite in Early Modern Philosophy: Berkeley's Objections to the Calculus and the Implications for Realism in the Philosophy of Mathematics

1. Introduction:

The early modern period was a time of great scientific (including mathematical) and philosophical development. One of the major achievements was the invention of the calculus, which was a considerable breakthrough in the solution of certain mathematical problems. However, Berkeley, a leading empiricist philosopher of the period, raised certain objections to the methodology of the calculus, specifically to its utilisation of infinitesimals, differences which are infinitely small. These objections were of both a metaphysical nature (as to the ontological status of these entities) and of a logical nature (as to their coherence or lack thereof in the proofs of the calculus). Further, the calculus, despite leading to true conclusions was not, according to Berkeley, genuine science. So, although the calculus worked, it raised various philosophical problems.

In the course of this thesis I shall look at the methodology of the calculus and describe and assess Berkeley's philosophical objections to it, and explore how far-reaching they are. I will then consider what consequences arise from these objections. Specifically, I will look at the consequences for mathematical realism and evaluate three leading positions from the contemporary metaphysics of mathematics to investigate whether any of them can answer Berkeley's objections.

In chapter 2, I research the history of the calculus as developed by Newton and Leibniz in separate but contemporaneous endeavours. In my introduction of the calculus

I talk briefly about why it was such an important mathematical development. I then go on to discuss the bitter priority dispute fought between Newton and Leibniz in their attempts to be recognised as the sole inventor of the calculus. In the course of this historical discussion I mention some of the wider philosophical output of both protagonists, and other philosophical disagreements which they have had. I briefly discuss their differing approaches to the calculus, as well as giving an account of the priority dispute itself. Then the methodology of the calculus is explained and an example is given from Newton's calculus.

In chapter 3, I introduce Berkeley's principal objections to the calculus. These form three basic patterns: metaphysical objections, logical objections and a compensation of errors thesis. The metaphysical objections deal with the nature of infinitesimals, and the primary focus is on the fluxions of the Newtonian calculus. Berkeley argues that fluxions are not finite, and are therefore inconceivable. He believes that it is hard enough to conceive of the smallest distance or time, and that it is impossible to conceive of increments of space and time before they become finite. The logical objections analyse two of Newton's proofs and it is argued that these proofs treat an infinitesimal as both having and lacking a quantity. The first proof breaks a standard rule for finding the difference in areas between shapes, and results in a different answer from the accepted one. The fact that it is different by an infinitesimal amount is not deemed acceptable as an explanation. The second proof contains premises that assume that the quantity of an infinitesimal is both greater than and equal to zero. As both premises can't be true the proof is flawed. The compensation of errors thesis argues that the calculus only yields true results due to the fact that there are two compensating errors in some of the proofs. In essence two of the expressed quantities in the proof are too large to the same degree, but they cancel each other out in the course of the calculations of the proof. For this reason, Berkeley argues that the calculus can get at truth, but not science. I then go on to argue that these objections raise some interesting metaphysical questions about the workings of the calculus, and more broadly about the nature of mathematical objects, (which I further address in chapter 5 of the thesis.) The chapter finishes with a brief look at some objections from Berkeley's contemporaries and his response.

In chapter 4, I look at Berkeley's motivations for raising the objections to the calculus. At least some of the motivations for raising the objections come from his broader philosophical outlook and his philosophy of religion. Firstly, Berkeley was concerned that some mathematicians had rejected the principles of Christianity, and some of their admirers were following suit. This rejection had been put forward on the grounds that the principles were incoherent, and Berkeley wanted to show that mathematics was not always metaphysically and logically so sound either. This strategy then could be used to argue that there was no greater reason for accepting mathematical truths than for accepting the principles of Christianity. The second motivation comes from Berkeley's brand of idealism; immaterialism. Berkeley believed that the world was essentially immaterial and that there was a (finite) limit to how small anything could be. Infinitesimals presented a problem for these beliefs. The third thing to consider is Berkeley's anti-abstractionism and the notion that abstract entities don't exist, and finally we look at his acceptance of the use of other mathematical entities that can be deemed to be at least as problematic: imaginary numbers.

In chapter 5, I introduce the modern debate about realism in the philosophy of mathematics and discuss three leading metaphysical positions; platonism, which accepts the existence of mathematical objects, fictionalism, which does not and truth-value realism, which is agnostic on the question. I briefly explain each position, first generally, and then consider how it may deal with infinity and infinitesimals more specifically. Next, I present some initial observations about how each position may be applied to the objections. I then go on to analyse the three positions, firstly in terms of Berkeley's underlying motivations, and then in terms of his objections themselves. I argue that none of the positions are wholly compatible with Berkeley's immaterialism, with the possible exception of fictionalism.

Turning to the objections, I argue that platonism fails to overcome both the metaphysical and logical objections, but does not specifically address the compensation of errors thesis. With fictionalism, the metaphysical objections are solved, but the logical objections still present a problem for the consistency of the 'story' of mathematics. The compensation of errors thesis, is, I argue not a problem for some varieties of fictionalism, which allow that some of the assumptions we build into our scientific theories are known to be not actually true, although there may be other difficulties. Truth-value realism, too, can cope with the metaphysical objections; it is ontologically neutral, so does not require the existence of mathematical entities. The logical objections remain problematic, for similar reasons than have been found in the case of fictionalism. It is less clear whether the compensation of errors thesis can be handled by truth-value realism, as the position makes no obvious provision for it. Finally, therefore, I conclude that while the compensation of errors thesis is only directly tackled by one approach, the metaphysical objections can be solved by two approaches, but that the logical objections potentially remain an obstacle for all three.

2. The Calculus:

The calculus was developed in the seventeenth and eighteenth centuries, independently by both Newton and Leibniz, in order to solve certain mathematical problems, and, especially in Leibniz's characterisation, it utilises infinitesimals. (However, the use of infinitesimal slices actually goes back to Archimedes who utilised them to calculate the volume of a sphere.)[1] The problems the calculus was designed to solve fall into two groups. The first relates to the measurements of curves: how to determine the area of curved figures, how to determine the slope of a tangent to a curve at a point, etc. The second relates to the idea of the continuous variation of one quantity with respect to another: how to analyse this variation, how to determine its rate, etc. An example of this continuous variation, given by A. W. Moore, is that if one object moves away from another, while constantly accelerating, then both its distance from the other object and its speed increase continuously with respect to time.[2]

The calculus was a considerable advance on what had gone before. Prior to its discovery solutions to such problems relied upon the theory of ratios and proportions found in Euclid, and would involve taking a given magnitude and constructing another magnitude bearing the desired ratio to it. This approach was finitistic in character and thus prevented the use of infinitesimal magnitudes. This gives rise to the method of exhaustion, whereby an unknown ratio between two magnitudes is determined by considering sequences of known quantities approximating the unknown to within a desired degree of accuracy. Such a technique is employed, for example, in the Euclidian proof that the ratio between the areas of two circles is the same ratio as the square of

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[1] Ian Stewart, *From Here to Infinity: A Guide to Today's Mathematics*, Third Edition, (Oxford and New York: Oxford University Press, 1996), pp.70-78
[2] A. W. Moore, *The Infinite*, Second edition, (Oxford and New York: Routledge, 2001), p. 57

their diameters. But this method has limitations, which make it difficult to apply to more complex curves, and these limitations inspired mathematicians to develop infinitesimal methods such as the calculus.[3]

2.1 The History of the Calculus: Newton, Leibniz and the Priority Dispute:

While it is now received wisdom that Newton and Leibniz invented the calculus independently of each other, this was not always the case. During their lifetimes, and for a period after, the credit for its invention was hotly contested between the two mathematicians and their followers, culminating in a priority dispute, which saw the Royal Society crediting Newton (who was by then its President) as the sole creator, in 1712. Although each man developed his own distinctive versions of the calculus they were both drawing on the work of other mathematicians who had gone before them, in particular Descartes and Fermat. Newton made his discoveries in the mid-1660s, about a decade before Leibniz, and completed most of a treatise-length account by 1671,[4] but Leibniz was first to publish. Leibniz published a brief outline in 1684, and Newton in 1704 (appended to his *Optiks*).[5] There were differences in their approaches, but both versions of the calculus were reliant on infinitesimals, which were disputed philosophically, not least, as we shall see in this thesis, by George Berkeley. There had also been some communication between them (some of which would become key material drawn on in the dispute), and therefore each was, at least to some degree, aware of the other's work in the field. Ill-feeling between Newton and Leibniz had been

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[3] Douglas M. Jesseph, Berkeley's Philosophy of Mathematics (Chicago: University of Chicago Press, 1993), pp. 124-129

[4] George Smith, "Isaac Newton", The Stanford Encyclopedia of Philosophy (Fall 2008 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/fall2008/entries/newton/ (Accessed 7 October 2015)

[5] A. Rupert Hall, *Philosophers At War: the Quarrel between Newton and Leibniz* (Cambridge: Cambridge University Press, 1980), p. *xii*

brewing for some time, but this escalated after John Keill, a follower of Newton, wrote an article, which appeared in *Philosophical Transactions* in 1710, a journal which would go on to publish Newton's review of the Royal Society's report, in which he accused Leibniz of plagiarising Newton, prompting Leibniz to seek redress.[6]

The dispute was situated in a very individualistic time when scientific prowess was rigorously defended, and achievements thought of as being purely a matter of personal merit, rather than being a social phenomenon. (Some institutions such as the Collège Royale in Paris even encouraged fierce competition where to be defeated in an argument could lead to one losing one's academic post.) The phenomenon of convergence, which recognises that it is in fact quite common for more than one scholar to arrive at an independent solution to the same problem in identical or closely similar ways, was not recognised at the time, despite its inevitability in an active research programme. All this meant that priority disputes over scientific and mathematical discoveries were quite common. Further, many people were working in this area, trying to find solutions to such problems as the general method of tangents and the quadrature of particular curvilinear areas, which the calculus was designed to solve. The dispute was however quite shocking at the time, due to the monumental status of the two protagonists, but it appears even more shocking to us today.[7]

2.1.1 Newton:

Newton of course is remembered primarily as a mathematician and a physicist, but his contribution to philosophy should not be understated. Janiak discusses his

[6] Smith
[7] Hall, pp. 3-7

importance to philosophy and of philosophy to his work. Firstly, Newton was working within the tradition of natural philosophy (before the notion of a physicist had been employed), which at the time was engaged in overthrowing many of Aristotle's ideas about the natural world, as well as the techniques employed by Aristotelians, and developing new mathematical, conceptual and experimental methods. Secondly Janiak tells us that Newton's work had a profound effect on eighteenth century philosophers, such as Berkeley and Hume. Further, Newton's most influential work, *The Principia*, is credited with effecting a branching within natural philosophy, which led to the development of the two separate disciplines of mathematical physics and philosophy.[8] George Smith also credits Newton's most influential work, *Principia*, as giving rise to the sub-discipline of philosophy of science.[9] Newton's work on gravity in the *Principia*, and Leibniz's views on it also came to bear in the priority dispute on the calculus.

Working in Cambridge and Lincoln, Newton began his work on the calculus in 1665-6, having previously mastered the binomial series expansion, which was an important step in this work. The motive for this step came from finding the area of a semi-circle, the first of the two types of problems the calculus was designed to solve, and his work enabled him to generalise the problem.[10] He also applied it to calculating the areas of the circle and hyperbola in infinite series and obtained expressions in an infinite series.[11] After turning certain expressions into infinite series he then went on to find the

[8] Andrew Janiak, "Newton's Philosophy", *The Stanford Encyclopedia of Philosophy* (Summer 2014 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/sum2014/entries/newton-philosophy/
(Accessed 7 October 2015)
[9] Smith
[10] Ivor Grattan-Guinness, The Norton History of the Mathematical Sciences, (London and New York: W. W. Norton and Company, 1998), p. 243
[11] W. W. Rouse Ball, *A Short Account of the History of Mathematics,* Third Edition (London: MacMillan and Co, 1901), pp. 337-338

inverse of such series. He also used infinite series to measure the quadrature of curves.[12] A proof for this will be discussed at length in section 2.1 of this thesis.

In his development of the calculus, first he obtained derivatives, went on to understand that integration was the inverse of differentiation, began to master the process of integration by means of infinite series, and by mid-1665, was setting down the standard procedures for integration. He had begun working on the idea of differences, in almost the same way that Leibniz later would, but this was dropped in favour of the notion of fluxions, a fluxion being a derivative of a continuous function. Hall explains that this notion comes from thinking of a variable as flowing from one value to another and considering its rate of flow, which is presented as a motion or speed, as opposed to thinking of a variable quantity proceeding by many infinitely small steps.[13] This fitted with Newton's interest in how the issues related to his scientific discoveries, especially his work in mechanics, and thus Newton approached the calculus as applied to motion and velocity, describing it as the method of fluxions, defining a fluent as a quantity that varies over time and a fluxion as the rate at which it does so.[14] Specifically Newton imagined a variable as flowing from one value to another and considered its rate of flow (a motion or speed). He then went on to write a short memoir 'How to Draw Tangents to Mechanical Lines'. Hall describes the procedure highlighted in the memoir as follows:

> Newton is generalising the Cartesian notion of coordinates so as to make the x axis and the y axis both change through infinitesimal intervals of time …... Any curve can be simulated by properly matching a changing flow of x to another changing flow of y: and if at any instant, we halt the double flow, the two (now static) rates define a straight line, which is the tangent to the curve at that point.[15]

[12] William Dunham, *The Calculus Gallery* (Princeton, N.J.: Princeton University Press, 2005), pp. 6-15
[13] Hall, pp.13-14
[14] Moore, p. 64
[15] Hall, p. 14

Then Newton went on, in a draft entitled 'To find the Velocities of Bodies by the Lines they Describe' to define the general rules for finding the fluxions related to given equations (the same rule that Leibniz would define as the foundation of the differential calculus). At this stage although the word 'fluxion' is not used, but the symbol which would go on to represent a fluxion, *o*, is.[16]

In 1666, a 'velocity' becomes a 'motion', and all the previous work on the calculus gets consolidated into an incomplete 48-page treatise, which Newton dates as October 1666. (The term, 'fluxion' is still not used). Thus, we have a record of sufficient, as then unpublished, documentation to demonstrate Newton's invention of the calculus by 1666, almost nine years before Leibniz.

Shortly after, 'fluxion' does begin to appear in his terminology: in 1671, he wrote, but again failed to publish, 'Treatise on the Methods of Series and Fluxions', the first work to explicitly state his concern with the calculus in the title, still a few years before Leibniz's work got going.[17] Eventually he publishes 'On the Quadrature of Curves' appended to *Optiks* in 1704.

2.1.2 Leibniz:

Leibniz is best remembered as a philosopher, although his opus encompassed many other disciplines including mathematics, and even law. He arrived on a diplomatic mission for a four-year stay in Paris in 1673, where he met many leading figures in the field of natural philosophy to which he had recently turned after gaining access to some

[16] Hall, pp 14-15
[17] Hall, pp.17-18

modern material. His philosophical work has been described as a reaction to two sets of modern opponents: Descartes and his followers, who focused on the Cartesian account of corporeal substance, and Hobbes and Spinoza who advanced, or were thought to advance materialism, necessitarianism and atheism.[18] [19] It was in Paris that Leibniz was first tutored by Christiaan Huygens (with whom he would go on to have a close association) in mathematics, philosophy and physics. And it was in Paris that he was able to read the unpublished mathematical manuscripts of Pascal, which inspired his differential calculus and his work on infinite series.[20]

Leibniz's approach to the calculus was also based on the work of Descartes. His interest in the calculus stemmed from his conviction that all change in nature is continuous, and thus he had a more analytic approach to the subject. It is his notation of the calculus that has survived to be used today.[21]

Steps en route to inventing and subsequently publishing his work on the calculus (again like Newton several years after discovery) included the transmutation theorem used to find the area under a curve and the Leibniz series. The Leibniz series applies the transmutation theorem to a particular curve: he considers a circle of radius 1 and centre (1,0) and lets the curve be the quadrant of this circle, whose area is $\pi/4$. This allows him via differentiation of the circle's equation to create the series which sums to $\pi/4$.

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[18] Brandon C. Look, "Gottfried Wilhelm Leibniz", *The Stanford Encyclopedia of Philosophy* (Spring 2014 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/spr2014/entries/leibniz/ (Accessed 7 October, 2015)

[19] Although it is not uncontroversial as to whether either man was truly an atheist, especially in a modern-day understanding of the term.

[20] Look

[21] Moore, p. 64

In 1684, he finally published 'A New Method for Maxima and Minima and also Tangents which is Impeded Neither by Fractional Nor by Irrational Quantities and a Remarkable Type of Calculus for this', his introduction to differential calculus, with a paper on integral calculus following two years later.[22]

2.1.3 The Priority Dispute:

In addition to the calculus (of which there already had been some murmurings) Newton and Leibniz quarrelled on the cause of gravity. Essentially this came down to whether the action of one physical object could have an effect on another without any physical contact. Newton believed that it could (although for this to obtain a non-material substance would be needed as an agent to gravity in order that the two physical substances could interact[23]) his work was therefore both ground-breaking and controversial. This quarrel was also underpinned by differences in methodology. Leibniz was working in what was known as the method of hypotheses whereby the hypotheses are given first and observable conclusions were deduced from them, which Newton opposed in favour of empirical study (or as Newton puts it in his review of the Royal Society's report 'experiments and phenomena') where each element of a theory was decided by specific phenomena.[24] This quarrel had been the main bone of contention between the two protagonists and had been raging since at least 1690,[25] long before Keill wrote his damning accusation of Leibniz in 1708.

[22] Dunham, pp. 20-34. Dunham also tells us that Leibniz's case in the priority dispute was not helped by earlier criticism over this series. Unbeknownst to Leibniz, Gregory had already discovered a similar series, meaning that Leibniz was already viewed with some suspicion in England.

[23] Isaac Newton, Original letter from Isaac Newton to Richard Bentley, (A 4th letter from Mr Newton), *The Newton Project*, (source 189.R.4.47, ff. 7-8, Trinity College Library, Cambridge, UK, ed. by Professor Rob Iliffe, 2007, http://www.newtonproject.ox.ac.uk/view/texts/normalized/THEM00258 (Accessed 24 January 2018)

[24] Janiak

[25] Hall, p. 152

Keill's accusation, which appeared in print in 1710, added fuel to the fire and brought the Royal Society into the dispute. In an article in *Philosophical Transactions* entitled 'On the Laws of Centripetal Force' Keill wrote:

> 'All these things follow from the nowadays highly celebrated arithmetic of fluxions, which Mr Newton beyond any shadow of a doubt first discovered, as any one reading his letters published by Wallis will readily ascertain and yet the same arithmetic was afterwards published by Mr Leibniz in the *Acta Eruditorium* having changed the name and the symbolism'[26]

It seems likely that Keill was aggrieved by Leibniz's criticisms of forces of attraction in the *Acta Eruditorium*, of which he himself as well as Newton had been on the receiving end.[27] It is also apparent that from 1710 the two quarrels would become fused together, until and even beyond the Royal Society's verdict.[28]

On 21 February 1711 Leibniz wrote a letter to the Royal Society in which he demanded that Keill apologise for his accusation. Crucially Leibniz stated that he had never heard the name 'calculus of fluxions', nor seen Newton's notation before they appeared in Wallis's *Works* over a decade after he had published his own work.

An apology, however, was not forthcoming. Newton by now believed that similar accusations against him could be found in the pages of *Acta Eruditorium*,[29] and after Keill had argued his case he was asked by Newton to vindicate himself in writing, rather than apologise. Newton's prior date of discovery not in doubt, Keill's justification of his accusations of plagiarism relied upon the contents of two letters sent by Newton to

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[26] Hall, p. 145
[27] Ibid
[28] Hall, p.164
[29] Hall, p. 169

Leibniz in 1676.[30] Crucially neither of these letters contained an explicit theory of fluxions, or even mentioned the word. Hall argues that this justification rested on two claims. Firstly the letters contained 'hints' which were sufficiently understood by Leibniz, and it was 'not contrary to reason that these gave him an entrance into the differential calculus'.[31] Secondly because his invention of the calculus enabled Newton to discuss certain mathematical procedures and examples with Leibniz, these procedures and examples would in turn enable a competent person to reconstruct the calculus, from which they were descended.[32] The first claim was so vague as to be hard to prove, but then it was also hard to disprove. The second claim, however, makes an assumption that there is only one general rule that could lead to the examples Newton gave in his letters, so anyone working backwards would arrive at it. This is by no means certain, and as a generalisation, manifestly false and tantamount to saying that it is as possible to deduce a general rule from particular cases as it is particular cases from a general rule.[33] A copy of Keill's vindication was sent to Leibniz, who not unsurprisingly responded with a second letter to the Royal Society, stating that he had never challenged Newton's right to claim the discovery independently, and that Newton should ask Keill to back down. An anonymous review of Newton's tract on quadrature written by Leibniz in 1704 belies this claim, however, implying that Newton got the idea from Leibniz.[34] In any case by this time Newton (who had previously not questioned Leibniz's independent discovery) was undergoing a change of attitude, and he was no longer willing to share the honours.[35]

[30] Hall, pp. 169-171
[31] Hall, p. 170
[32] Hall, pp. 170-173
[33] Hall, pp. 173-174.
[34] Ball, p. 369
[35] Hall, p. 177

Newton presented his side of the story to the Royal Society, claiming his right as 'first author'. Notes from a speech indicate that he claimed he had only recently been made aware of articles in the *Arta Eruditorum*, which claimed he had borrowed from others, and that he was owed as much of an apology over this as Leibniz was from Keill. In a draft, he asserted that Leibniz had learned from the 1676 letters to him that he had written a treatise of the methods of converging series and fluxions before he had heard of Leibniz's differential method.[36]

Following this the Royal Society appointed a committee to look into the dispute. The report, published (with extracts from 'relevant' documents) as the *Commercium Epistoclicum*, and written by Newton himself, concluded that he was 'the first inventor' of the calculus and found Leibniz guilty of concealing his knowledge of the prior, relevant achievements of others, and it was concluded that the 1670s correspondence had been of vital importance for his publication of the calculus. Newton's case relied on a letter of 1672 (which contained his tangent rule) to Collins, which he alleged, erroneously, that Leibniz had seen in early 1676, rather than the 1676 letters that Keill used. In reality argues Hall, by the time Leibniz did see the letter it was too late for it to have had an impact on is work. [37]

Leibniz responded with a 'anonymous' leaflet, known as the *Charta Volans*, which included a supposedly impartial opinion on the *Commercium Epistoclicum*, actually written by his friend and supporter, Johann Bernoulli. In this leaflet, Newton was now openly accused of plagiarism, with a claim that Newton's earlier work was not in fact the calculus, but rather was concerned with 'advancing geometry synthetically or

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[36] Hall, pp. 178-179
[37] Hall, pp. 179-181

directly by infinitely small quantities'. Leibniz's claim to primacy rested in the public history of the calculus's development in association with Leibniz himself.[38] By now both sides had dug their heels in and an impasse had been reached.

2.1.4 The Aftermath:

There was some more to-ing and fro-ing between the two sides with responses and remarks by various authors, including Keill and Bernoulli, published in journals.[39] But Newton and Leibniz carried on attacking each other indirectly too.

In February 1715 Newton wrote an anonymous review in *Philosophical Transactions*, of the Society's report on the priority dispute where he contrasted his methods with those of Leibniz:

> It must be allowed that these two Gentlemen differ very much in Philosophy. The one proceeds upon the Evidence arising from Experiments and Phenomena, and stops where such Evidence is wanting; the other is taken up with Hypotheses, and propounds them, not to be examined by Experiments, but to be believed without examination. The one for want of Experiments to decide the Question doth not affirm whether the Cause of Gravity be Mechanical or not Mechanical; the other that it is a perpetual Miracle if it be not Mechanical.[40]

This is obviously referring to the dispute about the cause of gravity and not the calculus, highlighting how the two issues have become interlinked, and is in turn a facetious dig responding to a criticism that Leibniz gave in a published letter of 1712 to Nicolas Hartsoeker.[41] Nonetheless it is all too easy to see Newton's central position in the

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[38] Hall, pp. 199-200
[39] Hall, pp. 202-212
[40] Smith
[41] Janiak

resolution of, and reporting on, the debate as an abuse of power, especially in light of current thinking on the invention of the calculus.

Leibniz meanwhile was resurrecting the gravity argument in an attack on Newton's philosophy, including a provocative attack on the religious consequences to Newtonian (and Lockean) thinking in a letter to princess Caroline of Wales in November 1715.[42] Although this does not appear to have presented any serious problems for Newton, it appears quite vindictive. This attack led to a correspondence between Leibniz and Samuel Clarke with Clarke defending Newton. Leibniz died in 1716, but the debate raged on after both his and Newton's death (in 1727). Finally, and certainly by the early 20th century it became commonplace to accept that they were both independent creators of the calculus. Ball, while not of this view, does however tell us in 1901 that this was indeed the prevalent opinion at the time.[43]

2.2 Methodology of the Calculus:

Ian Stewart explains the notion of an infinitesimal by appealing to the subdivision of a line. He asks if we can think of a line as being a sequence of points. If a line can be subdivided a definite amount, then the points would follow each other like beads on a string; after each point there would be a unique next point. But if we try to ask which real number corresponds to the point 'next to' the origin we get into difficulty. We want this number to be 0.00000……, with a 1 appearing in the very last place, but there is no last place. For any decimal number of the form 0.0000…01, we can always think of a smaller number.

[42] Janiak
[43] Ball, p. 357

Stewart says that there are two ways out of this dilemma. We can assert that there really is a next number larger than 0, but that it is *infinitesimally* larger than 0, which makes it smaller than anything of the form 0.0000…01. Or we can adopt the position that no such number exists, in which case we can think of the line as being subdivided indefinitely, and accept that there are no 'ultimate atoms'.

In the latter position the consequence is that we can't think of a line as being made up of points strung together in order, but at the same time we know, by Euclid, that any position on a line is a point. (We just draw two lines with one crossing the other, and where they meet is a point.)[44] Here we have a contradiction. But if we accept the former position we arrive at a contradiction too: if we take the variable x as the smallest number greater than 0, we end up in a position where $x/2$ must be smaller than x but greater than 0. Therefore, x can't be the smallest number greater than 0 on pain of contradiction.[45]

However, despite this difficulty, infinitesimals had proved useful. Long after Archimedes had used them to determine the volume of a sphere Nicholas of Cusa. in the fifteenth century, used a similar approach on the area of a circle. By slicing up a circle like a pie, he created a series of near triangles. If we then ignore the curves and work out the area of the triangles left by using the straight lines we can draw across the bottom of the triangles where the curves begin, to form the bases, and add the areas together the total will be approximate to πr^2, which we know to be the area of a circle (the height of each triangle is the radius of the circle). If we could make the 'triangles' infinitesimally

[44] Stewart, pp. 72-73
[45] Stewart, p. 73

small, then there would be no error.[46] Uses like this meant that infinitesimals were a long way from being abandoned, and they duly ended up being utilised in the calculus as it was developed by both Newton and Leibniz.

2.2.1 Newton's Calculus Applied to an Example of a Parabola:

Stewart demonstrates Newton's calculus as being put to use in solving the problem of how to draw a tangent to a given curve. He gives the example of a parabola whose equation is $y=x^2$ and shows how Newton's calculus allows us to work out the slope of the tangent at x. Stewart gives the following characterisation of Newton's work:

> Let x increase slightly to $x + o$. Then x^2 changes to $(x + o)^2$. The rate of change is therefore the ratio of the difference between the squares to the difference in the value of x, namely
>
> $$[((x + o)^2 – x^2]/[(x + o)– x],$$
>
> which simplifies to yield
>
> $$[2ox + o^2]/o = 2x + o.$$
>
> Let o approach zero; then the slope approaches $2x + 0 =2x$. This is the slope of the tangent, or, as Newton put it, the fluxion of the fluent x^2.[47]

This works on any parabola. If a parabola has the equation $y = x^3$, the fluxion will be $3x^2$.[48] Newton defined a fluent as a quantity that varies over time, and a fluxion as the rate at which it does so, (and which we now call a derivative or a differential).[49]

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[46] Stewart, pp.73-74
[47] Stewart, pp.74-75
Newton introduces the variable o (Leibniz used δ).
[48] Stewart, p. 75
[49] Moore, p. 64

3. Berkeley's Objections to the Calculus:

Douglas M. Jesseph separates Berkeley's objections into metaphysical and logical.[50] In addition, a compensation of errors thesis tackles the issue of how the calculus arrives at truth, but not science.[51] I will follow this distinction and will talk about each in turn. In 'The Analyst' the metaphysical objections occur first, so I shall begin with them.

3.1 Metaphysical Objections:

As Moore discusses, despite its brilliance the notion of an infinitesimal difference on which the calculus rests is flawed, an infinitesimal being not quite nothing, and not quite something. Leibniz and Newton both made use of the infinitesimally small, but were aware of the difficulties of relying on such a notion. Leibniz thought of it as a useful façon de parler, while Newton made suggestions for how to eliminate infinitesimals by considering them as limits, (foreshadowing modern-day calculus).[52] So, although the calculus worked, it raised problems, the infinitely small distances or infinitesimals were contradictory, and this was heavily criticised by Berkeley. This is an objection primarily about the nature of infinitesimals themselves and that an infinitely small object cannot exist and can therefore not be given an ontological status. A metaphysical mystery then obtains at the heart of the calculus.

Additionally, Berkeley would have seen this as a methodological problem, with the two fields of mathematics and the principles of Christianity subject to different

[50] Jesseph, pp. 185-189
[51] Jesseph, pp. 199-200
[52] Moore, pp. 63-66

conditions of proof. Mathematics was supposed to be rational and mystery free, unlike religion and the Principles of Christianity, which do, for Berkeley, have a mysterious nature, and can be underpinned by faith. As he states in the Philosophical Commentaries: 'for tho the Principles may be founded in Faith yet this hinders not but that legitimate Demonstrations might be built thereon'.[53]

Berkeley's attack can be found in his 1734 essay, 'The Analyst' in which he asks:

> And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?[54]

As Stewart puts it, Berkeley's arguments are that if *o* is not exactly 0, then the answer we get is very close, but wrong. On the other hand, if *o* is 0, then we can't divide by it, so the answer doesn't make sense[55]

Berkeley begins in §§3-8 of 'The Analyst' by describing the *object* of the mathematical analysis.[56] At the start of section §3, he identifies the method of fluxions as the key to how mathematics unlocks the secrets of geometry, and how it has been able to make progress since the Greeks, and he identifies fluxions as the main tool of contemporary geometry.[57] Following Newton's 'Introduction to the Quadrature of

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[53] George Berkeley, 'Philosophical Commentaries' 1871, Reprinted in Berkeley, *Philosophical Works Including the Works on Vision*, ed. by Michael R. Ayers, (London: Everyman, 1996), Reissue with Revisions and Additions, pp. 305-412, Commentaries 584, p. 376
[54] Berkeley, 'The Analyst; or, a Discourse Addressed to an Infidel Mathematician, wherein it is Examined whether the Object, Principles and Inferences of the Modern Analysis are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith', 1734, Reprinted in *The Works of George Berkeley Bishop of Cloyne*, Vol. 4, ed. by A. A. Luce and T. E. Jessop, (London: Thomas Nelson and Sons Ltd, 1951), §35, p. 89
[55] Stewart, p.75
[56] *The Works of George Berkeley*, p. 56
[57] 'The Analyst', §3, p. 66

Curves', Berkeley gives his description of the calculus by starting with the observation that lines are generated by the motion of points, planes by the motion of lines and solids by the motion of planes. In describing lines in this way, i.e. as generated by the motion of points, it is likely, following Newton as he does, that he is alluding to the representation of the progress of a single point by a line in a standard graph with an x and a y axis. This is one application of the calculus. Under this application we get the plane by extending this line through a third z axis, and the solid by projecting the plane through 3-dimensional space. Taking a line to represent the motion of an object (the y axis being distance travelled and the x axis being time), he says that the 'quantities', or distances travelled increase or decrease depending on their velocity.[58] (Quantities are Newton's 'fluents', and are described by him as being generated by a continual motion).[59] He goes on to tell us that we can determine these quantities from the velocities the line represents.[60] If this line has some curvature,[61] we will need the calculus to do this. Velocities which are subject to the calculus are called 'fluxions'(differentials) and the 'quantities' (distances travelled) are called 'flowing quantities', which is to say that they change depending on which position on the line we are analysing. The 'fluxions' are *nearly* the same increments as the 'flowing quantities', but they tend towards zero, and are in fact the same increments as can be generated in the immeasurable parts of time during which that part of the motion is taking place. Sometimes we will not be considering the velocities, but the instantaneous intervals of undetermined distance travelled; these we call moments.[62]

[58] Ibid.
[59] Newton, 'Introduction to the Quadrature of Curves', trans. By John Harris, ed. by David R Wilkins, 2002, http://www.maths.tcd.ie/pub/HistMath/People/Newton/Quadratura/HarrisIQ.pdf (Accessed 2 June 2015)
[60] 'The Analyst', §3, p. 66
[61] A line is in fact defined as a curve with null curvature.
[62] 'The Analyst', §3, p. 66

Fluxions are concerned with these moments. In §4, Berkeley tells us that these moments are not finite. Moments are only the burgeoning origins or increments of finite 'quantities' or distances travelled, before they become finite particles or increase by a finite amount, or their disappearing conclusions after they have been finite. What we are concerned with, the 'fluxions' or differentials, are 'celerities' or rates of change, not proportional to the smallest finite particles (which we can't quantify), but to the instantaneous moments, and we only consider the 'proportion' to these instantaneous increments and not their actual magnitude.[63] This seems to correspond to the steps in Newton's calculus where he introduces the quantity $x + o$ and lets o tend towards 0 in order to calculate the differential. The difference between the two quantities $2x + o$ and $2x + 0$, i.e. $2x$ in the example from the previous section, is the proportion, which is of course an infinitesimal amount in all but name. We can also get fluxions of fluxions (or further differentials) by starting with the first fluxion or differential (which is velocity) as the 'fluent' or 'quantity', and from there we can generate the second differential (which is acceleration), and then the third differential etc.[64]

Berkeley then objects that we cannot even perceive very minute particles and that our imagination which derives from our sense will struggle to allow us to conceive of such concepts as the least particles of time, or the least increments that they generate. The situation worsens when we try to conceive of 'moments' the instantaneous increments we consider before they become finite. Still harder, is an ability to conceive of the velocities of such 'entities', or differentials, which he describes as 'objects at first

[63] Berkeley, 'The Analyst', §4, p. 66
[64] Berkeley, 'The Analyst', §4, p. 67

fleeting and minute, soon vanishing out of sight', and conception of the second, third etc. differentials, he argues, is beyond human understanding.[65]

After analysing Newton's version of the calculus, Berkeley, in §5 and §6, moves on to Leibniz, whom he claims is considered more intelligible. But Leibniz fares no better under Berkeley's scrutiny. Where Newton gives us fluxions, Leibniz talks openly of infinitely small amounts, called differences and infinitesimals, which Berkeley finds equally perplexing.[66]

Douglas M. Jesseph points out that underlying these metaphysical objections (on the grounds of inconceivability) is a familiar trait of Berkeley's epistemology, and that he is making two assumptions; that 'extremely minute' objects cannot be clearly apprehended by sense and that our imagination is derived from our sensations. Jesseph accepts that the first claim is unproblematic, but the second claim contains an underlying assumption that our mental faculties consist only of sense and imagination with no faculty for framing independent ideas through 'pure intellect'.[67] Jesseph draws on §1 of 'Principles of Human Knowledge' for this observation.[68] (I shall return to this point when I consider the context of Berkeley's wider philosophy in relation to these objections in Chapter 4.)

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[65] Ibid.
[66] Berkeley, 'The Analyst', §§5-6, pp. 67-68
[67] Jesseph, p. 186
[68] Berkeley, 'A Treatise Concerning the Principles of Human Knowledge', Second Edition, 1734, Reprinted in Berkeley, *Philosophical Works Including the Works on Vision*, ed. by Michael R Ayres, (London: Everyman, 1996), Reissue with Revisions and Additions, pp. 71-153, 1, §1, p. 89

It may be the case that our imagination does depend in some way on our perceptions, but Berkeley's formulation of this idea is at odds with the standard mathematical epistemologies of his time.

An additional issue with this objection is that it may be at odds with Berkeley's views and apparent acceptance of other mathematical objects that could be considered ontologically suspect, such as imaginary numbers. In *Alciphron,* Berkeley accepts the role of imaginary numbers, 'for instance the algebraic mark, which denotes the root of q negative square hath its use in logistic operators'.[69] This objection is about the supposed nature of the objects, so why might imaginary numbers be different for Berkeley? It does not seem obvious that it wouldn't apply to both. I shall return to the issue of imaginary numbers when we consider Berkeley's philosophy beyond his objections in 'The Analyst'.

The metaphysical objection appears vulnerable then (and we shall go on to see that it is the most easily defused by considerations of approaches to realism in the modern philosophy of mathematics). More will need to be done to stand a chance of convincing mathematicians of his arguments. Thus in §8 of 'The Analyst', Berkeley alludes to the impossibilities and contradictions that will form the basis of his logical argument against the calculus.[70]

[69] Berkeley, 'Alciphron', partial reprint in Berkeley 'Alciphron in Focus', ed. by David Berman, (London: Routledge, 1993), The Seventh Dialogue, §14, p. 140
[70] Jesseph, pp. 186-189

3.2 Logical Objections:

In §§9-16 of 'The Analyst' Berkeley, using examples from Newton's *Principia* moves on to consider the principles of "this new analysis" using 'momentums', 'fluxions' or infinitesimals. (The metaphysical objections previously discussed are captured under what Berkeley calls considering its object).[71] Berkeley here introduces another term; 'momentum' which means the same as 'fluxion' (or what we call a differential today). No reason for introducing this new terminology, nor any indication as to why he is using the particular term 'momentum' is given, but it does seem plausible that this term is used because the quantities being considered, the fluxions, are derived from what Newton calls 'moments', so a 'momentum' is a 'fluxion', as it is derived from a moment. Further, Berkeley deliberately also uses the word infinitesimals here because he is going to argue that however Newton may choose to describe 'fluxions', they are in fact infinitesimals, and so he is introducing the correlation at the start of the argument. It is in this discussion, of the principles, that Berkeley's logical objections, which Jesseph considers much more compelling than the metaphysical ones, are brought to bear on the calculus.[72] These objections are about the inconsistency in the proofs in that the quantity assigned to the 'fluxions' or 'momentums' is supposed to be an (infinitely small) quantity at one stage of the proof and no quantity at all at another stage of the proof. It is therefore taken to be both something and nothing which is inconsistent.

Berkeley takes an example from Newton's *Principia* which he takes to be the main point of the method of fluxions: "to obtain the fluxion or momentum of the

[71] *The Works of George Berkeley*, p. 58
[72] Jesseph, p. 189

rectangle or product of two indeterminate quantities",[73] known today as the product rule. As Jesseph says in modern notation this rule states 'that given functions *f(x)* and *g(x)* the derivative of the product *f(x)g(x)* is *f'(x)g(x) + f(x)g'(x)*' '[74]

That is to say that the derivative of the product *f(x)g(x)* is (the derivative of the function *f(x)* multiplied by the function *g(x)*) plus (the function *f(x)* multiplied by the derivative of the function *g(x)*). From this rule we derive the rules for obtaining the derivatives of all other products and powers.[75]

In the *Principia* Newton gives a proof, which Berkeley describes as follows: Suppose the product of a rectangle AB is increased by continual motion and that the increments of the sides measure a and b respectively. We then consider the rectangle smaller by half the increment. At this point the sides measure (A – ½a) and (B – ½b), which means that (with expansion) the product or area of this smaller rectangle would measure AB – ½aB – ½bA + ¼ab. We then consider the original rectangle made larger by the other half of the increment. At this point the larger rectangle would measure AB + ½aB + ½Ba + ¼ab. We then subtract the area of the smaller rectangle from the larger rectangle, which leaves a difference of aB + bA, which is the increment (or moment) of the rectangle generated by the entire increments a and b.[76]

Berkeley then objects to this proof by pointing out that the correct method for calculating the increment of the rectangle AB is to take the incremented area and from it subtract the original area. This would give us (AB + aB + bA + ab) – AB, which leaves

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[73] 'The Analyst', §9, p. 69
[74] Jesseph, p. 190
[75] 'The Analyst', §9, p. 69
[76] 'The Analyst', §9, p. 70

us with aB + bA + ab. This is clearly larger, by the quantity ab. Berkeley says this has to be true in all cases regardless of the nature of the quantities involved. Claiming that ab is a quantity of infinitesimal size is, he argues, no defence for Newton's proof.[77]

Berkeley does, in §10, highlight the need for Newton to get rid of the quantity ab.[78] Newton's proof is also valid, and does demonstrate that the area derived from these calculations of the rectangles generated by using the two half a and b increments added together, as seen above, is the same as the area we get from the calculations using the increments a and b on the outside of the original rectangle, minus the area ab. In fact, if we visualise the rectangles we can quite easily see how these quantities match. But this comes at the cost of breaking the standard rule for finding the difference in areas between two shapes; i.e. that you simply subtract the smaller area from the larger one.

When Jesseph discusses this objection by Berkeley, he tells us that Newton in *Principia* declares that the moment of a flowing quantity is its momentary increment. Jesseph argues that if that is the case, then the moment of the product AB must be the difference between AB and (A + a) × (B + b), i.e. as Berkeley states it is the difference in area between the two original rectangles. Jesseph also criticises Newton for utilising the quantities ½a and ½b on two counts; firstly, that of taking the increment of the wrong product, and secondly, for introducing a supposition that we divide increments of

[77] Ibid

[78] Berkeley doesn't indicate why it is necessary for Newton to be rid of this quantity, but presumably it is because the distances represented by a and b in the various rectangles under consideration are designed to capture an instantaneous rate of change, so the full finite quantity would give a wrong representation of this and would fail to capture the fact that the rectangle is a representation of a continual motion. Therefore, the quantities generated by the increments are never going to be as great as those in the completed action.

negligible magnitude into parts, which he says is confusing. [79] In the latter case he is taking up a point that Berkeley makes in §11, where he criticises Newton's method.

These criticisms by Berkeley arise around Newton's use of infinitesimals. He points out that however Newton may describe these momentums or fluxions, they are in fact infinitesimals in the following argument:

1) Points or mere limits of nascent lines must be equal in magnitude as they are not actually quantities.

2) If a momentum is more than these initial limits, then it must either be finite or infinitesimal.

3) Newton has excluded the possibility that momentums can be understood to be finite.

4) Therefore, momentums must be infinitesimals.

Further, Berkeley argues that there just cannot be a quantity between finite and nothing without admitting of infinitesimals, and he states that if an increment is generated in a finite amount of time, it too must be finite.[80] Only an infinitesimal amount of time will allow you to generate an infinitesimal increment. These parts can have no magnitude

[79] Jesseph, p. 191

[80] This echoes Aristotle's discussion of the continuum in which a change happening corresponds to the time in which it happens. Berkeley's introduction of time is important because with the calculus we are measuring rates of change, and change is standardly regarded as needing time in which it can occur. In trying to measure an instantaneous rate of change, we are in essence capturing a snapshot of a certain change at a certain time, but without time there would be no rate of change to measure. Aristotle, *The Physics,* trans. by Robin Waterfield, (Oxford: Oxford University Press, 1996), (232b20), p.142

and therefore it makes no sense to say that they can be divided into smaller parts. In addition, Berkeley argues that it makes no sense to utilise quantities less than a and b to obtain the increment of AB and that Newton has failed to give his reasons for doing so.

Jesseph agrees that Berkeley is right in all of this, and that these arguments are pivotal in establishing Berkeley's claims that the calculus lacks rigour. This claim, Jesseph argues, is backed up by Newton's proof itself, which in fact demonstrates this lack of rigour. Berkeley's arguments in this section therefore correctly lead him to conclude that the procedures of the calculus are not properly demonstrated and that Newton's fluxions and moments are no different from the infinitesimals that Leibniz uses in his calculus.[81]

To further back up his arguments Berkeley goes on to look at another proof given by Newton; that of finding a fluxion of any power (what we today call the power rule). This rule states that the differential of a power of x is equal to the product of the exponent times x with the exponent reduced by 1, i.e. the differential of x^n is nx^{n-1}. In §12, prior to his discussion of this proof, Berkeley's adds an observation that you can't get rid of a point without getting rid of any points that depend on it, and introduces the following lemma to capture this observation:

> If with a view to demonstrate any proposition, a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be it self afterwards destroyed or rejected by a contrary supposition; in that case all the other points attained thereby, and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the demonstration.[82]

[81] Jesseph, pp. 193-194
[82] 'The Analyst', §12, pp. 71-72

As Jesseph points out this lemma is essentially saying that a proof must not contain contradictory premises. When moving on to the proof itself, in §§13 and14, Berkeley draws on his lemma to present his objection, accusing Newton of violating this principle, in his assumptions that the quantity *o* in the proof is both greater than zero and equal to zero.[83]

This proof utilises an infinite series, and starts with the premise that the quantity *x* flows uniformly, i.e. we are assuming a constant rate of change. The aim then is to find the 'fluxion' of x^n, (the 'fluxion' of any power of *x*). Next, we take the quantity $x + o$, which is what *x* becomes after some time. At this stage the power x^n becomes $(x + o)^n$, because it follows that as *x* increases every power of *x* will increase by the same proportion. So according to the method of infinite series:

$$x^n + nox^{n-1} + \frac{nn - n}{2} oox^{n-2} \text{ etc.}$$

And if we subtract the root and the power respectively, from the two incremented quantities:

$$o \text{ and } nox^{n-1} + \frac{nn - n}{2} oox^{n-2} \text{ etc.}$$

And if we divide the increments by the common divisor *o,* these quotients are yielded:

$$1 \text{ and } nx^{n-1} + \frac{nn - n}{2} ox^{n-2} \text{ etc.}$$

These are therefore the exponents of the ratio of the increments.

Up until now, Berkeley says, we have supposed that the variable *x* flows, that *x* has a real increment, that *o* has a quantity. This supposition has been necessary for us to make all the steps in the proof thus far. It is this assumption that allows us to compare

[83] Jesseph, p. 193

the increment of x^n with the increment of x, and find the proportion between the two increments. At this stage in the proof Newton now supposes that there are no increments of x or that *o* has no quantity, then the last ratio is 1 to nx^{n-1}.[84] Newton uses this to conclude that the fluxion of the quantity x is to the fluxion of the quantity x^n as 1 is to nx^{n-1}, [85] in other words, this give us the power rule.

The main thrust of Berkeley's objection is that Newton's last supposition that there are no increments violates the lemma he introduced in §12. This supposition is, argues Berkeley, contrary to the first supposition that Newton made, namely that there are such increments. By keeping the value nx^{n-1}, which Newton only got on the basis of a supposition he later drops, he has violated the principle of the lemma introduced by Berkeley. Specifically, Newton has assumed that *o* is greater than zero at the start of the proof, but later after dividing out the common term *o* he has assumed that *o* equals zero.[86]

Berkeley follows this argument by reiterating the similarity between Newton's fluxions and the differential calculus of Leibniz which openly utilises infinitesimals. Newton claims specifically in the *Introduction to the Quadrature of Curves* that his method means that he does not have to utilise infinitesimals (which makes it agreeable to classical geometry), but for Berkeley, this is not borne out in Newton's proofs, and there is no discernible difference between the two methods in this sense.[87]

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[84] 'The Analyst', §§13-14, pp. 72-73
[85] Newton
[86] Jesseph, p. 194
[87] 'The Analyst', §§17-18, pp. 74-75

Finally, Berkeley makes the point that he is not disputing the truth of Newton's claims, just the methods he uses to arrive at them. The truth of the conclusion cannot be used in logic to justify the soundness of an argument. As we learn in logic it is perfectly possible to have a true conclusion in spite of an argument where either a premise is false, or an inference is invalid. For example, the argument:

All cats are black

Lucky is a cat

Therefore, Lucky is black

where Lucky happens to be a black cat, is a valid argument that happens to have a true conclusion in spite of having a false premise. Berkeley asserts that in all other sciences the conclusions must follow from their principles, but the Newtonian approach is working the other way around by starting with the conclusion, and is thus unscientific.[88]

While Stewart and Moore, for example, accept that Berkeley's arguments were justified, some have tried to defend Newton against Berkeley. One such approach which argues that the tools of Newton's calculus are precursors to the limiting process and can therefore be understood in such terms. However, Jesseph tells us that this misses the point. Calculus has indeed developed and infinitesimals and fluxions have been replaced by first limits and then, in turn, real analysis has been introduced. But, Jesseph argues, we need to consider Berkeley's criticisms in the context of the mathematics of the day.[89] A charitable interpretation that is reliant on modern mathematical resources

[88] 'The Analyst', §§19-20, pp. 76-77
[89] It can also be argued that limits, while an improvement on the work of Newton and Leibniz, are still not clear, and lead to similar issues.

does not change the facts about the rigour, or lack thereof, of C17 and C18 calculus.[90] Moore even points out that Newton and Leibniz and their contemporaries were not unaware of the problems raised by the calculus at the time.[91] This is an important debate which will be explored when I consider the broader question of mathematical realism.

3.3 Objections: Truth and Reality - The Compensation of Errors Thesis:

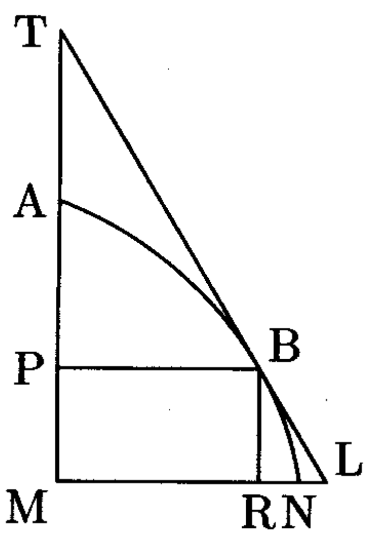
Berkeley has argued that the methodology of the calculus can get at truth but not science, so his next task is to explain this observation. This explanation is termed 'the Compensation of Errors Thesis' by Jesseph, who explains:

> He (Berkeley) claims that the calculus yields inexact results when its algorithms are applied to analytic expressions for curves, but that this analytic error is balanced by a compensating geometric error when the resulting equation is used in the solution of the problem.[92]

Berkeley then presents four problems to illustrate this point. Jesseph tells us that the problem which best highlights this point is the first one Berkeley gives. I will therefore only concentrate on this one, which can be found in §21-23 of 'The Analyst'. This is essentially an objection that while the method is clever in that it gets at truth, it does so at the cost of the required rigour that mathematics (as a science) ought to be adhering. The important point here to note is that Berkeley is taking issue with mathematicians using the tools of science to make judgements about the principles of Christianity, when in fact they don't adhere to the same rigour in parts of their own field. Further, Berkeley believes that religion should be mysterious, where science should not, so it is wrong to try to put the two fields on the same playing field.

[90] Jesseph, p. 198
[91] Moore, p. 65
[92] Jesseph, p. 200

For this critique, Berkeley takes one of Leibniz's proofs, that which solves the problem of finding a subtangent drawn to a parabola, and examines it as performed by infinitesimal differences. The proof is for finding the subtangent for the point B on the curve, which is the line PT, as illustrated in this diagram taken from 'The Analyst':[93]



Jesseph gives us the following distances which are also utilised in the proof: *x,* which is the distance AP, *dx*, which is the distance PM, *y*, which is the distance MR, *dy*, which is the distance RN and *z*, which is the distance NL. (As Jesseph points out, Berkeley says he is discussing a tangent, but the example swiftly moves on to discuss the subtangent instead).[94]

Firstly, we take a point M. For simplicity in describing the points needed for the proof I am assuming this point to be the origin in a system of Cartesian coordinates which has a value of (0, 0). Further, take the point M to be at the right angle of a right-

[93] 'The Analyst' §21, p. 77
[94] Jesseph, pp. 200-201

angled triangle, MLT where L is a positive point on the x-axis, i.e. (x, 0) and T is a positive point on the y-axis, i.e. (0, y). TL will also form part of the tangent at a point B on the curve which we will be considering. The proof then takes a parabola (a mirror symmetrical curve), AB,[95] where A is a point (0, y), and where the y value at A is less than the y value at T, and B is a point (x, y), and where the y value at B is lower than the y value at A. From this curve we then consider several functions:[96]

The abscissa (formerly abscisse); this is the line AP which is the line that can be drawn along the y-axis from point A to another point, P, which has the same value on the y-axis as point B of the curve. (This tracks the height of the curve.) This function has the value *x*.

The ordinate; this is the line PB. (This tracks the length of the curve). This function has the value *y*.

The difference of the abscissa; this is the line PM. This function has the value *dx*.

The difference of the ordinate; this is the line RN. (The differences are infinitely small differences that are designed to capture the rate of change along these lines as they relate to the rate of change of the curve.) This is a line that can be drawn along the x-axis, where the point R has the same x value as point B, and the point N forms a point

[95] The curve can be extended beyond point B (by an infinitely small distance) to a point N which lies between points R and L on the x axis. This point N will also be utilised in the proof.
[96] A function here meaning line segments or lengths that can be determined from each point of a curve relating it to a given line or axis. David Dennis and Jere Confrey, ‘Functions of a Curve: Leibniz’s Original Notion of Functions and its Meaning for the Parabola’, *The College Mathematics Journal*, Vol 26, No. 3, 1995, pp. 124-131, p.124

where an extension of the curve AB, namely BN, intersects with the x-axis, and thus has an x value less than RL. This function has the value *dy.* [97]

Thus AP = x, PB = y, PM = dx, RN = dy

The curve AB is expressed mathematically as $y^2 = px$, where p is a constant and x and y are variables. Next, we suppose the curve to be a polygon (with infinitely many sides), this allows us to think of BN, the difference or increment of the curve, as a straight line which coincides with the tangent. We also get a differential triangle BRN which we treat as similar to the triangle TPB. The subtangent for the point B on the curve is the line PT. The similarity of the triangles BRN and TPB allows us to establish that the ratio between RN and RB is the same as that between PB and PT. This means that the value of PT = ydx/dy. However, fully solving the problem means we need to be able to express PT in a way which will allow us to eliminate dx and dy from the equation. We can do this by taking $2ydy = pdx$ which is the derivative of $y^2 = px$ (the mathematical expression of the curve AB). dy can therefore be expressed by $pdx/2y$. PT than can be given in the following way that solves the problem:[98]

(1) $$\text{PT} = \frac{ydx}{pdx/2y} = \frac{2y^2}{p}$$

Having given this proof Berkeley then argues that it works by containing two errors which compensate each other. Firstly, the triangle RNB is not similar to PBT, RLB is. If we therefore give the line NL the value z, then the subtangent is actually expressed as:

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[97] 'The Analyst', §21, p. 77
[98] Jesseph, pp. 200-201

(2) $PT = \dfrac{ydx}{dy + z}$

This means the value of the subtangent at the end of this step was too large (by the quantity of *z*).[99]

Secondly, there is an error in the differentiation of $y^2 = px$. This was given as $2ydy = pdx$, which can also be rearranged and simplified as: $dy = pdx/2y$. But if we take the increments *dx* and *dy* of *x* and *y*, we should get $(y + dy)^2 = p(x + dx)$ or:

(3) $y^2 + 2ydy + dy^2 = px + pdx$

This means that the increment of the equation is $2ydy + dy^2 = pdx$. And when we rearrange and simplify this we get:

$$dy = \frac{pdx}{2y} - \frac{dy^2}{2y}$$

This also makes the value of *dy* appear to be too large, differing as it does by the term $dy^2/2y$. But if this term can be shown to equal *z*, then the two errors will cancel each other out leaving us with the same result as before.[100] Berkeley goes on to demonstrate that this is indeed the case.

In section §22 of 'The Analyst' Berkeley draws on the Conics of Appolonius to prove that $z = dy^2/2y$.[101] This states that the subtangent to a parabola is bisected at the vertex, implying that TP = 2AP or 2*x*.[102] Berkeley denotes BR (*dx*) by *m* and RN (*dy*) by

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[99] 'The Analyst', §21, p. 77, Jesseph, pp. 201-202
[100] Jesseph, p.202
[101] 'The Analyst', §22, p. 78
[102] Jesseph, pp. 202-203

n, and uses the Apollonian theorem and the similarity of the triangles TPB and BRL to derive the following:

$$\frac{2x}{y} = \frac{m}{(n+z)}$$

which can be expressed as:

$$\textbf{(4)}\ (n+z) = \frac{my}{2x}$$

We then go back to $y^2 + 2ydy + dy^2 = px + pdx$ (the expanded expression for the parabola), replacing dx by m and dy by n to get:

$$y^2 + 2yn + n^2 = px + mp$$

Removing y^2 and px which have the same value we get:

$$2yn + n^2 = mp$$

We then divide by p to get:

$$m = \frac{2yn + n^2}{p}$$

From ($y^2 = px$) we get $x = y^2/p$. We then substitute these values for m and x respectively into the equation **(4)** and get:

$$(n+z) = \frac{my}{2x} = \frac{2y^2np + yn^2p}{2y^2p}$$

We then cancel out the common term py, which leaves us with:

$$(n+z) = \frac{2yn + n^2}{2y}$$

And finally, when we reduce that to an expression for z, we get:[103]

$$z = \frac{n^2}{2y} = \frac{dy^2}{2y}$$

[103] Ibid

In section §23 of 'The Analyst' Berkeley lists four observations that he has drawn from his proof:

(i) The conclusion of Leibniz's proof comes out right not because the rejected square of *dy* was infinitely small, but because it is compensated by a contrary error that is of an equal value.

(ii) No matter how small a rejected value is, a proportional error will be present in the conclusion.

(iii) When a conclusion is in fact accurate but drawn from inaccurate premises, we can't say that the conclusion is accurate in virtue of the premises, but only in virtue of some other principles, of which even the person putting forth the argument may have been unaware.

(iv) This applies no matter the size of the quantities with which we are dealing. The rejected quantities are legitimately rejected not for their smallness (as infinitesimals), but because the compensating errors are the only things keeping them in the proof.[104]

Jesseph analyses Berkeley's proof and observations. The first point he makes is that Berkeley has demonstrated a way to find the subtangent without recourse to infinitesimals. He appears to interpret *dx* and *dy* as finite differences and this interpretation enables the Apollonian theorem to give the value of the subtangent correctly. This in turn hints at the possibility of developing the calculus without recourse to infinitesimals. Berkeley believed that this enterprise was possible, although he never attempted it. Further he believed in the importance of such a project, given that

[104] 'The Analyst', §23, pp. 78-79

he took the theorems to yield true results, despite the flawed nature of the methods employed.[105]

Such a view has something in common with certain types of nominalistic approaches to mathematical fictionalism, whereby philosophers of mathematics have attempted to recast some of mathematics without recourse to numbers. However, an approach that recast the calculus without recourse to infinitesimals would not appear to guarantee fictionalism of mathematics more generally, unless the project was extended to further nominalise the calculus. Rather it would be analogous, in that it might grant a fictionalist account of the calculus while still calling for a type of realism about mathematical objects more generally. (This comparison will briefly be discussed when we look at fictionalism in chapter 5 of this thesis.)

However, despite Berkeley's belief that the calculus could be developed along these lines his work only goes to show that some problems in the theory of conic sections could be solved by employing classical methods.[106] Further, if classical methods were sufficient or the simplest method of solving the problems of the calculus, there would surely have been no motivation to develop such a system in the first place.

The calculus wasn't developed along the lines that Berkeley utilised in his work on the compensation of errors theory. Instead the theory of infinitesimals developed and the notion of limits was employed. Jesseph explains that in the modern theory of infinitesimals we would treat z as an infinitesimal which obeys the law $p + z = p$ for any real number p. Under a theory of limits both steps take the limit of a quantity $k + z$ as z

[105] Jesseph, p. 204
[106] Jesseph, pp. 227-228

approaches zero. The other main point Jesseph makes in his discussion of the compensation or errors thesis in this example concerns an analysis of the assumption of the compensating errors themselves. He argues that employing these modern theories of the calculus allows us to deny that there are any such errors.[107] In the first case the real numbers are dense so there can be no real number between p and the next real number in the sequence, so *z* cannot specify a quantity. In the second case *z* approaches zero, but it never gets there, so it won't have the contradictory status of being both a quantity and zero. However, this is to take an anachronistic approach and does not really address the problem within the nascent calculus with which Berkeley is concerned.[108]

Berkeley's compensation of errors thesis specifically deals with the question of how we arrive at true results within a system which he sees as non-scientific. This issue is of central importance to the options I shall be exploring later in this thesis that specifically address the realism in philosophy of mathematics debate, and how we may best understand the role of mathematical entities.

3.4 Questions Arising from Berkeley's Objections to the Calculus:

Berkeley's main objections then follow three patterns: metaphysical objections based on the nature of infinitesimals and fluxions, logical objections based on the contradictory nature of the proofs and compensating errors that give us a version of 'truth' without a solid scientific foundation. The metaphysical objections appear to be based on in implicit understanding of realism, but as we shall see that is not the only option on the table. However, it does seem obvious that Berkeley wins the logical

[107] Jesseph, p. 205
[108] Ibid

argument. Newton's proofs are contradictory, in allowing, as Berkeley argues, a quantity to be assumed to be both something and nothing. Newton's proof of the product rule gives a method for calculating the difference in the area of two rectangles that yields a different answer to the established method. And the compensation of errors thesis highlights that at least some of the proofs of the calculus rely on the inclusion of false premises.

But despite all this the calculus worked; it enabled the calculation of forces, velocities, rates of acceleration, areas, volumes etc. in accordance with the observed data.[109] Further, these proofs have been instrumental in developing modern mathematics. And from the observation that we do get truth from these proofs, we can ask metaphysical questions, such as if Berkeley's philosophical arguments are valid how can we explain that the calculus enables us to derive useful results? What is the nature of such seeming non-entities as fluxions and infinitesimals? And how do they play a role in our understanding of mathematical truths? In addition, this raises issues for the debate on realism in the philosophy of mathematics. If we do or can rely on non-entities for our mathematical proofs, does this mean that platonism cannot be the true metaphysical picture? Or, do modern mathematical solutions where we can generate proofs that may not depend on non-entities render the question moot?[110] Can we have a position whereby we believe that any mathematical truth can in principle be justified without recourse to non-entities, even if such alternatives are to be developed with the tools of future mathematics?

[109] Moore, p. 65

[110] This is working from a position where we assume that platonism has not been ruled out, so we can accept that numbers and other mathematical objects are at least potentially real.

If all this, however, leads us to conclude that platonism is in fact the wrong picture of mathematical objects, does the dependence upon non-entities in these proofs mean that all forms of mathematical realism are off the table, or is this methodology consistent with non-platonic forms of mathematical realism such as truth-value realism? Or, must we look to an alternative picture, such as fictionalism? In what follows, I aim to answer these questions.

3.5 The Aftermath of Berkeley's Objections:

In this section I look at responses form one of Berkeley's contemporaries and briefly discuss the evolution of the calculus.

3.5.1 Responses from Berkeley's Contemporaries:

Only two of Berkeley's contemporaries who responded to his objections actually received replies from Berkeley; Jurin and Walton. In fact, Berkeley's last work on the calculus was his second of two replies to Walton.[111] But Jesseph argues that both Jurin's and Walton's responses do more to strengthen Berkeley's objections than dispel them.[112] Walton relies heavily on Newton's work itself, and while he does address some of the proofs the Berkeley criticises, he does so by not really engaging with Berkeley's arguments themselves.[113] Jurin's responses take more notice specifically of Berkeley's objections, so I will focus on those here.[114]

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[111] Luce and Jessop, pp. 105-106, 145
[112] Ibid
[113] Jesseph, pp. 250-254
[114] Interestingly, Berkeley believes that his response to Jurin will also in the main deal with Walton's objections too, so in his first reply to Walton, Walton's criticism is only given an appendix appended to

Jurin wrote his response, a pamphlet entitled 'Geometry no Friend to Infidelity' in 1734. Jurin, in particular, took issue with what he saw as the hypocrisy of Berkeley, who was objecting to practices in mathematics, after he had criticised mathematicians for getting involved with theology.[115] This criticism, and what Berkeley thought of it, can also be seen in the title of Berkeley's response, 'A Defence of Free-Thinking in Mathematics', which is a reference to what Berkeley saw as a right to participate in the mathematical debate. Jurin firstly alleged that Berkeley was accusing the mathematical community of infidelity with regards to the Christian religion, and trying to make other infidelity. As we have seen there is some truth to this claim, (although Berkeley does not specifically say that there is a deliberate intent from mathematicians to turn their followers away from Christianity.) He also argues that Berkeley accuses mathematicians of using error and false reasoning in their own science. Again, this claim has some justification.

Turning to the criticism of mathematicians using error and false reasoning there are principally two specific charges that relate to Berkeley's objections directly, rather than just praising Newton at Berkeley's expense. The first of these is that Berkeley argues that the doctrine is obscure. Jurin accuses Berkeley of overcomplicating the notion of fluxions, especially with regard to derivatives. He argues that 'velocities of velocities' is not a Newtonian expression, and instead that Berkeley is putting an unacceptable gloss on Newton's true position. This Jesseph agrees, on the face of it, may have some justification, as Newton does not use the phrase. But at the same time,

the response to Jurin. Walton later responds to this appendix, which does elicit another reply of sorts from Berkeley, entitled 'Reasons for not Replying to Mr. Walton's *Full Answer*' (which is a pun on Walton's second criticisms of Berkeley). This reply, which addresses Walton in the third person, however, takes the form of a letter addressed to P.T.P.

[115] Luce and Jessop, p. 105

he argues that Berkeley's characterisation is not in fact an unreasonable one, as it does capture the essence of what is going on in the calculus. [116]

The second charge is that Berkeley accuses Newton of using false reasoning. This relates to Berkeley's logical objections to the calculus and the compensation of errors thesis. Firstly, Jurin tries to prove that Newton's product rule is rigorous. To do this he claims that the difference ab, by which Berkeley's calculation of the increment of AB differs from Newton's moment of the rectangle AB, is inconsiderable and makes no difference in practice. He argues that the quantity ab is properly rejected and concludes that what Newton tries to obtain by his suppositions is simply the increment of the rectangle (A – ½a) × (B – ½b), arrived at by a direct and true method. But all that Jurin actually does to get this result is restate Newton's proof, except Jurin uses (A + a) etc. and divides by two at the end, whereas Newton uses (A + ½ a) etc., so the division does not need to be made later. Jesseph argues that all this does is make Berkeley's point for him.[117] And Berkeley's response is to point out the weakness of the argument,[118] and reiterate his claim that we can't judge the calculus by the application of the results in practice. This further demonstrates Berkeley's concern with the rigour, of the calculus, rather than the fact that it worked in practice.[119]

Jurin next goes on to attack the arguments against the power rule, in which Berkeley argues that a fluxion first has an increment and then lets that increment vanish, which means that the proof contains two inconsistent premises. This attack is based upon Jurin's translation of Newton, in which he takes the fluxions to be on the point of

[116] Jesseph, pp. 234-235
[117] Jesseph, pp. 235-237
[118] 'A Defence of Free-thinking in Mathematics' §§26-30, pp. 120-123
[119] Jesseph, pp. 243-244

evanescence rather than being either evanescent or not. But this misses the point; in this proof the point is not the ontological status of the fluxions themselves, but the issue that the proof is illegitimate, in that it takes fluxions to retain a quantity even after a step in the proof where we have already taken the quantity to have vanished.[120] And Berkeley responds accordingly to address this point.[121]

Jurin also makes an attempt to refute the compensation of errors thesis, which as Jesseph points out, is the easiest way to attack Berkeley's arguments, but even this attack is not successful. Jurin's method is to calculate the value for the subtangent twice, each time retaining one of the errors which Berkeley identified. The first calculation 'corrects' the error of substituting RN for RL, by using RL, but retains the second error of assuming that $dy = pdx/2y$. In the second calculation, he keeps the discarded infinitesimal dy^2 in the differentiation of the equation $y^2 = px$ removing the second error, but he reintroduces the first error of using RN for RL. He then concludes that all three results, his two new results and the result of the original proof by Leibniz are valid and amount to the same answer. Jesseph points out there are two ways of taking this claim, either we take it that they are literally equal, in which case $dy = 0$, and the calculus is inconsistent, or we take the exactness of the results to mean that the differences are insignificant, and we allow small errors and in so doing remove rigour from mathematics.[122] Clearly this is no good as a response to Berkeley's objections and, tellingly, Berkeley doesn't even address it in his reply.

[120] Jesseph. pp. 237-238
[121] 'A Defence of Free-thinking in Mathematics' §33, pp. 124-125
[122] Jesseph, pp. 238-240

So, we have seen that initially at least, Berkeley's objections were not overcome. Later we will see if modern metaphysics and philosophy of mathematics can fare any better.

3.5.2 The Developments of the Calculus:

Infinitesimals plagued seventeenth and eighteenth century analysis, but despite the philosophical arguments against them, in particular Berkeley's, they were still utilised throughout this period.[123] That is not to diminish the potency of Berkeley's criticisms, but opinion is divided on how effective Berkeley actually was. Hilary Putnam goes as far as to cite Berkeley's attack on actual infinitesimals as an example of philosophy making a discovery that requires changes in science,[124] and Luce and Jessop share this line of thinking,[125] while Stewart remains less convinced that any shifts were due to Berkeley. So, while it may be fair to say that Berkeley had indeed pointed out something that would require a change in science, the objections raised by Berkeley may not have been the motivation for the change, or at least not the only motivation.[126]

The notion of infinitesimals in the calculus was formally replaced by limits in the nineteenth century.[127] This was refined by Cauchy and Weierstrass. An infinitesimal was a limit as it tends towards 0 or infinity. An infinite sequence of rational numbers can have a limit that is not rational, such as $\sqrt{2}$. The calculus was carried out purely in

[123] Stewart, p.71
[124] Hilary Putnam, 'Mathematics without Foundations' *The Journal of Philosophy*, 64, 5-22, Reprinted in *Philosophy of Mathematics*, ed. by Paul Benacerraf and Hilary Putnam, (Englewood Cliffs, N.J.: Prentice-Hall, 1964), pp. 295-311, p. 296
[125] *The Works of George Berkeley*, p. 58
[126] Stewart, p.76
[127] Stewart, p.71

analytic terms.[128] Limits were an improvement on infinitesimals, but they remained imprecise, and arguably only slightly more intelligible. It wasn't until the subsequent development of real analysis that the problem of relying on entities that didn't appear to exist was finally solved in mainstream mathematics. However, in the work of mathematicians such as Abraham Robinson and H. Jerome Keisler, who utilises Robinson's ideas, infinitesimals have made a comeback. Keisler even uses them in teaching the calculus, although these are different in character from the ones used by Leibniz and from Newton's fluxions.

[128] Moore, pp.66-69

4. Contextualisation of Berkeley: The Objections in Relation to Berkeley's Overall Philosophy:

While Berkeley's objections to the calculus are largely well considered on their own terms as metaphysical and logical objections, his reasons for putting the argument forward stem from his broader philosophical outlook, his philosophy of religion, and the theological consequences that he believed could be drawn from the principles of the calculus.

Berkeley was specifically motivated in his arguments by contemporary mathematicians' objections to principles of Christianity on the grounds of incoherence. Berkeley's arguments are designed to show that the principles of the calculus are no less mysterious or hard to comprehend.[129] Therefore there is no greater reason to accept infinitesimals and fluxions than there is to accept the principles of Christianity which were no more incomprehensible.[130] Luce and Jessop, editors of *The Works of George Berkeley*, point out that his immaterialism is another motivating factor.[131] As a consequence of his immaterialism Berkeley, believed that only what we could perceive existed. Therefore, infinitesimlals, which we are unable to perceive could not really exist. There are also considerations to be taken from Berkeley's anti-abstractionism and his acceptance of the practical uses of some abstract entities in mathematics, such (as we have seen) as imaginary numbers.

In this section I shall look at these four factors in turn.

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[129] Moore, p. 65
[130] Jesseph, p.178
[131] *The Works of George Berkeley*, p. 55

4.1 Incoherence of the Principles of Christianity

In the first two sections of 'The Analyst' Berkeley directly addresses 'the infidel mathematician'. Here Berkeley lays the charge that mathematicians by dint of being regarded as masters of reason attract followers not only in mathematics but in other fields and matters on which they may offer an opinion. Berkeley believes that these mathematicians have no business presenting themselves as authorities in subjects outside their field, and that this practice should stop, as it leads to the creation of further 'infidels'. He is especially concerned that the perception that those who use more reason and judgement are also less religious, and seemingly fearful that this will encourage the admirers of such men to follow suit.[132] Berkeley believes that those pre-eminent in their fields have a responsibility to those who follow their teachings, something we also see at the beginning of 'Three Dialogues between Hylas and Philonous where the two protagonists worry that the wrong views will be spread by those that heed the philosophers of the day to the detriment of mankind.[133]

To counteract the trend of mathematicians to make forays over to the field of religion, Berkeley allows himself 'the same freedom' that these men of reason have given themselves in exploring what he sees as his field of religion, by exploring for himself their field of mathematics. He sets himself the task of showing that they have no right to lead and expect followers in fields that are beyond their area of expertise.[134] In the course of 'The Analyst' then, he argues that certain postulates and axioms of mathematics are not so underpinned by logic and reasoning as is supposed. Showing this would have the effect of demonstrating that the reason and logic employed by the

[132] 'The Analyst', §§1-2, p. 65
[133] 'Three Dialogues between Hylas and Philonous', First Dialogue, pp.161-162
[134] 'The Analyst', §2, p. 65

mathematicians is not necessarily of a greater calibre than that of those in the field of religion, and consequently that there is no more reason to accept certain aspects of mathematics than there is to accept Christianity. In fact, the principles of Christianity are matters of faith, and should hold some mystery, unlike mathematics which should not as a field of science.

This would serve to allow, and perhaps compel, people to turn to mathematicians for accepting the axioms and principles of the field of mathematics, and to men of the Church for accepting the principles of Christianity. But this is not simply to say that the same degree or lack thereof of rationality can be applied to both disciplines. Crucially mathematics and religion are very different in their natures. Berkeley argues that there is a demand that Christianity should have some mystery, and he regards such mystery as beyond human reason, but not contrary to it. But science including mathematics, on the other hand, must be rigorous, and objects of true science must be properly conceived, and science must be limited to things which are evident to reason.[135]

In the queries at the end of 'The Analyst' Berkeley suggests that we may want to go further than that in redeveloping the calculus in terms of finitary mathematics, to presumably eliminate any mystery from it.[136] But, as we shall see he only does this himself in a very limited capacity, and it is not entirely clear that this could be done for the whole of the calculus, or at least not demonstrated.

[135] Jesseph, p. 180
[136] 'The Analyst', queries 1-3, p. 96

Additionally, historical curiosities arise from this debate such as who the infidel in question is, and the precise attitude towards Christianity that he holds. The information about this is sketchy but it is explored by Luce and Jessop, as well as Jesseph. It appears that Berkeley does have a particular person in mind. However, despite how it may appear on a reading of 'The Analyst' that individual is not Isaac Newton himself. This is seen in Berkeley's appendix to his 'A Defence of Free-thinking in Mathematics' in which he addresses a pamphlet written by Walton in response to 'The Analyst', in which Berkeley rejects the charge that Newton is the intended target.[137] What we can glean about the identity of the 'infidel' from Berkeley's own words is that it is a noted mathematician still living (in 1735), whose 'infidelity' has been brought to the attention of Berkeley by Addison. Berkeley's early biographer Stock identifies Edmund Halley as the 'infidel' apparently based on communication that Addison had with Berkeley. Addison is alleged to have told Berkeley of a story of a Dr. Garth who when close to death claimed that he did not believe it necessary to prepare for it 'since my friend Dr. Halley who has dealt so much in demonstration has assured me that the doctrines of Christianity are incomprehensible and the religion itself an imposture'[138] This fits the description of Berkeley's target of a mathematician whose reputation for reasoning has enabled him to influence others with his views about religion. However, this information about Halley would have had to have been conveyed by letter as Addison and Berkeley were in different countries during the relevant time-frame. Luce and Jessop appear to accept this, believing in Stock's testimony and considering it not unreasonable that such a letter had been sent.[139]

[137] Berkeley, 'A Defence of Free-thinking in Mathematics', 1735, Reprinted in *The Works of George Berkeley*, pp. 103-141, p. 139-140
[138] Stock, Joseph, *An Account of the Life of George Berkeley, D. D. Late Bishop of Cloyne in Ireland. With Notes, Containing Strictures Upon his Works,* 1776, Reprinted in *George Berkeley: Eighteenth Century Responses*, ed. by David Berman, (New York: Garland, 1989), Vol. 1, pp. 5-85, pp.29-30
[139] *The Works of George Berkeley*, pp. 56-27

Jesseph, however, writing considerably more recently regards Stock as an unreliable source, and finds the lack of any evidence of such a letter troubling.[140] So while Halley would fit the bill, the true identity of the 'infidel' along with the extent to which he fails to accept the doctrines of Christianity remain in some doubt. But clearly Berkeley was agitated by the influence such masters of reason had on others with regard to topics beyond mathematics, notably religious belief.

Additionally, Berkeley's proposed enterprise to rewrite the calculus in terms of finitary mathematics can be interpreted in part as a reaction to issues of reconciling God and infinity with the use of the infinite in mathematics, or the notion of the infinite more generally. For the prevailing Christian faith at that time, only God could be thought of as truly infinite. Further, Berkeley would argue that given the way in which we understand ideas as being planted in our minds by the divine mind, (which will be explained more fully in the next section) any ideas must be possible to be experienced by the finite human mind. This would make the notion of infinitesimals impossible. But what the belief that only God could be truly infinite meant for the notion of infinity considered beyond this was interpreted by different philosophers in different ways. The rationalists found ways to accommodate both notions, but for the empiricists (with their experience-dependent philosophy), such as Locke, Hume and Berkeley, the infinite, which could not be directly experienced, was inevitably going to be a problem. While he does tackle the infinitely small by commenting on infinitesimals, Berkeley, along with Hume, never properly discussed the infinitely big at all, which gives an indication of their level of discomfort with the idea.[141]

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[140] Jesseph, pp. 179-180
[141] Moore, pp. 75-83

4.2 Immaterialism

Berkeley's immaterialism is a form of idealism. Specifically this is metaphysical or ontological idealism, which is described in the Stanford Encyclopedia of Philosophy as the concept that 'something mental (the mind, spirit, reason, will) is the ultimate foundation of all reality, or even exhaustive of reality'.[142] This is contrasted with formal or epistemological idealism where the mind independent world is taken to exist, but that all we know about this world is taken to be given to us by mental activities (entailing that all knowledge is a form of self-knowledge). And while epistemological idealism does not entail ontological idealism, ontological idealism does entail epistemological idealism. For Berkeley, then, only ideas and the minds that have them exist. His position of immaterialism starts with epistemological idealism, with the addition of the more radical ontological idealism in a move to avert scepticism. In this move he is challenging the ontological agnosticism which accompanies Locke's epistemological idealism.[143]

In his 'A Treatise Concerning the Principles of Human Knowledge' Berkeley puts forward his arguments for his immaterialism. Guyer and Horstmann identify three key steps in this argument.[144] Firstly, Berkeley argues in accordance with Locke that ideas exist only in the mind. This is developed from some underlying principles that Berkeley takes for granted.[145] All ideas or objects of knowledge are perceived by something which knows them, and is willing, imagining or remembering them. And this

[142] Paul Guyer and Rolf-Peter Horstmann, Rolf-Peter, "Idealism", The Stanford Encyclopedia of Philosophy (Fall 2015 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/fall2015/entries/idealism/
(Accessed 5 August 2016)
[143] Ibid
[144] Ibid
[145] Ibid

thing is a mind which is distinct from the things that it is perceiving. He then argues that the sensations of the ideas and the objects that they compose cannot exist without a mind perceiving them. He therefore famously concludes that to be is to be perceived.[146] Guyer and Horstmann posit that at this stage in the argument, in light of his conclusion, Berkeley is already in a position to cast doubt on the notion that unperceived things could exist. In the first place the only things that can exist are mind-dependent. So, if the only things that exist for a mind are ideas, due to the fact that nothing else besides ideas can exist for the mind, and nothing can exist that is mind-independent, then the very notion that there could be something in existence which is not for the mind, or is not perceived, is contradictory.[147]

Secondly, Berkeley makes a claim that an idea cannot be like anything other than another idea. In this assertion, Berkeley is attacking the notion that ideas resemble things that can exist without a mind. The reasoning behind this is assumed to be as follows. Two things that stand in a likeness relation must have common traits. But if ideas are mind-dependent and are all that there can be for the mind, then they have to be totally different from anything that is not mind-dependent. Therefore, there can be no likeness relation between two such things.[148] Berkeley then challenges anyone who disputes this to look into their thoughts and to compare likenesses. He maintains that it is only possible to consider ideas, because if the external objects that these ideas represent are perceivable, then they have to be ideas too, for that is all that the mind can

[146] Berkeley, ‘A Treatise Concerning the Principles of Human Knowledge’, Second Edition, 1734, Reprinted in Berkeley, *Philosophical Works Including the Works on Vision*, ed. by Michael R Ayres, (London: Everyman, 1996), Reissue with Revisions and Additions, pp. 71-153, 1, §§2-3, pp. 89-90
[147] Guyer and Horstmann
[148] Ibid

perceive. Therefore, all we have been able to do is compare idea with idea, which proves Berkeley's point.[149]

Thirdly, Berkeley claims that ideas are passive and causally inert. Again, Berkeley calls on the introspection of the reader. He argues that when we consider our ideas we will not be able to consider them to hold any power or activity. This means that they must be passive and inert, because an idea cannot do anything or cause anything to happen.[150] Guyer and Horstmann argue that while this section has a primary function of arguing against Locke's distinction between primary and secondary qualities, it also designed to support the notion that agnosticism about the existence of mind-independent objects is an untenable position for someone who holds a Lockean epistemological position.[151] This appears to be because secondary qualities are not generally considered to be mind-independent, they are not thought of being intrinsic to the object and are only relevant when we are perceiving an object. Thus, if primary qualities are reduced to secondary qualities, then the same must go for them.

Guyer and Horstmann go on to say that this criticism of Locke is borne out of Berkeley's belief in the unavoidable metaphysical conclusions that come from such epistemological idealism, an epistemological positon that he shares. They present a brief sketch of the argument 'If existence is restricted to ideas (and minds) and if, what is undoubtedly the case, things or substances exist, then things or substances must be ideas (or minds) too.'[152]

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[149] 'Principles of Human Knowledge', 1, §8, pp. 91-92
[150] 'Principles of Human Knowledge', 1, §25, p. 98
[151] Guyer and Horstmann
[152] Ibid

The final step to fully establish the nature of this immaterialism is to consider the nature of the mind in which these ideas exist. This is done by first considering the fact that our ideas change and something must be causing that to happen. As Berkeley has argued in the previous section this cannot be the ideas themselves, so it must be a substance of some sort. He has already previously argued that there is no material substance, so Berkeley concludes that this substance must be incorporeal and therefore a spirit. This must be one undivided being, a divine mind. Berkeley than gives it two functions; the understanding when it perceives ideas and the will when it produces them or makes them do anything. This divine mind and its functions of will and understanding together with the notion of substance or being in general cannot stand for an idea.[153]

4.3 Anti-Abstractionism

In this section I shall look at the assumption by Berkeley, pointed out by Jesseph (which is mentioned in Chapter 3 of this thesis and derived from anti-abstractionism), that the imagination is derived from our sensations, and the underlying assumption that our mental faculties consist only of sense and imagination with no faculty for framing independent ideas through 'pure intellect', as well as the issue of Berkeley's acceptance of other abstract mathematical entities.

This actually goes beyond mathematical entities to the problem of thing of anything abstracted from its qualities, but it is perhaps in mathematics where the appeal to abstract away from properties it at its most self-evident. Berkeley's argument for his

[153] Principles of Human Knowledge', 1, §26-27, pp. 98-99

assumption that our mental faculties consist only of sense and imagination with no faculty for framing independent ideas through ‘pure intellect’ comes from his anti-abstractionism and can be found in the first dialogue between Hylas and Philonous, when he talks of the impossibility of framing abstract ideas.[154] This follows on from a discussion of the supposed distinction between perceived and actual extension and the question of whether the former can be taken to be a secondary quality, while the latter can be taken to be a primary one. (Berkeley argues in this work that there is in fact no distinction between primary and secondary qualities). Berkeley argues (via Philonous) that while mathematicians consider quantity removed from any other sensible quality, they are not in fact thinking of pure abstracted ideas of extension. He then introduces (via Hylas), the notion of ‘pure intellect’, asking if so called abstracted ideas (like pure extension) cannot be framed in this way. This notion is rejected by Berkeley. Firstly, figures and extension are originally perceived through our senses, by sense, which means that they don’t belong to pure intellect. Secondly, he argues that we cannot frame the idea of a figure abstracted from all sensible qualities, (qualities that we frame ideas of through our senses). That is to say that if we could frame abstract ideas they would have to be by dint of our senses, as we would only get to them via our perceptual experience, but we actually find that we can’t frame abstract ideas at all, because we just can’t abstract away the sensible qualities form our idea of an object.[155] We can’t therefore conceive of extension, without thinking of it as the extension of some ‘thing’.

[154] Berkeley, ‘Three Dialogues between Hylas and Philonous’ Third Edition, 1734, Reprinted in Berkeley, *Philosophical Works Including the Works on Vision*, ed. by Michael R. Ayers, (London: Everyman, 1996), Reissue with Revisions and Additions, pp. 155-252, pp. 183-184

[155] ‘Three Dialogues Between Hylas and Philonous’ pp. 183-184

4.4 Other Mathematical Entities

Berkeley's views on the usefulness of imaginary numbers as discussed in the previous chapter may present a problem for the so-called metaphysical objection. If the nature of something not quite finite is at issue, then surely something that is a placeholder for a number not contained within the real numbers is as much of a problem. An imaginary number doesn't change its value; it always represents the same value 5i is always the square route of -25 for example, but that would not appear to give it an ontological status while an infinitesimal lacks one. It may also be worthwhile to consider the other objections in light of the other two objections. I don't think it is a problem for the logical objection for the very fact an imaginary number does always represent the same nominal value, so it won't change its 'value' in the course of a proof, it will remain consistent unlike an infinitesimal which changes value from something (infinitely small) to nothing in the proof outlined in the previous chapter. With the compensation of errors thesis, again if the value of an imaginary number remains constant it is hard to see why a proof containing one would need to rely on compensating errors on the way shown by the proof that

The point about the usefulness of imaginary numbers comes from a discussion about the use of signs such as letters and mathematical symbols in the course of our lives (from within a boarder discussion about the relationship between science and religion). These signs are a means to suggest ideas to the mind although they don't always, but when they do they are not of abstract ideas. When they don't suggest ideas, they can still be helpful in allowing us to reach a conceived good. This is the case of imaginary numbers in an equation; they help us get the right answer, but don't suggest an idea. But here Berkeley also makes the point that mathematical sciences can fall short of the rigour they are thought to possess, and mistakenly think should also apply to the mysteries of religion.[156]

This exploration of the context for Berkeley's arguments in 'The Analyst' allows us to better understand Berkeley's reasoning and motivations and enables us to appreciate the issues from their situation in history. It also helps us to see how these arguments fit into Berkeley's wider philosophical picture. And it is not the case that Berkeley is saying we should abandon the calculus because of these objections, rather mathematicians need to mindful of the limitations of their discipline and not to assume that the discipline of religion should be subject to the same kind of scrutiny (as a certain degree of mystery is rightly part of religion and the principles of Christianity. However, independently of these factors the objections Berkeley are well thought out, informed and worth taking seriously.[157]

While we have been pondering the ideas behind the understanding of the nature of mathematical entities for a long time, the developments in the field of philosophy of mathematics in the C20 and early C21 have brought different interpretations and ways of potentially explaining the true nature of mathematical entities that we find so indispensable to our lives. Berkley's objections to the calculus are an interesting example of how in even early stages of modern infinitary mathematics the understanding of how to incorporate infinite quantities into mathematics raised questions, which feed into a broader concern about the nature of mathematical entities themselves. The calculus worked, but it raised questions as to why it worked, given its

[156] 'Alciphron' §§14-15, pp. 140-142
[157] Stewart, p.76

lack of scientific rigour. Reconciling the issues surrounding Berkeley's position (particularly of seeing the calculus as truth without science) may therefore be found in more current metaphysics of mathematics, as we still ask similar questions, and this will be the focus of the next section.

5. Berkeley's Objections and the Debate concerning Realism in the Philosophy of Mathematics:

Having considered Berkeley's objections, I will proceed to examine them as a case-study in relation to the philosophical debate about realism in the philosophy of mathematics. While the arguments Berkeley has put forward in his objections are derived from external issues such as his philosophy of religion and his immaterialism, the objections, as previously stated are nevertheless well thought out and not easy to dismiss out of hand. Therefore, I shall take them on board as a serious worry for the notion that we can be realists about mathematical objects. In so doing I shall consider how these objections relate to three positions that are held by participants in the debate, and consider the impact they have on these positions. These are:

Platonism – the realist idea that mathematical objects are abstract and have an independent existence.

Fictionalism – the idea that mathematical objects don't exist and are merely useful fictions, and therefore that statements which purport to be about them are not literally true.

Truth-value Realism – an alternative version of realism that does not depend upon the actual existence of mathematical objects in order for statements which purport to be about them to be true.[158]

[158] Another alternative form of realism that has been put forward is the set theoretic realism of Penelope Maddy, which she put forward in 1990, and in which the accepted existence of sets provides a starting point for accepting the existence of mathematical objects. See her *Realism in Mathematics* (Oxford:

In this section I shall describe each position in turn, and briefly discuss how it may or may not help with overcoming Berkeley's objections, before summing up and making the case for the metaphysical picture that may best succeed in doing so.

5.1 Platonism:

Mathematical platonism is a realist position within the philosophy of mathematics. Øystein Linnebo has defined mathematical platonism with the following three theses:

There are mathematical objects.

Mathematical objects are abstract.

Mathematical objects are independent of intelligent agents, their language, thought and practices.[159]

Platonism is not simply the view that Plato himself held, although it is inspired by his theory of abstract and eternal forms. Contemporary platonism, as characterised above is, unlike some older versions, standardly taken to be a purely metaphysical position. It excludes epistemological and modal claims that earlier definitions may have included.[160]

Clarendon Press, 1990). Maddy has however subsequently changed her position. See her *Defending the Axioms* (Oxford: Oxford University Press, 2011)

[159] Øystein Linnebo, "Platonism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Winter 2013 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/win2013/entries/platonism-mathematics/ (Accessed 14 August 2016)

[160] Ibid

According to Linnebo the most important argument for the first thesis comes from Frege and is put forward in *the Foundations of Arithmetic*, in which Frege attempts to analyse the concept of number. Linnebo sums up this argument, which can be found in sections §§26-27 and §§61-62 of *the Foundations of Arithmetic*.[161] Frege argues that when we use mathematical language we talk as if we are referring to, and quantifying over, mathematical objects, and that many of our mathematical theorems are true. In addition, in order for a sentence to be true the sub-expressions contained within it must actually do what they purport to do. Therefore, mathematical objects must exist otherwise it would not be possible for the statements of mathematics to be true.[162] So, according to Frege, when we accept that a mathematical proposition is true, the only way that it can be true is if the objects that the sentence is about actually exist.

From this first thesis Linnebo argues that we can get to platonism by adding the abstractness and independence conditions. Abstractness, he suggests, can be added reasonably uncontroversially. For instance, we could add a constraint that we should avoid ascribing any features to mathematics that makes its practice inadequate or misguided. So, if we were to ascribe concreteness to mathematical objects we would regard the fact that mathematicians show no interest in the spatio-temporal location of these mathematical objects to be inadequate and misguided. This is because we expect that those who study particular concrete objects are concerned about those objects' whereabouts. Zoologists, after all are concerned about the locations of the animals they study.[163] Importantly, this notion of abstractness and concreteness is not the same as Berkley and what he is getting at in his anti-abstractionism. Berkeley's notion of

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[161] Gottlob Frege, *The Foundations of Arithmetic*, 1884, trans. by J. L. Austin, Second Revised Edition (New York: Harper and Brothers, 1953), §§26-27 & 61-62, pp 72-73
[162] Linnebo
[163] Ibid

abstract is the idea that we can't perceive an object of any kind abstracted from its properties, as illustrated in 'Three Dialogues between Hylas and Philonous', and the human mind would be regarded as concrete albeit non-material. Whereas the platonist notion of abstract is that entities which exist, but we don't 'bump into' in the material world must be abstract, that is to say non-concrete or non-physical. Hence, mathematical objects, if there are any, are abstract.

Linnebo also argues that the independence thesis, 'that mathematical objects, if there are any, are independent of intelligent agents, their language, thought and practices',[164] while maybe not as uncontroversial as the abstractness thesis, is tacitly accepted by most analytic philosophers (although objections to it have been raised by constructivists and intuitionists). This may not be a thesis that is loudly endorsed. In fact, its acceptance may come more from a worry that we don't understand what it would really mean for the independence thesis to fail. Further, Linnebo believes that our conceptions of ordinary physical objects may provide the template for this tacit acceptance of the thesis.[165]

Standard platonism deals with the question of infinity and accepts infinite numbers. Frege tells us that the truths of arithmetic and thus the natural numbers sequence can be derived by logic alone.[166] And even the transfinite numbers discovered by Georg Cantor, shortly before Frege was writing, are accepted into the ontology. Frege argues that they too have been logically introduced and that we can quantify over them and refer to them in the same way as we do finite numbers by using the names and

[164] Ibid
[165] Ibid
[166] Frege, §§77-83, pp. 90-96

symbols that are used for them in mathematics.[167] Infinitesimals too are discussed, but they are an example that needs a little further explanation. Frege does not appear to take them to be an unsurmountable problem to his theory, but he does think that we lack a true idea of them. But he argues the we do sometimes even with ordinary physical objects lack a true idea of them, such as when we consider the Earth, but this does not mean that we don't give the words we use to refer to such objects a meaning, or stop using the words. The problems arise when we try to consider the words in isolation and not in the context of an expression.[168] When it comes to infinitesimals we can make sense of the equations such as $df(x) = g(x)dx$, but not be required to be able to draw a specific line segment to indicate the infinitesimal distance alluded to in the equation.[169] The notion of an infinitesimal then can still be understood as the word has meaning. But arguably this is not entirely satisfactory and may even reduce infinitesimals to a mere turn of phrase.

Subsequently to Frege, a major argument for platonism was the Quine-Putnam indispensability argument. This argues that mathematical sentences form an indispensable part of our empirical theories of the physical world, that we have good reasons for thinking that these empirical theories are true and therefore we have good reasons to think that our mathematical sentences are true.[170] The position I shall consider in the next section, that of fictionalism, arose mainly in response to this argument. There have also been moves to preserve realism that recognise some of the objections to platonism. In the twentieth century, alternative varieties of realism began

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[167] Frege, §§84-85, pp. 96-98
[168] Frege, §60, pp. 71-72
[169] Frege, footnote 1, p. 72
[170] Mark Balaguer, "Fictionalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Summer 2015 Edition), Edward N. Zalta (ed.), URL = http://plato.stanford.edu/archives/sum2015/entries/fictionalism-mathematics/ (Accessed 8 December 2015)

to be put forward, and a non-platonist variety of realism, truth-value realism will be explored too.

Berkeley clearly would have issues with platonism that come from his motivations for his arguments about the calculus, specifically his idealism, as well as those that derive from the arguments themselves. As we have seen, Berkeley had a thesis of anti-abstractionism and was at pains to show that there were no abstract ideas. A specific example showing how this relates to mathematical objects is his argument against Locke that we can form no general idea of a triangle. All our ideas of triangles come from our experiences of actual triangles that we perceive, and we are not able to abstract away from them. Specifically in order to form a general idea of a triangle it would have to come from our observations of triangles with various properties, and the abstract triangle would have to have all and none of these properties in order to be truly abstract. It would thus have to be 'neither oblique nor rectangle, neither equilateral, equicrural, nor scalenon, but *all and none* of these at once.'[171] This even is framed in terms resonant of 'fluxions'; being both something and nothing. When it comes to numbers themselves ideas of numbers for Berkeley, his view is at odds with the notion that numbers exit. They too cannot be abstracted away from ideas of the things that are numbered and can therefore not be framed in the abstract.[172] The final premise would be rejected because under idealism nothing can have an existence independent of thought. Thus, for Berkeley, all three of the premises of platonism would be contested.

Focusing on the arguments themselves, the obvious sticking point to platonism is going to be the very notion of infinite numbers and infinitesimals, and despite Frege's

[171] 'Principles of Human Knowledge, §13, pp. 80-81.
[172] 'Alciphron' §5, p. 126

logicism and contextualisation which inform his platonism even he appears to recognise that infinitesimals are not quite so straightforward. Infinitesimals do appear to present a unique problem to platonism. So even if we accept the abstract nature of mathematical objects and further accept that numbers we can't properly apprehend exist, as in the case of very large or transfinite numbers, there is a further concern with infinitesimals as used in the calculus in that they are in Berkeley's view not quite something and not quite nothing. Here an objector would argue that the first premise, applied to infinitesimals, that they exist is going to be contradictory: we first imagine them to have a positive quantity and then imagine that quantity to be zero. The argument that we do get at truth is not going to be enough to compel us to accept platonism, when the road to truth relies on contradictory premises.

This then raises questions for the truth we do arrive at when we use the calculus. Should we accept that statements made by using the calculus are not literally true, as in mathematical fictionalism? Or can the notion that we get literal truth be preserved by recourse to a version of realism that does not have the ontological demands of platonism? These positions must too be evaluated before a final conclusion is made.

5.2 Fictionalism:

Mathematical fictionalism is a metaphysical position that argues that mathematical objects do not exist, and thus statements that purport to be about them are not literally true. Instead talk of mathematical objects should be interpreted as concerned with useful fictions. This means that statements about them can only be true in the sense that statements about fictional characters are true. Hartry Field who introduced modern fictionalism argues that '2 + 2 = 4' is not literally true. It is only true in the sense that the sentence 'Oliver Twist lived in London' is true. That is to say the statement about Oliver Twist is true only in the sense that it is true according to the story by Charles Dickens. In a similar way, the sum 2 + 2 = 4 is true only in the sense that it is true according to standard mathematics.[173] As explained by Michèle Friend a fictionalist will accept standard mathematics but deny that mathematical objects exist. For a fictionalist standard mathematics is standard because it can be readily applied to the world, but this is not enough to warrant an ontological claim that mathematical entities exist.[174] We have no more reason to believe in the independent existence of mathematical entities than we do fictional characters, so they are treated the same in terms of truth.

Broadly speaking there are two varieties of fictionalism, a nominalisation, or 'hard-road', version where recourse to mathematical objects is recast in non-mathematical language, and a no-nominalisation, or 'easy-road' version in which we maintain reference to mathematical objects.[175] As stated fictionalism was introduced by Field with reference to the Quine-Putnam indispensability argument, which states that talk of mathematical objects is indispensable to our scientific theories, because he saw it as the only serious argument for the existence of mathematical entities.[176] But that does not preclude it being applied to other arguments for platonism. In his version of the indispensability argument Putnam argues that scientific language cannot be nominalised, i.e. it cannot be recast without recourse to abstract entities.[177]

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[173] Hartry Field, *Realism, Mathematics and Modality*, (Oxford: Basil Blackwell, 1989), p. 3
[174] Michèle Friend, Introducing Philosophy of Mathematics, (Trowbridge, Acumen Publishing, 2007), p. 135
[175] Balaguer
[176] Field, *Science without Numbers*, (Oxford: Basil Blackwell, 1980), p.5
[177] Stewart Shapiro, *Thinking About Mathematics* (New York: Oxford University Press, 2000), p. 216

Indispensability is therefore rooted in Putnam's belief that nominalistic language is inadequate for science. He argues that being restricted to such a language would involve giving up most of mathematics due to the fact that mathematics presupposes the existence of abstract entities.

Putnam gives a particular example of Newton's law of gravitation. He argues that it presupposes certain things; the existence of forces, distances and masses as things that can be measured by real numbers. He explores the notion of using points, but he argues that we don't really have a way of measuring the distances between these points without recourse to numbers. Numbers are therefore needed for us to understand the law.[178] Field argues, *contra* Putnam, that empirical theories can be reformulated without recourse to abstract objects such as numbers, and he attempts the enterprise with the Newtonian gravitational theory.[179] It is arguably not clear that Field's nominalisation can fully achieve what it sets out to do. Friend argues that the expression of Newtonian mechanics that Field uses is still reliant on points, which as Putnam discovered need to be differentiated by numbers, so mechanics is not actually expressed without relying on numbers.[180] It also raises issues of how hard it would be to establish more generally. To determine whether this nominalisation project really works it seems that we would need to carry it out for each of our empirical theories.

In contrast the more recently developed type of fictionalism known as easy-road fictionalism sidesteps Putnam's claims that scientific language cannot be nominalised and is therefore a no-nominalisation response. Balaguer who holds this kind of view thinks that mathematics functions as a descriptive or representational aid. It gives us an

[178] Hilary Putnam, *Philosophy of Logic*, (New York: Harper and Row, 1971), pp. 36-43
[179] Science without Numbers, pp. 61-91
[180] Friend, p. 136

easy way to make claims about the physical world, and our use of numbers in these claims gives us an easy way to describe such things as the temperature states of physical systems.[181] We therefore have no need to dispense with mathematical entities, we just use them as tools to make science more comprehensible, but this is not the same as accepting that they exist. Mary Leng, in her brand of easy-road

fictionalism discusses how science utilises fictions in some theories and presents the argument that numbers could be playing a similar role.[182]

On the face of it, fictionalism may offer a useful metaphysical account of infinitesimals under a Berkeleyan analysis of the calculus. There are two factors to this idea as a potential solution. Firstly, the objections themselves offer a comparison with fictionalism as a general position, and secondly the project to redevelop the calculus without infinitesimals, suggested by Berkeley has some similarities with the hard-road version of fictionalism.

Taking the objections themselves, if we consider the best argument for the compensation of errors thesis, for example, detailed in section 3.3 of this thesis, Berkeley argues that we get at truth, but not science. For him truth is reached in the long term only because two erroneous steps in the process cancel each other out, as both utilise quantities that are too large by the same value. Therefore, although the final result is true, true claims have not been made at every step of the proof. Perhaps these steps could be thought of as useful fictions that get us the right result. But this approach

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[181] Balaguer

[182] Mary Leng, 'Truth, Fiction and Stipulation' in New Perspectives on the Philosophy of Paul Benacerraf: Truth, Objects, Infinity, ed. by Fabrice Pataut, (Springer) Forthcoming, available at: https://www.academia.edu/3852067/Truth_Fiction_and_Stipulation
(Accessed 19 October 2015)

may be at odds with aspects of the relationship between fictionalism and science which will be explored when the three approaches presented in this section are evaluated.

The suggestion that the calculus of Newton and Leibniz may be redeveloped without the use of infinitesimals is echoed in Field's project of reformulating Newton's gravitational laws without recourse to numbers. As previously stated, the nominalisation of infinitesimals in the calculus would not amount to full fictionalism of mathematics, because mathematical objects would remain in other branches of mathematics, and even within the calculus itself. However, the nominalisation of a fundamental element of the calculus, namely the use of infinitesimals could serve to render the calculus as fictionalist as a whole. Under such a scenario its methodology would be reliant on the inclusion of entities we take to be fictional, and would have been demonstrated to be useful tools for getting the right results, without being a direct road to truth. This would serve to preclude a realist view of the calculus, and it may enable a Berkeleyan to be a fictionalist about the calculus, but a realist about mathematics in general. (This would of course mean that the Berkeleyan would have to adhere to a non-platonist form of realism, i.e. one that did not require them to postulate the actual existence of mathematical objects, and, as we shall see in the discussion of truth-value realism, such possibilities exist, that is not to say that truth-value realism is the only non-platonist realism available.) It is interesting, for instance, to note that a non-platonist version of realism could be made compatible with idealism, as the immaterial nature of numbers does not present an obvious problem to an idealist picture, wherein nothing we experience is material.

However, such an enterprise as described would be open to similar charges as Field's nominalisation process. Berkeley was able to nominalise a small portion of the calculus along classical lines for the proofs he utilised in 'The Analyst', specifically for problems in the theory of conic sections. However, we don't know if such a programme is possible across the calculus as a whole. It also seems unlikely that any attempt will ever be made. Hard-road fictionalism has fallen out of favour as a way of solving these kinds of metaphysical problems, so it is difficult to imagine anyone prepared to take up the challenge for a smaller branch of mathematics, and which would do little to help broaden the nominalisation of mathematics more generally. Further, the direction in which the calculus was developed, first by using the notion of limits and then with real analysis, has removed any potential need to worry about recasting it along the classical lines such that Berkeley suggested.

Fictionalism may appear to have something to offer the Berkeleyan, but some of the issues identified here indicate that a realist picture, which can tackle Berkeley's objections, particularly one that can be non-platonist in nature, would still be the preferred option. I now therefore turn to a metaphysical picture that can be presented as non-platonist, that of truth-value realism, in order to assess if it can tackle the objections.

5.3 Truth-Value Realism:

Truth-value realism holds that every well-formed mathematical statement has a unique and objective truth-value that is independent of whether it can be known. Importantly this view gives no ontological viewpoint and thus is not dependent on the existence thesis in platonism, and on its own does not hold it. The independence thesis, which truth-value realism preserves, then relates to mathematical statements, which have independent truth-values. Without the existence thesis truth-value realism is not a form of platonism, but the theory can become a form of platonism by adding the existence thesis.[183]

Linnebo tells us that truth-value realism involves a mathematical language and a philosophical language, so there is no contradiction between a mathematician asserting the existence of mathematical objects and a philosopher denying it. Someone who holds this view can then say that a statement that refers to the existence of abstract entities is true in the mathematical language, but the assertion that there are no mathematical objects is made in the philosophical language. The statements in the mathematical language are then translated into the philosophical language.[184] Shapiro describes these 'translations' as a systematic way of interpreting mathematical language, so that it does not reference mathematical objects, but grants that mathematical sentences preserve their standard truth-values.[185] Truth-value realism comes in different varieties and has no one formulation. Thus, there are different philosophical languages available. Truth-value realism may give an account of how mathematical sentences can be true, but it differs widely from platonism in that it can be used in systems that deny mathematical objects exist.

[183] Linnebo. Although we might naturally assume that platonism entails truth-value realism Linnebo tells us that neither view in fact entails the other; platonism does not rule out the possibility of indeterminacy in mathematical statements.
[184] Ibid
[185] Shapiro, p.227

I shall focus on the variety of truth-value realism developed by Charles Chihara in his book, *Constructibility and Mathematical Existence*.[186] The purpose of the focus on Chihara is not to endorse one particular form of truth-value realism, but rather to briefly present a system within the position, in order that we can gain an understanding of how the truth of mathematical statements may be accepted without any recourse to the existence of mathematical objects. Hellman's structuralism could equally have provided the template.[187] Interestingly both offer a modal approach to mathematical statements.

Chihara is presented by Shapiro as an ontological anti-realist. That is to say that he is anti-realist about the existence of mathematical objects. (Mathematical ontological anti-realism does not in itself have to deny the existence of mathematical objects, it can merely be agnostic about it, and this explains why Linnebo argues that the existence claim of platonism can be added to the position). In his project Chihara replaces talk of sets with talk of open sentences. For instance, the English sentence '*x* is a sloth' is an open sentence which is satisfied by Gallina, who is a sloth living in Colchester Zoo. An open sentence is one in which a singular term gets replaced by a variable, and in this example the singular term, or name, 'Gallina' has been replaced by the variable *x*.[188] Gallina satisfies this sentence because it is true of Gallina that she is a sloth, and it is this relationship of 'true of' between objects and open sentences which is described as satisfaction. Using Chihara's approach we would replace talk of the set of all sloths with talk about the open sentence 'x is a sloth'. However, no natural language contains enough open sentences to cover all mathematical objects, so Chihara focuses on the possibility of writing open sentences, where such possibilities are not limited to all of

[186] Charles S. Chihara, *Constructibility and Mathematical Existence,* (Oxford: Clarendon Press, 1990)
[187] See Geoffrey Hellman, *Mathematics Without Numbers* (Oxford: Clarendon Press, 1989)
[188] Chihara, p. 41

our natural languages. That is to say he employs modality.[189] Specifically Chihara utilises the Kripkean semantics of possible worlds, but he explains that this is intended as a heuristic device, and that this is not intended to actually be foundational to his theory.[190]

Since Chihara accepts Quine's notion of ontological commitment for the ordinary quantifiers of first order logic ($\forall$ and $\exists$) but disputes it when applied to mathematical objects,[191] his programme begins with the introduction of a 'constructibility' quantifier (C) which formally behaves like the existential quantifier, so if ϕ is a formula then $(Cx)\phi$ is a formula, read as 'it is possible to construct an *x* such that ϕ', or 'it is possible that there be an x such that ϕ'. Chihara introduces a system of languages, L and L* in Chapter 2, and Lt in Chapter 4. L is a language based on the first-order language of Benson Mates in *Elementary* Logic, with a modal semantics to accommodate the constructability quantifiers. In L the only quantifiers are the constructability quantifiers. L* is a development of L in which both ordinary and constructability quantifiers are accommodated which finally leads to the more complex Lt in which his which the full constructability theory is developed.[192]

The constructability quantifier does not carry ontological commitment. As Shapiro points out this appears to be a common-sense approach.[193] (Although presumably it will carry a commitment to the possibility of the examples that we employ.) We don't, however, take talk about the possibility that something, say a house, can be constructed in a particular spot to include an assertion that such a house exists, or

[189] Shapiro, pp. 239-240
[190] Chihara, p. 25
[191] Chihara, p.53
[192] Chihara, pp. 25-37, 55-58
[193] Shapiro, p. 240

even that an abstract object such as a 'possible house' exists.[194] This is a weaker claim given by the constructability quantifiers, one which the ordinary quantifiers already make about objects that do actually exist, for to exist entails the possibility of existence. The account can therefore maintain its ontological neutrality but in line with Linnebo's characterisation the existence thesis can be added to turn it into platonism, if, for example, we assume that possibility implies existence. Chihara argues that the literalist (which includes, but is not limited to, the platonist) may believe what they do about the existence of mathematical objects precisely because mathematical language is standardly analysed in terms of the ordinary quantifiers of first order logic (which is a crucial part of Frege's logicism), meaning that they are analysed to assert the actual existence of mathematical objects.[195] This observation is clearly motivating Chihara to introduce his alternative quantifiers which don't get analysed in this literal way.

Chihara introduces the language Lt, in which his theory will be developed.[196] His theory, Ct, he describes as a simple type theory.[197] Therefore following type theory, Lt has infinitely many sorts of variables. These begin with level 0 variables which range over ordinary objects like sloths and houses and are bound by our standard universal and existential quantifiers. The constructability quantifiers are introduced at level 1, whose variables range over the open sentences satisfied by ordinary objects, 'x is a sloth' etc. These variables can only be bound by constructability quantifiers, and these open sentences correspond to sets. The language makes no provision for *existing* open sentences, all open sentences are merely considered *possible* (Shapiro's emphasis). If we wanted to linguistically pick out a specific object that any natural language lacks the

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[194] Chihara, p. 39
[195] Chihara, pp. 3-4
[196] Chihara, pp. 55-58
[197] Chihara, p. 43

resources to do, we imagine an expansion of that language that gives us the resources to do so. Level 2 variables range over open sentences that are satisfied by level 1 open sentences, which correspond to sets of sets, and so on.[198]

We can explain levels 1 and 2 in the following way. Suppose I have several pairs of earrings in my jewellery box. On level 1 I might want to pick out one earring in a pair, so I could say *x* is an earring that is one of the rightmost pair in my jewellery box. If I take *c* to be the one on the left, instead of thinking of it as a member of the set that forms that pair of earrings I would think of it as satisfying the given open sentence 'x is an earring from the rightmost pair of earrings in my jewellery box'. When we move to level 2 whose open sentences correspond to sets of sets (e.g. the set of pairs of earrings in my jewellery box), we would use an open sentence like 'α is an open sentence describing two matched earrings in my jewellery box'. Here the open sentence is in the range of level 2 variables because the variable α is of level 1. These level 2 variables are again bound by constructability quantifiers.[199]

But unlike the nominalisation approaches of fictionalism, Chihara is not attempting to revise mathematics. He is still aiming to ensure true statements of contemporary mathematics come out as actually true. His system grants impredicative definitions at each level. Shapiro explains:

> If $\Phi(\alpha)$ is any formula in which the level 1 variable α occurs free, there is an axiom asserting that it is possible to construct an open sentence (of level 2) which is satisfied by all and only the level 1 open sentences that would satisfy $\Phi(\alpha)$ (if only they existed).[200]

[198] Shapiro, pp. 241-242
[199] Shapiro, p. 241
[200] Shapiro, p. 241

This impredicativity is argued for by Chihara due to the nature of the modality involved. Shapiro accepts that the system Chihara developed has some similarity with ordinary type theory, but he argues that there are important differences. If, using type theory a mathematician proves the sentence ‘there is a type 3 set x such that $\Phi(x)$’then the type theorist is presupposing the existence of a set of sets of sets. But in Chihara’s system, his equivalent sentence instead has the form ‘it is possible to construct a level 3 open sentence s such that $\Phi^*(s)$’.

After presenting the theory Ct, Chihara uses the approach to develop mathematics such as arithmetic and analysis, similarly to the way they are developed in simple type theory.[201]

Shapiro points out that Chihara’s brand of truth-value realism is a nominalist position. This is of course something it has in common with fictionalism. However, unlike fictionalism, it accepts the actual truth of mathematical sentences and does not categorise them as merely useful fictions. But it does not accept the independent existence of mathematical objects. The sentences come out true, but only because they can be reformulated along the lines of open sentences. The possibility of construction gives these sentences their truth, rather than an ontological commitment to the actual existence of any mathematical objects.

[201] Chihara, pp. 80-121

Chihara's truth-value realism can give us a principle asserting the possibility of constructing infinitely many objects. As the approach is fully equipped to deal with the infinite (as demonstrated in terms of the infinitely large of set theory), it should also be able to deal with the infinitely small, giving us infinitesimals as a merely possible construction; so sentences that utilise them come out true, but to refer to one is not to presuppose its existence, although this language may not be that straightforward to formulate. Assuming that this can be overcome, it remains a theoretical possibility that allows us to get to the truth of certain mathematical statements. However, it is not clear that this will solve the objection of inconsistency that arises from the use of infinitesimals in Newtonian or Leibnizian calculus whereby the distances are postulated first to have a positive quantity and then presented as being zero, at different stages in certain proofs. There are also some issues again which would be problematic for Berkeley's motivations for his objections; namely that the statements of mathematics exist independently.[202] The position is therefore in conflict with the mind-dependence of immaterialism.

I shall in the next section end this chapter by evaluating how these three positions might relate to Berkeley's objections and determining if any of them can explain the success of the calculus in view of the objections.

[202] It is of course the case that in some theories the notion of construction depends on a mind being able to do the constructing, but in this instance because we are only dealing with the possibility of construction and owing to the iterative nature of the type theory that Chihara employs, this restriction would appear not to be in place here, allowing the independence thesis to be reached.

5.4 Analysis of the Metaphysical Positions in Relation to Berkeley:

I will consider the metaphysical accounts discussed in this chapter both in terms of the arguments that motivate Berkeley's objections and in terms of the objections themselves. The second considerations are the more important, as the validity of these objections has been the primary focus of this thesis. However, it seems to me that in the pursuit of a deeper review of Berkeley's philosophy of mathematics, the question of whether Berkeley's broader philosophical outlook presents a problem for these metaphysical positions is of interest.

Firstly, it is important to remember that even the inventors of the calculus had concerns about the precise nature of infinitesimals and fluxions. As noted, Leibniz thought of infinitesimals as a useful façon de parler. This would perhaps best accord with a fictionalist metaphysical account of mathematical entities. Mikhail G. Katz and David Sherry argue as much, taking Leibniz's own description of infinitesimals as mental fictions in a letter to Des Bosses dated 1706, as a justification for their viewpoint. The fictional nature that Leibniz attributed to infinitesimals is described by them as giving them an ideal ontological status along the lines of complex numbers.[203]

It has also been pointed out that Newton made suggestions for the elimination of infinitesimals by considering them as limits, which is of course an approach that was taken later in the development of the calculus. However, the proofs of his calculus that Newton develops and which have been presented in this thesis as the targets of Berkeley's objections still utilise fluxions.

[203] Mikhail G. Katz and David Sherry, 'Leibniz's Infinitesimals: Their Fictionality, Their Modern Implementations, and Their Foes from Berkeley to Russell and Beyond', *Erkenntnis* 78, 571–625, p. 572-576

5.4.1 The Motivations in Relation to the Metaphysical Positions:

Platonism is plainly at odds with Berkeley's idealism. The first and third thesis do not hold for idealism. The existence of numbers is problematic in that they don't exist in their own right only in relation to ideas of objects that employ the concept. Independence is also a problem according to Berkeley, nothing can exist independently of a mind, and God who gives everything existence is described as a mind—an infinite mind, but a mind nevertheless. Numbers then cannot be mind-independent. This would mean that the second thesis that numbers, if they exist, are abstract (in the sense employed by platonism rather than a Berkeleyan sense), would be at best vacuously true given the status of premise one.

Fictionalism fares better here, in that numbers don't exist on their own for Berkeley and they don't exist at all for fictionalism. They are just part of the story for the fictionalist, and only can be perceived when they are part of the idea of something else for Berkeley. So there appears to be no inherent problem with postulating fictionalism as a method of dealing with mathematical entities in Berkeley's idealism.

Truth-value realism as a general position is agnostic on existence, however Chihara's version of truth-value approach does appear to eschew the idea that mathematical objects do actually exist. The existence claim can be added to truth-value realism, so there is nothing inherently wrong with a truth-value realist position that allows non-abstract mathematical objects to exist as an idea. As the theory in general terms is agnostic on the existence of numbers, then this does not present an obvious problem. But we would still have to overcome the independence thesis, and truth-value realism would still be faced with the same objections to the independence thesis as platonism.

5.4.2 The Objections in Relation to the Metaphysical Positions:

Platonism as a metaphysical picture is not going to be a suitable means for dealing with the infinitesimals of the calculus. In a sense it is key to the problem. Although platonism as we understand it had not been formalised at the time Newton and Leibniz were developing the calculus, mathematical practice was undertaken, and to a large extent still is, as if mathematical objects do exist, at least while mathematics is being carried out. Mathematical objects are utilised because they are needed for the work of mathematicians. Today we call this working platonism. But as Shapiro points out this is simply a theory of how mathematics is, and should be, done and is in fact silent on ontology.[204] However this tacit presupposition about the existence of mathematical objects appears to be at play in Berkeley's programme. And it is therefore against this backdrop that the logical and metaphysical objections of incoherence that Berkeley finds in the use of infinitesimals in the calculus are brought to bear. Two aspects of the nature of infinitesimals are problematic for platonism. In the first instance the very notion of something existing that is infinitely small is difficult to quantify. In other words, platonism can't overcome the metaphysical objections. In the second, the problem of assuming infinitesimals to have a quantity and then for them to have a value of zero at different points in a proof presents itself as a hard metaphysical position to hold, especially when that position is taken to presuppose existence. Therefore, it also can't overcome the logical objections. Whether or not it can cope with the compensation

[204] Shapiro, pp. 39-40

of errors thesis is left undetermined. There is nothing built into the theory that is specifically equipped to deal with this objection, but it is not obviously precluded either.

Moving on to Fictionalism, it can defeat the metaphysical objections, simply by holding that mathematical objects don't exist. The question of the logical objection is more complex and may be dependent on the particular type of fictionalism employed. Fictionalism treats mathematics like a 'story', but the sentences of the 'story' would include some claims that are contradictory, again by assuming infinitesimals to have a quantity and then for them to have a value of zero at different points in the 'story'. While fiction does not have to be consistent, (although arguably we prefer our stories not to contain plot holes) the stories of mathematics, in so far as they are supposed to be similar to mathematics would presumably be best thought to be consistent. However, the fictionalism of Mary Leng (as we shall see) may be able to sidestep this objection. Berkeley's claims that the calculus can get at truth but not science is also important for considering an appropriate metaphysical picture, and here fictionalism is in the strongest position. There is some resonance with the easy-road fictionalism of Mary Leng, in which she accepts that science itself contains some useful fictions, generally idealisations of how things would behave in perfect circumstances.[205] But her arguments have a key difference. She accepts the fictions as part of science. Science then is true, in so far as it can be, in spite of the inclusion of fictions. The contrast between truth and science is not being made. Berkeley argues that the truth that we get in the compensation of errors thesis is due to the two errors which me make in certain proofs of the calculus. These would be analogous to the fictions which are the stipulations that Leng cites. But for Berkeley it's precisely the presence of these fictions that mean that

[205] Leng

the calculus is not science. This may be an indication that easy-road fictionalism is not a suitable match. The hard-road version of fictionalism first proposed by Hartry Field, of course, does have some similarities with Berkeley's proposal to nominalise the calculus. But, as has been argued earlier in this chapter, hard-road fictionalism faces strong objections that would have to be overcome before it could seriously be put forward as a solution.

Truth-value realism too has an interesting relationship with Berkeley's suggestion for nominalising the calculus because it nominalises mathematical language. Further, in order for it to be a successful candidate, that nominalisation would have to include infinitesimals. However, as an approach truth-value realism has in place the capacity to handle infinity, through its ability to recast set theory in terms of open sentences. Thus, dealing with infinitesimals should not prove too much of a challenge. Of course, we would be transferring our language from talk of the infinitely big to talk of the infinitely small but it is not obvious that this cannot be done. Truth-value realism therefore overcomes the metaphysical objections. It can talk about infinitesimals without having to make a claim that they actually exist. The logical objections, however, are not overcome. As a linguistic approach, truth-value realism means that some of the sentences in some of the proofs are inconsistent, so the proofs will contain two contradictory premises. Therefore, there remains a worry that we end up saying something both exists and doesn't. Truth-value realism may be able to account for open sentences of the form 'x is an infinitesimal' but it would probably still struggle to explain how x is a quantity and x is nothing. It is not clear that there is another way to recast the language in a way that can deal with sentences that contain the two contradictory claims. Perhaps the suggested existence of infinitesimals themselves would cease to be a problem, but the sentences in the proofs still prove to be problematic by dint of the contradictions. This issue is not easily solvable using an approach that depends upon nominalism. Again, as with platonism, there is nothing in the theory that specifically deals with the compensation of errors thesis, so it Is not clear if truth-value realism overcomes it either.

Berkeley's philosophical landscape, specifically his idealism, is at odds with all the metaphysical positions we have considered. Further, his objections in 'The Analyst' do show up the issues that platonism has in dealing with quantities such as infinitesimals, when we set his motivations aside. The first thesis that mathematical objects exist is deeply problematic in the case of infinitesimals. Approaches that countenance the nominalism, as opposed to the nominalisation,[206] of infinitesimals, either by describing them as fictional entities or by expressing the literal truth in a new formal language, provide a way of denying their actual existence, while still appreciating the veracity of the results derived from their use. However, the two-step process in the proofs, whereby infinitesimals have a quantity which is also taken to be zero, and the objections Berkeley raises to that process are not overcome by this approach.

Perhaps all this demonstrates why Berkeley's objections have the resonance that they do. It appears that they stand up against the major metaphysical viewpoints that try to explain the nature of mathematical entities, even today.

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[206] This refers to the distinction that Balaguer makes between nominalism and nominalisation. Nominalism is just the belief that mathematical objects should be replaced ontologically with alternatives whether we take the language of mathematics to be the correct one or not. Nominalisation on the other hand suggests that there should be an alternative language constructed that allows us to make the same statements that we make when we use mathematical language. Nominalisation approaches include Field's hard-road fictionalism and Berkeley's suggestion for a geometric language for the calculus. Leibniz's view of infinitesimals as a façon de parler would I think be nominalism, but not nominalisation.

Fortunately, the progression from infinitesimals to real analysis in the methodology of the calculus means that we have options that can account for how we arrive at truth when we employ the calculus today. For modern calculus, we can take an easy-road fictionalist approach, if we are so motivated, or, if we want to argue that a form of realism can be employed, real-analysis has been translated into Lt, the language introduced by Chihara.[207] We can't after all, it seems, present a metaphysical picture that solves the past problems of the calculus, even with a more developed metaphysical picture that we have today, unless it does prove possible to nominalise the 17th century calculus. Some of our understanding of why the Newtonian and Leibnizian calculus worked would appear to remain philosophically mysterious

[207] Chihara, pp. 95-121

6. Conclusion:

In the course of this thesis I have considered the calculus of the seventeenth and eighteenth centuries from a historical and philosophical viewpoint. In the discussion of the history of the calculus and the priority dispute we have seen how, in the course of this dispute, each participant accused the other of plagiarism and attempted to inflict some damage on his rival's reputation. In part this behaviour was symptomatic of the way science was practised and the way in which people were credited with scientific discoveries, prior to the acceptance of the phenomenon of convergence. But the levels of bitterness on both sides, the attention the dispute received and the prevalence of so much documentation of it, both at the time, and subsequently, also highlights how significant the mathematical discovery was, and how important the calculus is to modern mathematics. This significance is a reason that we should take objections to it seriously. Even though the methodology of the calculus may have developed in such a way that means these problems may no longer arise in the calculus as practised today, the calculus as it was practised then still yielded true results. Understanding how this is so is therefore an important endeavour.

To that end, I have highlighted key objections which were introduced by Berkeley and particularly focused on the use of infinitesimals in the calculus. These objections raise philosophical questions which are still central to the debates in the metaphysics of mathematics, in particular, what metaphysical picture can best explain the ontological status of mathematical objects, and how, and in what sense mathematics is true. These objections, as we have seen, are of three types; metaphysical objections, logical objections and a compensation of errors thesis. Broadly, the metaphysical objections can be taken to be concerned with the nature of the objects of the calculus, infinitesimals. Berkeley takes their nature to be inconceivable, arguing that we can't conceive of increments of space and time before they become finite. These objections appear to be based on at least an implicit understanding of realism in mathematics. As other metaphysical positions have subsequently been put forward, these objections remain the easiest to defuse.

The logical objections argue that some of the proofs of the calculus treat an infinitesimal as both having and lacking a quantity. Certain premises within these proofs assume that an infinitesimal has a quantity, while others assume that that quantity is zero. Therefore, as the proof relies on the two contradictory premises, it must be flawed. This remains the most intractable objection.

The compensation of errors thesis argues that the calculus only yields true results due to the fact that there are two compensating errors in some of the proofs, which cancel each other out in the course of the calculations of the proof. For this reason, Berkeley argues that the calculus can get at truth, but not science. The overcoming of this objection is harder to determine, as only one position appears obviously equipped to handle an objection of this sort.

The objections have been considered in relation to three major theories of the twentieth and twenty- first centuries from the debate about realism in the philosophy of mathematics; platonism, fictionalism and truth-value realism. First, each position is explained, and then the question of infinity and infinitesimals is considered in relation to the position. Each position is then assessed to see if it can overcome the three major objections.

Platonism fails to overcome both the metaphysical and logical objections. The nature of infinitesimals presents a unique problem to platonism, and here more than anywhere we can see how the metaphysical and logical objections are interlinked. Berkeley argues in the logical objections that the proofs do assume at one stage that an infinitesimal has a certain positive quantity and later assume that it has a quantity of zero. This, as well as the notion of something infinitely small, in itself makes infinitesimals very difficult to quantify and conceive. Platonism is also one of the two positions that doesn't directly deal with the compensation of errors thesis.

Fictionalism overcomes the metaphysical objection by holding that mathematical objects don't exist. However, it is not clear that all fictionalists can defeat the logical objections. Fictionalism treats mathematics like a 'story', but the 'story' then contains some claims that are contradictory. It is, though, the only position that, at least in some forms, has a method to overcome the compensation of errors thesis directly, by observing that in scientific theories we actually utilise theses that we know not to be literally true. Thus, on this score at least, it also directly argues against Berkeley's argument that the calculus is not science.

Truth-value realism overcomes the metaphysical objections as it does not depend upon the existence of mathematical objects. Further the particular version of truth-value realism considered in this thesis appears to take it that they don't exist. The logical objections, however, are not overcome. As a linguistic approach, truth-value realism means that some of the sentences in some of the proofs are inconsistent, so the proofs will contain two contradictory premises. It is less clear whether the compensation of errors thesis can be handled, as the position makes no obvious provision for it.

In each case at least some of the objections have held up. The metaphysical objections may be solved by two of the options, but the logical objections appear to remain unresolved by all three approaches, and as stated, the compensation of errors thesis remains at best unanswered by two of them. This highlights the potency of Berkeley's arguments, and the difficulty we still have in explaining why the calculus worked, and derived true results, when it was first introduced. The robustness of Berkeley's arguments indicates that these three theories of contemporary mathematical metaphysics cannot fully deal with all the objects that are, and have been, used by mathematicians. In order to provide a theory that can account for all mathematical objects and answer the specific question of how we can explain that the calculus enables us to derive useful results, we need to look further to develop one of these positions, or perhaps to find a new metaphysical position altogether.

As a secondary concern, in order to contextualise the objections, Berkeley's motivations for raising the objections to the calculus have been examined, namely his concern about the arguments of at least one mathematician that the principles of Christianity are incoherent, and his immaterialism. The three approaches to the metaphysics have been analysed in terms of his immaterialism, and only fictionalism has been found to be compatible with it.

Finally, it remains to address the questions I presented in section 3.4 of the thesis, to assess whether the modern metaphysical analysis has been able to offer any answers:

If Berkeley's philosophical arguments are valid, how can we explain that the calculus enables us to derive useful results?

This is at the centre of the enquiry and remains elusive. At least some of Berkeley's arguments have been shown to be valid, and have been able to withstand our best contemporary theories in the metaphysics of mathematics. The fact that we have been unable to resolve the objections satisfactorily, in particular the logical objections, means that there is still an explanatory gap preventing us from fully giving an account of how the calculus worked. Perhaps this helps to demonstrate why the calculus evolved in a way that it did, moving away from the use of infinitesimals as understood in the early modern period. (This sets aside the modern use of infinitesimals, which as we have seen uses a different approach, and is taken by Jesseph to sidestep the compensation of errors thesis.)[208]

If we do, or can, rely on non-entities for our mathematical proofs, does this mean that platonism cannot be the true metaphysical picture?

It would appear that the use of infinitesimals in the calculus of the early modern period does present a problem for platonism. The notion of a quantity that is not quite something and not quite nothing, or is infinitesimally small, is hard to grasp as an

[208] Jesseph, p. 205

existent entity. This makes appealing the attempt to find an explanation of mathematical truth that is not platonist.

Do modern mathematical solutions where we can generate proofs that may not depend on non-entities render the question moot?

As discussed in section 3.3, Jesseph argues that to appeal to modern mathematics would be an anachronistic approach that does not fairly attempt to address the objections that Berkeley raised.[209] Therefore we cannot dodge the objections by saying that they don't arise when we practice the calculus today.

It may be asked that if we cannot look to modern mathematics for the answers to Berkeley's objections, then can we really be justified in appealing to twentieth and twenty-first century metaphysics in pursuit of a means to overcome these same objections? I would argue that the anachronism of using contemporary mathematics derives from an approach that would develop the mathematics in such a way as to sidestep the objections rather than meet them. The application of metaphysics does not take this approach. It tries to deal with the mathematics on its own terms, and instead takes the position that it may be possible to reconcile the fact that the early calculus worked with the situation that there are legitimate questions to be asked about its methodology. In this sense the problem can be treated as if we are trying to fill an explanatory gap between the use of a successful mathematical system and our full understanding of how the system works. The fact that the system may have been subsequently developed in such a way that we no longer have to worry about the issues

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[209] Ibid

that Berkeley raised does not detract from the fact that the system worked in the seventeenth and eighteenth centuries in a way that makes such objections pertinent. Nor does it preclude the resolution of questions from the early modern age by modern developments in philosophical enquiry.

Can we have a position whereby we believe that any mathematical truth can in principle be justified without recourse to non-entities, even if such alternatives are to be developed with the tools of future mathematics?

On the face of it modern calculus provides an example that could be in line with this claim. If we are not fictionalists and we accept the literal truth of mathematics, we could point to the fact that we have rid the calculus of the necessity of recourse to non-entities such as infinitesimals and fluxions. However, this approach is unconvincing for two reasons. Firstly, a theory cannot be based on one example, so other instances besides the calculus would have to be found. Secondly, this again is an anachronistic approach that relies upon modern mathematics to diffuse a philosophical issue from an earlier time.

What is the nature of such seeming non-entities as fluxions and infinitesimals? And how do they play a role in our understanding of mathematical truths?

The question of the nature of these seeming non-entities is still not fully resolved. The notion that they could be abstract independently existing entities in line with platonism is not an easy position to maintain. An answer may be best pursued by looking towards a different metaphysical picture. Fictionalism and truth-value realism go some way to showing how we might approach the question of their nature, by essentially agreeing with Leibniz that they are façons de parler. As shown by the attempts in this thesis to explain how we understand mathematical truths in light of the objections to the calculus and its use of infinitesimals the second of these questions remains somewhat mysterious.

If all this, however, leads us to conclude that platonism is in fact the wrong picture of mathematical objects, does the dependence upon non-entities in these proofs mean that all forms of mathematical realism are off the table, or is this methodology consistent with non-platonic forms of mathematical realism such as truth-value realism?

We have in the course of this thesis concluded that platonism is the wrong picture of mathematical objects, but this does not in and of itself preclude other forms of mathematical realism from offering possible alternatives. Truth-value realism fares slightly better in that it can see off the metaphysical objections by denying the need for mathematical objects to exist. However, it cannot refute the logical objections, and possibly not the compensation of errors thesis either.

Must we look to an alternative picture, such as fictionalism?

Fictionalism does appear to come out strongest in the debate, because as well as being able to overcome the metaphysical objections, some varieties of easy-road fictionalism clearly do have a mechanism for overcoming the compensation of errors thesis. Further, it is in accord with at least one interpretation of how Leibniz saw the calculus. However, it still does not obviously overcome the logical objections. All of this means that more work will have to be done in order for us to be able to satisfactorily answer all of Berkley's objections, and for us to fully understand the mysteries of the early modern calculus and the infinite in early modern philosophy.

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