Monetary and fiscal policies near the zero lower bound

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of Birkbeck is solely my own work, except where stated otherwise.

Abstract

This thesis contributes to the ongoing debate on the conduct of monetary and fiscal policies near the lower bound on the policy rate.

Chapter 1 studies optimal monetary policy in a model with portfolio adjustment costs. Central bank purchases of long-term debt (quantitative easing; 'QE') influence the average portfolio return and hence aggregate demand and inflation. It is optimal to adopt QE rapidly with large scale asset purchases triggered when the policy rate hits the lower bound, consistent with observed policy responses to the Global Financial Crisis. Optimal exit is gradual.

Chapter 2 examines the effects of money-financed fiscal transfers at the lower bound. It is assumed that money may earn interest so that money-financed transfers at the lower bound are feasible, while the short-term policy rate is used to stabilize the economy in normal times. A simple financial friction generates a wealth effect on household spending from government liabilities. This encourages households to spend rather than save a transfer from the government. While temporary money-financed transfers to households can stimulate spending and inflation at the lower bound, similar effects could be achieved by bond-financed tax cuts.

Chapter 3 explores optimal monetary policy in a model with long-term government debt and 'active' fiscal policy. This means that stabilization of government debt is a binding constraint on monetary policy. Away from the lower bound, policy cannot fully offset the effects of shocks to the natural rate of interest, reducing welfare. At the lower bound, recessionary shocks increase debt and generate the anticipation that inflation will be higher in future, to stabilize the real value of debt. This mechanism mitigates the effects of recessionary shocks. For sufficiently long debt duration, improved performance at the lower bound may outweigh welfare losses in normal times.

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Though this thesis is my own work and does not necessarily reflect the views of the Bank of England or any of its policy committees, I am fortunate to be surrounded by so many thoughtful and talented people in my day job. I am particularly grateful to Gareth Ramsay and James Bell, my managers, for their flexibility and understanding. Chapter 2 was written jointly with Ryland Thomas, who has been a constant source of enthusiasm and challenge.

Throughout my life I have been very fortunate to have the wholehearted encouragement of my parents. They have done everything in their power to help me pursue my academic goals, build a career and ultimately (finally!) take on the challenge that has been tempting me and scaring me in equal measure for two decades. I am forever grateful.

My wife has been a constant support throughout my PhD experience and, perhaps more importantly, the many years of uncertainty and indecision beforehand. Her unwavering belief in me has been a source of great comfort and strength, particularly in the inevitable times of self doubt.

Karen, I dedicate this thesis to you.

Contents

List of Figures

List of Tables

Introduction

The interaction of monetary and fiscal policy has been the subject of extensive study for many years. The wide range of important questions in this area has generated several strands of literature. For example, Blinder (1982) examines the appropriate level of coordination between monetary and fiscal policies using Tinbergen-Theil analysis, large scale macroeconomic models and game-theoretic approaches.

Indeed, analysis of these questions predates modern economics, in part because many central banks were formed to assist in the financing of the central government (for example, in France, Spain and the United Kingdom). For the United Kingdom, Ricardo (1824) discusses the incentives for the government to use monetary policy for fiscal purposes.

However, the recent experience with interest rates at the zero lower bound has raised new questions.

Shortly after the Global Financial Crisis in 2007–8, short-term policy rates in the United States and United Kingdom reached their lower bounds. Policymakers embarked on large scale purchases of assets (primarily government debt) financed by the creation of central bank reserves. This unconventional policy – commonly known as quantitative easing (QE) – was introduced in a bid to stimulate the economy, when conventional instruments were unavailable. Such policy measures were introduced without a clear understanding of some key questions. For example, how should QE be optimally used alongside the short-term policy rate? Is there scope for QE to become part of the conventional monetary policy toolkit?

Unconventional monetary policies were accompanied by discretionary fiscal stimulus in a number of countries. However, many of these programs were reversed, at least in part because of concerns about rapidly rising government debt. How should monetary policy be optimally conducted if the government is unwilling or unable to stabilize the debt stock using taxes and spending? How should monetary policy respond to a risk of such a situation arising?

While the monetary and fiscal responses to the financial crisis were bold and unprecedented, some economists and commentators have proposed even more radical policy options. In particular, some have argued that financing government spending increases, tax cuts or direct monetary transfers to households by printing money would boost spending when in an economy trapped at the zero bound. Others argued that such policies risked creating runaway inflation. What are the possible effects of such 'monetary financing' policies in a standard macroeconomic framework?

This thesis addresses these questions.

Chapter 1 looks at the optimal use of quantitative easing ('QE'): the purchase of long-term government bonds by the central bank, financed by the creation of reserves. It adds to the literature by studying the optimal deployment of QE in a model containing a 'portfolio balance channel'. This is the mechanism through which most monetary policymakers believe that QE operates.

Portfolio rebalancing effects arise from the assumption that the relative demand for assets in an investor's portfolio will depend on their relative returns. The mechanism that captures this effect in the model follows the ideas of James Tobin and William Brainard in the 1950s and 1960s. Different assets are imperfect substitutes because they have individual non-pecuniary properties that investors value. In the model, investors face a particular type of portfolio adjustment costs that give rise to asset demand functions similar to those studied by Tobin and Brainard.

The presence of a portfolio balance effect creates the potential for a policymaker to influence relative asset prices by altering the relative supplies of assets available to investors. In particular, QE reduces the available supply of long-term government bonds, which increases their price and hence reduces long-term interest rates. A reduction in long-term interest rates can be used to stimulate spending and hence inflation when the short-term interest rate is stuck at its lower bound.

The model is calibrated to match evidence on the effects of QE in the United States. I use the model to study how QE should be optimally used alongside the short-term policy rate when both instruments may be constrained. The short-term policy rate is constrained by an effective lower bound (ELB). The quantity of long-term government debt purchased by the central bank cannot be negative or exceed a pre-specified upper bound. This upper bound can be interpreted as a proxy for central bank solvency concerns highlighted by some monetary policymakers.

The analysis delivers three main results. First, QE is only adopted when the economy hits the zero bound and in such cases asset purchases are often large and occur rapidly. Second, exit from QE is gradual. Third, QE policy generally starts to tighten before the policy rate rises from the ELB.

The first two results are consistent with both observed QE policies and communications about exit plans by policymakers in the United Kingdom and United States. But the third result is not consistent with the observed behavior of both the FOMC and the MPC: 'liftoff' from the zero bound has preceded a QE unwind. I demonstrate that the third result could be reconciled with actual policy behavior by accounting for the fact that total total government debt was rising during the period of large scale asset purchases, a factor that is abstracted from in the simple model.

Chapter 2 studies fiscal policy actions financed by the creation of money. Variants of this type of policy have been advocated in light of the slow recovery from the Global Financial Crisis in many economies, despite the deployment of unconventional policies, such as QE considered in Chapter 1. The use of monetary financing as a policy tool is regarded as even more unconventional or extreme than QE, given historical experiences of high inflation (or even hyperinflation) when governments have relied on money creation to finance their deficits.

I use a small sticky price model to examine the effects of two types of money-financed fiscal policies: using money creation to finance government spending and using money creation to finance direct transfers to households. The latter has some similarities to Milton Friedman's famous 'helicopter drop' experiment.

With respect to money-financed government spending increases, I verify existing results regarding their apparently powerful effects. Specifically, money-financed government spending has large effects on output and inflation because this policy amounts to adopting a policy rule for the *short-term nominal interest rate* that responds very weakly to inflation. This implies that monetary policy accommodates the inflationary effects of a government spending increase.

I contribute to the literature by extending existing results to demonstrate that the adoption of a money-financing rule may generate very poor macroeconomic outcomes in response to other economic shocks, such as an unanticipated change in private sector demand.

These results show that the effects of adopting a monetary financing rule stem from the implications of that rule for the behavior of the short-term nominal interest rate, rather than a special role for money.

The subsequent analysis focuses on money-financed transfers to households, relying on two model features. First, that money may earn a strictly positive rate of return. Second, a simple financial friction implies that households regard government liabilities as net wealth.

The fact that money earns interest allows control of the stock of money independently of the short-term bond rate. So monetary transfers can be used without altering the monetary policy response to shocks away from the zero bound. The financial friction creates a special role for money (and other government liabilities). Importantly, it introduces an incentive to spend, rather than save, a monetary transfer from the government.

Simulations of money-financed transfers to households demonstrate that they can increase spending and inflation when the short-term nominal interest rate is temporarily stuck at the lower bound. Moreover, such transfers increase household wealth and hence spending and inflation, even if they are implemented in the form of a *temporary* increase in the stock of money.

However, the results also suggest three reasons to be cautious about the use of money-financed transfers to stimulate spending and inflation. First, the scale of the monetary transfers required to deliver a meaningful increase in aggregate demand and inflation is likely to be extremely large. Second, the frictions in the model suggest that equivalent effects could be achieved by an increase in conventional government debt, without requiring interest-bearing money. Finally, robustness analysis shows that the stimulative effect is likely to be sensitive to both the precise nature of the frictions giving rise to a meaningful role for money and the policy rule used to set the short-term bond rate.

While Chapters 1 and 2 examine alternative monetary and fiscal policy options close to the zero bound, they share a common assumption about the broad configuration of monetary and fiscal policies. In particular, fiscal policy operates in a so-called 'passive' fashion: taxes and/or spending are adjusted to ensure that the real government debt stock is stabilized for any path of prices. This behavior means that the government's intertemporal budget constraint is irrelevant for the monetary policymaker.

Chapter 3 studies the opposite case: fiscal policy is 'active' and monetary policy decisions must ensure that the real value of (nominal) government debt is stabilized. This policy configuration is sometimes called 'fiscal dominance'.

I use a textbook New Keynesian model extended to include long-term nominal government debt. While the presence of this debt has no implications for optimal monetary policy under the textbook assumption of passive fiscal policy, it plays an important role when fiscal policy is active.

I first ignore the lower bound on the short-term interest rate and demonstrate three key results.

First, the duration of government debt plays a key role in determining the equilibrium behavior of output and inflation and underpins the extent of the so-called 'debt stabilization bias'.

Second, the monetary policymaker cannot fully offset disturbances to the natural rate of interest under active fiscal policy, giving rise to costly fluctuations in output and inflation (that are fully offset when fiscal policy is passive).

Third, welfare losses generated by shocks that generate a trade-off between stabilizing the output gap and inflation may be *smaller* under active fiscal policy than passive fiscal policy. A cost-push shock that reduces inflation today increases the real value of outstanding nominal government debt. Stabilizing the real debt stock requires future policymakers to deliver higher inflation. Higher inflation expectations cushion the impact of the cost-push shock on current inflation.

These results also provide intuition for the behavior of the model in the presence of a lower bound on the short-term interest rate. My key result is that when the lower bound is accounted for, welfare losses may be *smaller* when fiscal policy is active than for the textbook model with passive fiscal policy.

This result is driven by the balance between two effects. Away from the zero bound, welfare losses are larger under active fiscal policy, since shocks to the natural rate of interest cannot be fully stabilized. But when the short-term interest rate is constrained by the lower bound, the combination of active fiscal policy and long-duration debt reduces welfare losses. Deflationary shocks that drive the policy rate to the lower bound raise the real value of government debt. This requires future policymakers to generate higher inflation to stabilize the debt stock, thus increasing inflation expectations. Higher inflation expectations at the lower bound reduce the real interest rate, stimulating spending and mitigating the recessionary effects of the deflationary shock. If the duration of government debt is long enough, the reduction in welfare losses at the zero bound outweighs the higher welfare losses from poorer performance away from the zero bound.

Finally, I consider the effects of a *risk* that fiscal policy behavior switches from passive to active during a debt reduction program. These experiments are motivated by recent debates over fiscal sustainability and so-called austerity programs in many countries. While the debt reduction program has no implications for the output gap or inflation under passive fiscal policy, even a small risk that fiscal policy becomes active can generate sizable effects. In the event that fiscal policy does become active in the future, delivering the debt reduction program will require higher inflation. The risk of higher future inflation increases expected longer-term inflation expectations and the optimal time consistent policy is to allow inflation to rise in the near term.

Chapter 1

Optimal quantitative easing

Abstract

I study optimal monetary policy in a simple New Keynesian model with portfolio adjustment costs. Purchases of long-term debt by the central bank (quantitative easing; 'QE') alter the average portfolio return and hence influence aggregate demand and inflation. The central bank chooses the short-term policy rate and QE to minimize a welfare-based loss function under discretion. Adoption of QE is rapid, with large scale asset purchases triggered when the policy rate hits the zero bound, consistent with observed policy responses to the Global Financial Crisis. Optimal exit is gradual. Despite the presence of portfolio adjustment costs, a policy of 'permanent QE' in which the central bank holds a constant stock of long-term bonds does not improve welfare.

1.1 Introduction

Central bank purchases of long-term government debt – often called quantitative easing (QE) – have been deployed as a monetary policy tool since the depth of the Great Recession, when short-term policy rates became constrained at their effective lower bounds. The widespread use of an unconventional monetary policy instrument has spawned much research. Perhaps surprisingly, however, there has been relatively little investigation of the optimal conduct of monetary policy when QE is a policy instrument, though recent contributions include Cui and Sterk (2018), Darracq Pariès and Kühl (2016), Harrison (2012), Reis (2017) and Woodford (2016).

In this chapter I study the optimal use of QE alongside the short-term policy rate using a model that captures a 'portfolio balance' mechanism. This is the predominant channel through which most monetary policymakers believe that QE affects the economy. I extend the textbook New Keynesian model (Galí, 2008; Woodford, 2003) to include a bond market friction, following Andrés, López-Salido, and Nelson (2004) and Harrison (2012). The representative household faces portfolio adjustment costs when allocating its assets between short-term and long-term bonds.

Portfolio adjustment costs create a wedge between returns on short-term and long-term bonds that can be influenced by changes in the relative supplies of assets, thus providing a role for QE as a policy instrument. In addition to the direct ('stock') effects of asset purchases on relative bond yields, the adjustment cost specification also captures 'flow effects' of QE purchases (the effects on *changes* in the stocks of long-term and short-term bonds held by households). The adjustment costs are calibrated to match estimates of the stock and flow effects of QE on US long-term bond yields by D'Amico and King (2013).

The monetary policymaker acts under discretion to minimize a loss function derived from a quadratic approximation to the welfare of the representative household. In addition to the standard New Keynesian terms in inflation and the output gap, the loss function includes terms in the quantitative easing instrument. These arise because the portfolio adjustment costs through which QE has traction are welfare reducing.

The model is solved using projection methods, accounting for the non-linearities generated by the zero bound on the short-term interest rate and the possibility that bounds may also apply to the QE instrument (for example, the central bank's holdings of long-term debt must be non-negative).

I study entry into and exit from a 'QE regime', defined as a period during which the central bank holds a positive stock of long-term bonds on its balance sheet. I find that entry into QE regimes can be rapid, with large scale asset purchases commencing as soon as the short-term policy rate hits the zero bound. Exit from QE is slower in order to mitigate the costs of changes in the portfolio mix.

Relative to the case in which the only policy instrument is the short-term interest rate, use of QE reduces the welfare costs of fluctuations by around 50%. In this 'active QE' case, the central bank holds a positive stock of long-term bonds on average. On average, the long-term interest rate is below the short-term rate.

These observations suggest that a policy of 'permanent QE' in which the central bank is instructed to hold a constant stock of long-term bonds on its balance sheet may mitigate the effects of the zero bound on the short-term interest rate, by increasing the average short-term interest rate. However, I show that this is *not* the case.

While permanent QE does succeed in 'twisting' the term structure on average (the long-term rate falls and the short-term policy rate rises), this has little effect on average inflation expectations. Raising average inflation expectations requires agents to expect that the central bank will cushion the effects of future deflationary shocks by purchasing assets if those shocks are sufficiently large to force the short-term policy rate to the zero bound. A permanent QE policy does not have this property. The welfare gains of 'active QE' are therefore generated by an expectation effect.

I also study the effects of delegation schemes by allowing the central bank to use both instruments, but instructing the central bank to minimize a loss function that differs from the one derived from household welfare. Consistent with similar analysis using textbook New Keynesian models, a small increase in the inflation target does improve welfare. However, increasing average inflation beyond a small amount generates welfare costs that outweigh the benefits associated with hitting the zero bound less frequently.

Allowing active use of QE but instructing the central bank to target a positive average quantity of long-term bonds on its balance sheet does not improve welfare relative to the case in which the central bank may freely choose the scale of QE. As in the case of permanent QE, this result stems from the fact that the most powerful effects of QE arise from the expectation that it will be deployed when necessary, rather than the direct effects of central bank asset holdings on long-term bond returns.

Several papers have studied QE using larger models featuring similar portfolio frictions: for example, Chen, Cúrdia, and Ferrero (2012), Darracq Pariès and Kühl (2016), De Graeve and Theodoridis (2016), Hohberger, Priftis, and Vogel (2017) and Priftis and Vogel (2016). However, all of these papers assume that agents' expectations satisfy a certainty equivalence assumption.[1] With the exception of Darracq Pariès and Kühl (2016) and Quint and Rabanal (2017), these papers do not consider the optimal design of QE policies. Neither Darracq Pariès and Kühl (2016) nor Quint and Rabanal (2017) consider potential bounds on the QE instrument or use a welfare-based loss function.

The rest of this chapter is organized as follows. Section 1.2 discusses the 'portfolio balance effect' through which QE operates in the model and relates it to the broader literature on QE. Section 1.3 presents the model. Section 1.4 analyzes the optimal policy problem of a central bank tasked with using the short-term interest rate and QE to minimize a welfare-based loss function in a time-consistent manner. The results from the baseline parameterization of the model are presented in Section 1.5. Section 1.6 examines the effects of delegating alternative loss functions to the central bank. Section 1.7 assesses the robustness of the results to alternative assumptions about key parameter values and Section 1.8 concludes.

1.2 The portfolio balance mechanism

In an oft-quoted remark, former FOMC Chairman Ben Bernanke stated that "the trouble with QE is that it works in practice, but not in theory".[2] In this section, I argue that the so-called 'portfolio balance' mechanism has become the predominant channel through which most monetary policymakers believe that quantitative easing affects asset prices and the wider economy.

When quantitative easing was introduced as a response to the global financial crisis, there was uncertainty among policymakers about the channels through which the policy might operate and skepticism among academics that it would have any

[1]While the rational expectations assumption specifies that shocks are zero in expectation, the certainty equivalence assumption specifies that these shocks are assumed (by agents) to be zero with certainty.

[2]The comment was made during a discussion session at the Brookings Institution: Bernanke (2014).

effect at all.[3] For example, Benford, Berry, Nikolov, Young, and Robson (2009) document several possible channels through which quantitative easing might stimulate spending and inflation.[4] Academic skepticism over the likely effects of the policy was typified by the analysis of Eggertsson and Woodford (2003), who demonstrated that a change in the composition of households' portfolios would have no effect on equilibrium asset prices or allocations in a widely studied benchmark model.

A wide range of studies provided evidence that the quantitative easing policies enacted in response to the financial crisis increased asset prices and reduced longer-term interest rates.[5] Other studies attempted to estimate the macroeconomic effects of these changes in asset prices and yields, with a general consensus that central bank asset purchases were successful in increasing output and inflation.[6]

Alongside the accumulating empirical evidence, economists explored possible theoretical mechanisms that could give rise to such effects. From an asset pricing perspective, King (2015) notes that the neutrality results of Eggertsson and Woodford (2003) rely on the (common) assumption of an additively time separable utility function. This implies that the stochastic discount factor used to price assets depends only on consumption allocations across time. A broader class of utility functions imply that the stochastic discount factor also depends on the return on wealth (or the average portfolio return). In such cases, shifts in the composition of agents' portfolios can affect the average portfolio return and hence individual rates of return via the stochastic discount factor.

King (2015) demonstrates that Epstein-Zinn-Weil preferences[7] and the 'preferred

[3]I focus on quantitative easing measures of the type introduced by several central banks in response to the Global Financial Crisis. The Bank of Japan introduced a range of (somewhat different) balance sheet measures much earlier, given that it encountered the zero bound in the late 1990s.

[4]As well as the portfolio balance effect discussed in this section, the authors argue that the expansion of bank reserves generated by asset purchases may create conditions that encourage bank lending and that asset purchases may help to anchor inflation expectations by signaling the central bank's resolve to return inflation to target.

[5]Notable examples include D'Amico and King (2013), Greenwood and Vayanos (2010, 2014), Joyce, Lasaosa, Stevens, and Tong (2011) and Krishnamurthy and Vissing-Jorgensen (2012).

[6]See, among many others, Baumeister and Benati (2013), Lenza, Pill, and Reichlin (2010), Kapetanios, Mumtaz, Stevens, and Theodoridis (2012), Pesaran and Smith (2016) and Weale and Wieladek (2016).

[7]This specification of preferences has become a benchmark model for the case in which the

habitat' investor framework set out by Vayanos and Vila (2009) fit into the wider class of models in which portfolio composition affects asset prices.

The Vayanos and Vila (2009) model is an important contribution, as it provides a link with the strand of the macroeconomics literature, described below, to which this chapter contributes. The model features two types of agents, one of which has preferences for assets of a particular maturity which give rise to a downward sloping demand curve for the asset. The second type is an arbitrageur, trading in all assets. The interaction of the two agents gives rise to an equilibrium in which changes in the supply of an asset of a particular maturity affects the price of that asset (through the downward sloping demand of preferred habitat investors) and the prices of other assets with similar maturities (through the effect of arbitrage).

In macroeconomics, there is a long tradition of studying the effects of portfolio allocations on asset prices (and vice versa), dating back at least to the work of James Tobin and coauthors.[8] The key assumption underpinning this theory is that the relative demand for alternative asset classes depend on their relative prices or returns, because of imperfect substitutability:

> [A]ssets are assumed to be imperfect substitutes for each other in wealth-owners' portfolios. That is, an increase in the rate of return on any one asset will lead to an increase in the fraction of wealth held in that asset, and to a decrease or at most no change in the fraction held in every other asset. (Tobin and Brainard, 1963)

These models assumed a (primitive) relationship between relative yields and relative asset demands. Frankel (1985) showed that this type of asset demand could be derived as the solution to a Markowitz portfolio problem.[9] As King (2015) notes, this approach does not incorporate rational expectations, because the portfolio problem does not account for the fact that future asset prices (which determine the rates of return on some assets) will be determined in the same way as current asset prices.

elasticity of intertemporal substitution is distinct from the coefficient of relative risk aversion. See Epstein and Zin (1989) and Weil (1990).

[8] See, for example, Tobin (1956, 1969) and Tobin and Brainard (1963).

[9] The investor's objective function is the expected return on the portfolio, less a term in the covariance across returns that captures risk aversion.

The seminal work of Andrés et al. (2004) embedded portfolio adjustment costs into a New Keynesian rational expectations model to provide a more microfounded treatment of imperfect substitutability. The model echoes the finance approach of Vayanos and Vila (2009) and also features two types of agents.[10] Unconstrained households have access to both short-term and long-term bonds, paying a portfolio adjustment cost when investing in the latter. Arbitrage by these households equates the returns (accounting for adjustment costs) of the two bonds. Constrained households only have access to long-term bonds. The consumption of constrained households is influenced by changes in the price of long-term bonds, which can be driven by changes in their relative supply via the portfolio adjustment costs paid by unconstrained households.

The Andrés et al. (2004) model has been modified and extended in several directions. Harrison (2012) builds a representative agent model in which all households face portfolio adjustment costs. In such a setting, aggregate demand depends on the average returns of short-term and long-term bonds as in Andrés et al. (2004). However, there is no heterogeneity, so that the effect of long-term returns on aggregate demand depends on the (average) shares of long-term and short-term debt held by households rather than on the fraction of constrained households. This representative agent framework is arguably more tractable, in particular facilitating welfare analysis.[11] Ellison and Tischbirek (2014) uses an indirect utility argument to directly impose portfolio balance terms in the asset pricing equations of the banks who manage portfolios on the behalf of households.[12]

Chen et al. (2012) develop a medium-scale model based on the Andrés et al. (2004) setup, estimate it on US data and use it to study the effects of the FOMC's

[10]Note, however, that the portfolio adjustment cost role for QE is somewhat different from the role generated by the effects on portfolio risk studied in the finance context by, for example, King (2015).

[11]Welfare analysis is possible in models with heterogeneous agents, given an assumption about how to measure social welfare. For example, Cui and Sterk (2018) study the welfare implications of QE in a state of the art heterogeneous agent New Keynesian (HANK) model by adopting a utilitarian definition of social welfare.

[12]In some ways, this approach has more similarities with the early models of Tobin and others, though the indirect utility approach does deliver cross equation restrictions on the asset pricing relationships.

Large Scale Asset Purchase programs.[13] Carlstrom, Fuerst, and Paustian (2017) adopt a market segmentation approach in which households invest in long-term government debt via leveraged financial intermediaries.

Once QE programs had been implemented and their effects observed, a consensus among monetary policymakers on the portfolio balance transmission channel seemed to emerge. For example, Bernanke (2010) argues that:

> The channels through which the Fed's purchases affect longer-term interest rates and financial conditions more generally have been subject to debate. I see the evidence as most favorable to the view that such purchases work primarily through the so-called portfolio balance channel, which holds that once short-term interest rates have reached zero, the Federal Reserve's purchases of longer-term securities affect financial conditions by changing the quantity and mix of financial assets held by the public.

Of course, while there may be near consensus among monetary policymakers on the transmission channel of QE, the portfolio balance effect is not without critique.[14] Thornton (2014) challenges the empirical evidence on the effects of QE, finding little evidence of that QE operations had economically important effects on long-term bond yields.[15]

One alternative theory for the efficacy of QE is that it contains signals about the likely path for the short-term policy rate Bauer and Rudebusch (2014) provide some empirical evidence for this channel and Bhattarai, Eggertsson, and Gafarov

[13] Canzoneri and Diba (2005) and Canzoneri, Cumby, Diba, and López-Salido (2008, 2011) have explored models in which government bonds provide liquidity services so that the mix of assets held by private agents affects relative returns via liquidity premia. This complementary strand of the literature does not focus on the effects of quantitative easing *per se*.

[14] The discussion here focuses on quantitative easing operations in which the central bank purchases long-term government debt (the focus of this chapter). In response to the financial crisis, some central banks also engaged in the purchase of private debt instruments. Such policies may be expected to operate through different channels and represent a complementary line of research. Important contributions include Cúrdia and Woodford (2010), Del Negro, Eggertsson, Ferrero, and Kiyotaki (2017) and Gertler and Karadi (2011).

[15] These findings chime with the argument that Federal Reserve purchases of US government debt constituted such a small fraction of total debt holdings that any portfolio balance effects would likely be very small (Bauer and Rudebusch, 2014; Cochrane, 2011).

(2015) provide a theoretical framework. Another theory is that changes in the composition of the central bank balance sheet can be used to reduce the risk exposure of private agents (Farmer and Zabczyk, 2016). Other authors have focused on the liabilities side of the central bank balance sheet, arguing that QE operates through the expansion of central bank reserves associated with asset purchases (see, for example, Aksoy and Basso, 2014; Reis, 2017).

The transmission mechanism in Cui and Sterk (2018) implies that both sides of the central bank balance sheet matter since QE involves a swap of relatively illiquid assets for more liquid assets (interpreted as reserves or bank deposits). Providing more liquid assets to households increases their ability to maintain consumption when they become unemployed, stimulating overall spending. This approach could be regarded as a microfounded variant of the 'demand for liquidity' narrative used by Andrés et al. (2004) to motivate the type of portfolio friction used in their (and my) model.

My model can be seen as complementary to many of those discussed above, as QE may operate via several channels. However, my focus on a portfolio balance mechanism is prompted by the views of monetary policymakers cited above. So this chapter can be viewed as an exploration of what the portfolio balance mechanism implies for the optimal conduct of monetary policy.

1.3 The model

The model is a simple extension to the textbook New Keynesian model (Woodford, 2003; Galí, 2008). This highlights the marginal implications of introducing an additional friction (portfolio adjustment costs) and hence the possible value of an additional monetary policy instrument, relative to a widely studied benchmark. Given the widespread use of the textbook model, in this section I focus on the additional features and relegate details of the derivation to Appendix 1.B.

1.3.1 Short-term and long-term bonds

There are two assets in the economy: short-term and long-term nominal government bonds. Following Woodford (2001), long-term government bonds are infinite maturity instruments, paying a geometrically declining coupon. Specifically, a bond issued at date t pays nominal coupons χ^s in dates $t+1+s$, $s \geq 0$. This modeling assumption is convenient because it implies that a one dollar holding of a bond issued j periods ago is equivalent to a χ^j dollar holding of a bond issued today. The fact that the values of long-term bonds issued at different dates can be linked in this way means that it is possible to write budget constraints in terms of a single bond price and a single stock of long-term bonds.[16]

Consider first the nominal budget constraint of a representative household:

$$V_t \widetilde{B}^h_{L,t} + B^h_t = (1 + \chi V_t)\, \widetilde{B}^h_{L,t-1} + R_{t-1} B^h_{t-1} + W_t n_t + T_t + D_t - P_t c_t - \Psi_t \quad (1.1)$$

The right hand side of the budget constraint captures income from working n_t hours at nominal wage W_t, net transfers/taxes T_t from the government and dividends D_t from firms, less spending on consumption goods c_t at price P_t and portfolio adjustment costs Ψ (discussed in Section 1.3.2). The household decision problem will be analyzed in detail below: here I focus on the role of short-term and long-term bonds.

The household holds one-period bonds B^h, which pay a gross rate of return R. The budget constraint with respect to short-term bonds is standard: bonds purchased at date $t-1$ mature in date t with a nominal payoff of R_{t-1} per bond.

The household also holds long-term bonds, where $\widetilde{B}^h_L$ denotes the number of bonds held, measured in terms of the equivalent quantity of newly issued bonds. V is the nominal value (price) of each bond. The right hand side of the budget constraint contains the current value of existing holdings of the long-term bond. The quantity of long-term bonds purchased at all previous dates can be summarized in terms of a quantity of bonds (newly) issued in the previous period by virtue of the pricing relationship discussed above. The bond holdings from the previous period

[16] See Woodford (2001) and Chen et al. (2012) for further discussion.

$\widetilde{B}^h_{L,t-1}$ pay a coupon of 1 per bond in period t and have value χV_t, reflecting the fact that the quantity $\widetilde{B}^h_{L,t-1}$ of date $t-1$ issued bonds has the same value as a quantity $\chi \widetilde{B}^h_{L,t-1}$ of date t issued bonds.

The budget constraint can be conveniently re-written in terms of the one-period return on long-term bonds:

$$B^h_{L,t} + B^h_t = R^1_{L,t} B^h_{L,t-1} + R_{t-1} B^h_{t-1} + W_t n_t + T_t + D_t - P_t c_t - \Psi_t \tag{1.2}$$

where:

$$B^h_{L,t} \equiv V_t \widetilde{B}^h_{L,t} \quad ; \quad R^1_{L,t} \equiv \frac{1 + \chi V_t}{V_{t-1}}$$

This formulation treats the choice variables of the household as the *value* of long-term bond holdings. Because households take bond prices as given this is isomorphic to the original formulation, but simplifies the model derivation. Similarly, the one period return is simply a definition expressed in terms of other asset prices. While the one-period return on long-term bonds is a sufficient statistic to characterize household behavior in the model, it is possible to map the implications for the one-period return back to bond yields that are more readily compared with the data, as shown below.

1.3.2 Households

The optimization problem of the representative household is

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left\{ \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right\}$$

where c is consumption and n is hours worked. A preference shock ϕ_t is included and will serve as the 'demand shock' that generates a persistent decline in the natural real interest rate considered in the simulation experiments examined below.

Maximization is subject to the budget constraint (1.2), including an explicit formulation of portfolio adjustment costs, Ψ:[17]

$$B_{L,t}^h + B_t^h = R_{L,t}^1 B_{L,t-1}^h + R_{t-1} B_{t-1}^h + W_t n_t + T_t + D_t - P_t c_t$$
$$- \frac{\tilde{\nu} P_t \left(b^h + b_L^h\right)}{2} \left[\delta \frac{B_t^h}{B_{L,t}^h} - 1\right]^2$$
$$- \frac{\tilde{\xi} P_t \left(b^h + b_L^h\right)}{2} \left[\frac{B_t^h / B_{L,t}^h}{B_{t-1}^h / B_{L,t-1}^h} - 1\right]^2 \tag{1.3}$$

The portfolio adjustment costs have two components. The first component is a function of the deviation of the households 'portfolio mix', $\frac{B_t^h}{B_{L,t}^h}$ from their desired level, δ^{-1}. These adjustment costs are intended to capture 'stock effects': shifts in the supply of these assets can have a direct effect on their price. Following Andrés et al. (2004), δ is set equal to the steady-state ratio of long-term bonds to short-term bonds so that these portfolio costs are zero at the non-stochastic steady state.

The second component of the portfolio adjustment costs is a function of the *change* in the household's portfolio mix. This adjustment cost is motivated by the empirical evidence that changes in asset supplies associated with the auctions that implement asset purchases have an effect on the prices of assets purchased and their close substitutes (see D'Amico and King, 2013). In the context of my model, such 'flow' effects may be interpreted in terms of frictions in adjusting portfolios including transactions costs.

The tractability of this type of adjustment costs has led to their adoption in several monetary models.[18] In reality, transactions costs are likely to be low, so the portfolio adjustment costs in the model are a stand in for a broader range of frictions. Andrés et al. (2004) argue that they represent a perception by households that longer-term bonds are riskier than short-term bonds, such that households' require a greater quantity of liquid assets (in their model, money) as compensation. Cast in this way, these costs may be better suited to inclusion in the utility function. However, Harrison (2012) demonstrates that taking this approach gives rise

[17] Here b^h and b_L^h denote the steady state *real* levels of short-term and long-term bonds.

[18] See, for example, Andrés et al. (2004), Chen et al. (2012), De Graeve and Theodoridis (2016), Gertler and Karadi (2013), Darracq Pariès and Kühl (2016) and Priftis and Vogel (2016).

to isomorphic expressions for the model equations and welfare functions. Similarly, portfolio frictions in financial intermediation can give rise to very similar behavioral equations (Carlstrom et al., 2017; Harrison, 2011).

1.3.3 Firms

There is a set of monopolistically competitive producers indexed by $j \in (0,1)$ that produce differentiated products that form a Dixit-Stiglitz bundle that is purchased by households. Preferences over differentiated products are given by

$$y_t = \left[\int_0^1 y_{j,t}^{1-\eta_t^{-1}} dj\right]^{\frac{1}{1-\eta_t^{-1}}}$$

where y_j is firm j's output. The elasticity of demand among consumption varieties η_t is assumed to be time-varying, which generates a 'cost push' shock in the Phillips curve that characterizes log-linear pricing decisions.

Firms produce using a constant returns production function in the single input (labor):

$$y_{j,t} = An_{j,t}$$

where A is a productivity parameter.

A fixed subsidy is assumed to ensure that the steady state is efficient. Calvo (1983) staggered pricing gives rise to a New Keynesian Phillips curve derived in Appendix 1.B.2 and discussed below.

1.3.4 Fiscal and monetary policies

To focus on the role of monetary policy, fiscal policy is highly simplified. There is no government spending and net transfers to households are lump sum. Of course, quantitative easing, by its nature, is a prime candidate for study from the perspective of monetary and fiscal policy interactions.[19] My assumptions abstract from these considerations entirely.

[19] For example, Del Negro and Sims (2015) and Benigno and Nistico (2015) take such an approach to examine the potential importance of the government and central bank intertemporal budget constraints.

Two aspects of these assumptions (detailed below) are intended to make quantitative easing an exclusively monetary policy operation. The assumption of a 'neutral' debt management policy (so that relative bond supplies are kept always in line with the desired holdings of households) gives the monetary policymaker maximal control over the debt stocks held by households. The assumption that the total value of debt is fixed implies that fiscal policy is 'passive' (in the sense of Leeper, 1991).[20] As a result, the only non-neutrality from QE operates through the portfolio balance channel.[21]

To the extent that fiscal policy does not deliver the optimal mix of assets for households, the model would imply a role for QE in normal times (away from the zero bound). However, my assumptions are an attempt to capture the key elements of institutional arrangements in practice. For example, government treasury departments (or their agents) are tasked with actively managing the maturity structure of government debt. Their mandate is typically expressed in terms of achieving favorable financing conditions for the government and ensuring adequate liquidity in government debt markets. In the context of my model, debt issuance in line with household portfolio preferences would (other things equal) minimize portfolio adjustment costs and hence the (social) costs of financing a given debt stock.

My assumptions also require that debt management policy remains unchanged when monetary policy uses QE at the zero bound. There is an active debate on the extent to which US government debt issuance may have offset some effects of FOMC asset purchases (see, for example, Greenwood, Hanson, Rudolph, and Summers, 2015). However, my assumptions are consistent with the institutional

[20] Tax revenues are adjusted to hold the debt stock constant which ensures that the government's intertemporal budget constraint is always satisfied, for any policy choices of the central bank. In particular, losses and gains on the central bank's asset portfolios are immediately financed/rebated to private agents via lump sum taxes/transfers. As Benigno and Nistico (2015) point out, such a setup implies that only the consolidated government/central bank budget constraint matters for allocations.

[21] This focuses the analysis on the implications of the portfolio balance channel separately from other mechanisms through which QE may operate. For example, Bhattarai et al. (2015) analyze the case in which the stock of long-term debt is a state variable in the model, because the government budget constraint is a constraint on policy actions. This setup gives rise to the possibility that QE can be used to provide a credible signal that interest rates will remain low in the future (the 'signaling channel'). By assuming that government debt stocks are held fixed for all realizations of the short-term policy rate, this channel is eliminated from the model.

arrangements for QE in the United Kingdom, where the Debt Management Office was instructed to ensure that debt management operations "be consistent with the aims of monetary policy" including the asset purchases implemented by the Bank of England's Monetary Policy Committee.[22]

Given the specification of the long-term bond discussed in Section 1.3.1, the nominal government budget constraint is:

$$B_t + V_t \widetilde{B}_{L,t} = R_{t-1} B_{t-1} + (1 + \chi V_t) \widetilde{B}_{L,t-1} + Z_t - P_t \tau_t$$

where B and $\widetilde{B}_L$ represent stocks of short-term and long-term debt, Z denotes net asset purchases by the central bank and τ represents net tax/transfer payments from/to households. The inclusion of Z reflects the assumption that QE is financed by the central government, as discussed below.

Applying the same change of variables introduced in Section 1.3.1 allows the constraint to be expressed in terms of the value of long-term bonds and their one-period return:

$$B_t + B_{L,t} = R_{t-1} B_{t-1} + R^1_{L,t} B_{L,t-1} + Z_t - P_t \tau_t \tag{1.4}$$

The government implements the following debt issuance policies:

$$\frac{B_t}{P_t} \equiv b_t = b > 0, \ \ \forall t \tag{1.5}$$

$$\frac{B_{L,t}}{P_t} \equiv b_{L,t} = \delta b, \ \ \forall t \tag{1.6}$$

As discussed above, these issuance policies ensure that – absent QE operations by the central bank – households achieve their desired portfolio positions. Conditional on these issuance policies and QE by the central bank, net transfers to households T are pinned down by the government budget constraint (1.4). In particular, changes in the value of the total government debt stock are transferred to/from households (lump sum) in order to keep the overall value of debt constant over time (a form of balanced budget financing).

[22]The quotation is from the letter from the Chancellor of the Exchequer to the Governor of the Bank of England, 3 March 2009: `http://www.bankofengland.co.uk/monetarypolicy/Documents/pdf/chancellorletter050309.pdf`.

Net purchases of long-term government bonds by the central bank are:[23]

$$Z_t = V_t \widetilde{Q}_t - (1 + \chi V_t)\, \widetilde{Q}_{t-1}$$

where the quantity of long-term bonds purchased by the central bank is denoted by $\widetilde{Q}$ and it is assumed that coupon payments are paid to the central bank. Defining $Q_t \equiv V_t \widetilde{Q}_t$ implies that:

$$Z_t = Q_t - R^1_{L,t} Q_{t-1} \tag{1.7}$$

The QE policy instrument is defined as the fraction of the market value of long-term bonds purchased by the central bank, denoted q:

$$Q_t = q_t B_{L,t}$$

1.3.5 Market clearing and aggregate output

Market clearing for short-term and long-term bonds implies that:

$$b^h_t = b_t = b \quad ; \quad \frac{Q_t}{P_t} + b^h_{L,t} = b_{L,t} = b_L$$

where lower case letters denote real-valued debt stocks (for example, $b^h_{L,t} \equiv B^h_{L,t}/P_t$).

Combining the government debt issuance policy with the specification of the QE instrument q gives:

$$b^h_{L,t} = (1 - q_t)\, \delta b$$

Goods market clearing implies that:

$$c_t = y_t - \frac{\tilde{\nu}\left(b^h + b^h_L\right)}{2}\left[\delta \frac{b^h_t}{b^h_{L,t}} - 1\right]^2 - \frac{\tilde{\xi}\left(b^h + b^h_L\right)}{2}\left[\frac{b^h_t}{b^h_{t-1}} \frac{b^h_{L,t-1}}{b^h_{L,t}} - 1\right]^2$$

where total output satisfies:

$$y_t = \frac{A n_t}{\mathcal{D}_t}$$

and $\mathcal{D}_t \equiv \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta} \mathrm{d}j$ is a measure of price dispersion across firms.

[23] In a model with money, the net expansion in the monetary base would also be included in this expression. Here, QE is financed by a loan from the central government, which must ultimately be financed by lump sum taxes on households.

1.3.6 Model equations

As shown in Appendix 1.B, the log-linearized model can be reduced to an Euler equation for the output gap ($\hat{x}$) and a Phillips curve for inflation ($\hat{\pi}$):[24]

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \sigma \left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \gamma q_t + \xi q_{t-1} + \beta \xi \mathbb{E}_t q_{t+1} - r_t^* \right] \tag{1.8}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{x}_t + u_t \tag{1.9}$$

where $\gamma \equiv \nu + \xi (1 + \beta)$, $\nu \equiv \tilde{\nu} (1 + \delta)$ and $\xi \equiv \tilde{\xi} (1 + \delta)$.

The 'natural rate of interest' is $r_t^* \equiv -\mathbb{E}_t \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right)$ and the cost push shock is defined as $u_t \equiv -\frac{(1-\alpha)(1-\beta\alpha)}{\alpha} \frac{\eta}{\eta - 1} \hat{\eta}_t$. These variables follow exogenous processes given by:

$$r_t^* = \rho_r r_{t-1}^* + \varepsilon_t^r \tag{1.10}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u \tag{1.11}$$

where $\varepsilon_t^r \sim N(0, \sigma_r^2)$ and $\varepsilon_t^u \sim N(0, \sigma_u^2)$.

As shown in Appendix 1.D, the yield to maturity of the long-term bond is given by:

$$\begin{aligned} \hat{\mathcal{R}}_t =& \chi \beta \mathbb{E}_t \hat{\mathcal{R}}_{t+1} \\ &+ (1 - \chi\beta) \left(\begin{array}{c} \hat{R}_t - \delta^{-1} (1 + \delta) \gamma q_t \\ + \xi \delta^{-1} (1 + \delta) q_{t-1} + \beta \xi \delta^{-1} (1 + \delta) \mathbb{E}_t q_{t+1} \end{array} \right) \end{aligned} \tag{1.12}$$

1.3.7 Parameter values

Table 1.1 shows the baseline parameter values.[25] It is convenient to scale the model by 100 to convert log-deviations into (approximate) percentage deviations.[26] To do this, the standard deviations of the natural rate and cost-push shocks are scaled by 100. The portfolio adjustment cost coefficients, ν and ξ, are also scaled by 100, since the model equations are derived by linearizing (rather than log-linearizing) with respect to q.[27]

[24] Here, $\hat{z}_t \equiv \ln(z_t / z)$ denotes the log-deviation of variable z_t from its non-stochastic steady state, z. The equations are linearized (rather than log-linearized) with respect to q.

[25] The productivity parameter A is chosen to normalize output to unity in the steady state.

[26] So that an output gap of $x = 1$ corresponds to a gap of one per cent.

[27] The approach to scaling effectively multiplies all model equations by 100 to convert log deviations to percentage units. However, q represents an absolute deviation of q from zero, so does

Table 1.1: Parameter values

	Description	Value		Description	Value
σ	Intertemporal substitution elasticity	1	χ	Long-term bond coupon decay rate	0.975
κ	Slope of Phillips curve	0.0516	δ	Ratio of long-term to short-term bonds	0.3
β	Discount factor	0.9918	$b + b_L$	Total debt stock (relative to *quarterly* GDP)	2
ρ_r	Autocorrelation, natural rate	0.85	100ν	Adjustment cost (portfolio mix)	0.105
$100\sigma_r$	Standard deviation, natural rate	0.25	100ξ	Adjustment cost (change in portfolio mix)	3.2
ρ_u	Autocorrelation, cost push shock	0	$\underline{q}$	Lower bound on QE	0
$100\sigma_u$	Standard deviation, cost push shock	0.154	$\bar{q}$	Upper bound on QE	0.5
η	Elasticity of substitution	7.66			
α	Probability of *not* changing price	0.855			
ψ	Inverse labor supply elasticity	1			

The key parameters of the aggregate demand and pricing equations are σ and κ. Setting $\sigma = 1$ is a standard assumption in the literature. Many studies that examine optimal policy at the zero bound use a much higher value (Adam and Billi, 2006; Bodenstein, Hebden, and Nunes, 2012; Levin, López-Salido, Nelson, and Yun, 2010, among others, use a value of 6 or above). As Levin et al. (2010) point out, such calibrations are often required to generate significant effects on output at the zero bound under optimal *commitment* policy in a canonical New Keynesian model. As I focus on the case of optimal discretionary policy, the zero bound has substantial effects even with a value for σ that is more in line with empirical evidence (such as that presented by Guvenen, 2006).

The slope of the Phillips curve ($\kappa = 0.0516$), though larger than the values used in similar studies (typically around 0.02–0.024), is consistent with my choice of a lower value for σ, given the values for the other parameters.[28]

The value of β is chosen to be consistent with a real interest rate of 3.35% in the non-stochastic steady state. As shown by Adam and Billi (2007), as β increases, the steady-state real interest rate falls and so the chances of encountering the zero bound (and the costs associated with hitting it) increase.[29]

not require scaling in this way. To preserve the model relationships, the coefficients multiplying q are therefore multiplied by 100.

[28]Given the assumed elasticity of disutility of labor supply ($\psi = 1$), achieving this value of κ requires setting $\alpha = 0.855$. This high degree of price stickiness is consistent with estimates from macroeconomic models such as Smets and Wouters (2005). More plausible estimates of average contract length can be obtained by adopting more flexible formulations of the demand for alternative product varieties as demonstrated by Smets and Wouters (2007). The value of $\eta = 7.66$ is commonly used in the canonical New Keynesian model (see, for example, Adam and Billi, 2006; Bodenstein et al., 2012).

[29]My calibration implies a lower non-stochastic steady-state real interest rate than previous

Other parameters that are important in determining the incidence of the zero bound are those governing the shock processes. The process for the natural real interest rate is assumed to be persistent, with $\rho_r = 0.85$, following Levin et al. (2010). The standard deviation of the shock is roughly in line with the value used by Adam and Billi (2006) in their 'RBC calibration' and the values of the parameters governing the cost push shock are also taken from that calibration.

The parameters related to long-term and short-term bonds deserve particular attention. The value of χ is chosen to imply that the long-term bond has a duration of between 7 and 8 years in the non-stochastic steady state (see Appendix 1.D). This corresponds to the average duration of 10-year US Treasuries at the time of the first large scale asset purchase programme (D'Amico and King, 2013). I therefore interpret the long-term bond as a 10-year bond for the purposes comparing the model predictions with the data.

The steady-state ratio of government debt to (quarterly) GDP (i.e., $b + b_L$) is set to 2, in line with the findings of Reinhart, Reinhart, and Rogoff (2012). They estimate an average debt to (annual) GDP ratio of around 50% for advanced economies over the pre-crisis period. The steady-state ratio of long-term to short-term bonds (δ) is set to 0.3 on the basis of the data presented in D'Amico and King (2013).[30]

The values for the parameters governing the portfolio adjustment costs, ν and ξ are designed to capture the empirical effects of quantitative easing, defined as 'stock effects' and 'flow effects' by D'Amico and King (2013). To arrive at these parameter values the model was solved on a grid of $\{\nu, \xi\}$ pairs and the values that

studies, including Adam and Billi (2007). Nevertheless, this value may be considered rather high, even by pre-crisis standards. The calibration is best thought of as an assumption about the non-stochastic steady-state *nominal* interest rate, because the efficient inflation rate in the model is zero.

[30]The ratio can be inferred from the data on the dollar amounts and percentages of stock purchased displayed in D'Amico and King (2013, Fig 1) where short-term bonds are interpreted as those with an outstanding maturity of six years or less. Debt management strategies differ over time and across countries. Kuttner (2006, Figure 3) suggests that the average fraction of short-term (less than five-year maturity) bonds held by the private sector was around 25% over the period from 1965 to 2006, suggesting $\delta \approx 3$. The data underlying Figure 1.1 suggests $\delta > 1$ for the United Kingdom.

generated stock and flow effects closest to those estimated by D'Amico and King (2013) selected. This procedure and the results are discussed in Section 1.5.1.

Andrés et al. (2004) estimate a parameter similar to ν (relating the long-term bond premium to household's relative holdings of money and long bonds) using US data. Their estimate implies a value of $100 \times \nu$ of around 0.035, though the long-term rate in that study is a three-year bond, a somewhat shorter maturity than the focus of my model. The evidence presented in Bernanke, Reinhart, and Sack (2004) would, using a simple back of the envelope calculation, suggest a much larger value for $100 \times \nu \approx 2$.[31] Of course, such calculations ignore the fact that asset purchases will have effects on other asset prices (in particular, expected short-term rates). The simulation approach discussed in Section 1.5.1 attempts to overcome these issues.

The finding that flow effects appear to be more important than stock effects (since $\xi > \nu$) is consistent with the results of De Graeve and Theodoridis (2016). They estimate a flexible functional form for the mapping between maturity structure and the long-short bond spread and find that the data prefers a specification close to a first difference specification (implying $\nu \approx 0$ in the context of my model).

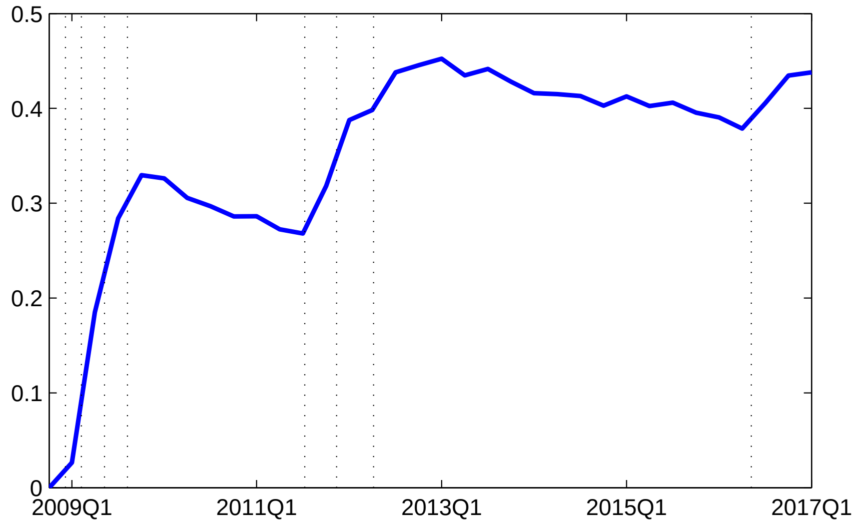
Finally, the parameters $\underline{q}$ and $\bar{q}$ represent the lower and upper bounds on the scale of QE operations that the central bank may undertake. Recall that q represents the fraction of the total quantity of outstanding long-term bonds held by the central bank. Under the assumption that the central bank cannot issue long-term bonds that are perfect substitutes for long-term government bonds, $q_t \geq 0$, and I set $\underline{q} = 0$.

It must also be the case that $\bar{q} \leq 1$, since the central bank cannot purchase more than 100% of the existing stock of long-term bonds. There may be practical reasons why the upper bound on asset purchases is less than 1, for example if there are some financial institutions that must hold long-term safe assets for regulatory

[31] This is calculated using a steady-state version of equation (1.12), assuming that an asset purchase operation of size $q = 0.1$ is permanent and there are no effects on short-term rates long-term bond returns. To see this note that, since $100\gamma \equiv 100\nu + 100\xi\,(1+\beta)$, equation (1.12) can be written as
$\hat{\mathcal{R}}_t = \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1} + (1-\chi\beta)\left(\hat{R}_t - \left(1+\delta^{-1}\right)100\nu q_t - 100\xi\left(1+\delta^{-1}\right)\Delta q_t + \beta 100\xi\left(1+\delta^{-1}\right)\mathbb{E}_t\Delta q_{t+1}\right)$.
A 'steady state' version of the equation sets $\hat{z}_t = \hat{z}, \forall t$ so that $\hat{\mathcal{R}} = \hat{R} - \left(1+\delta^{-1}\right)100\nu q$ and hence $\frac{\partial\hat{\mathcal{R}}}{\partial q} = -\left(1+\delta^{-1}\right)100\nu$. Since $\hat{\mathcal{R}}$ is measured in quarterly units, we require $0.1 \times \left(1+\delta^{-1}\right)100\nu = 0.25$ which implies $100\nu \approx 2$.

purposes. In addition, if the central bank balance sheet is considered independently from the government, then the size of the balance sheet may be limited by a solvency constraint.[32]

Figure 1.1: Approximate measure of q for the United Kingdom



Notes: The figure shows the ratio of the value of the Bank of England's asset purchase facility (APF) to the value of outstanding medium-term and long-term UK government debt. Vertical dotted lines indicate dates at which the Monetary Policy Committee voted to increase the size of the APF.
Sources: Bank of England; UK Debt Management Office.

These types of friction are not explicitly incorporated in the model. My assumption that $\bar{q} = 0.5$ is set with reference to the QE programs observed since the financial crisis. QE in the United Kingdom resulted in purchases amounting to around half of the long-term debt stock (Figure 1.1).[33] Quantitative easing programs in the United States, while substantial, represented a much smaller share of the long-term government debt market. At the time of writing, the ECB's quantitative easing is limited to purchasing no more than one third of the eligible sovereign debt of any member state.[34] Section 1.7.2 examines the robustness of the results to

[32]The issue depends on the financing agreement between the central bank and government, as explored by Benigno and Nistico (2015).

[33]Figure 1.1 is consistent with the results in Daines, Joyce, and Tong (2012), who estimate that the first phase of QE in the United Kingdom purchased around 30% of the long-term government debt stock.

[34]See https://www.ecb.europa.eu/mopo/implement/omt/html/pspp-qa.en.html.

the assumed value of $\bar{q}$ and the implications for profits and losses associated with the central bank's asset portfolio.

1.4 The monetary policy problem

In this section, I consider the optimal use of QE alongside the short-term policy rate. I assume that the monetary policymaker sets both instruments to minimize a loss function based on an approximation to the utility of the representative household. The loss function includes terms in the QE instrument (q), reflecting the fact that the portfolio frictions that give QE traction impose costs on households.[35] Specifically, Appendix 1.C demonstrates that a loss function based on a quadratic approximation to household utility is given by:

$$\mathcal{L}_0 = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\omega_x \hat{x}_t^2 + \omega_\pi \hat{\pi}_t^2 + \omega_q q_t^2 + \omega_{\Delta q}\left(q_t - q_{t-1}\right)^2\right) \tag{1.13}$$

where the weights are related to the model parameters according to:

$$\omega_x \equiv \left(\psi + \sigma^{-1}\right); \quad \omega_\pi \equiv \frac{\alpha\eta}{(1-\alpha\beta)(1-\alpha)}; \quad \omega_q \equiv \tilde{\nu}\left(b^h + b_L^h\right); \quad \omega_{\Delta q} \equiv \tilde{\xi}\left(b^h + b_L^h\right)$$

The loss function specifies that the policymaker seeks to stabilize the output gap, inflation and the extent of (and changes in) its quantitative easing policy. The first two terms in parentheses appear in the welfare-based loss function of the textbook New Keynesian model.[36] The third and fourth terms appear because of the introduction of imperfect substitutability between assets. This additional friction can be mitigated by stabilizing the relative supplies of assets and the rate at which portfolio shares change. Because the maturity structure of government debt issuance is matched to households' preferred portfolio mix, deviations in the relative supplies of assets are due entirely to quantitative easing, q_t.

[35] Indeed, Alla, Espinoza, and Ghosh (2016) argue that welfare-based loss functions for models that feature a wide range of unconventional policy instruments (for example, including foreign exchange intervention) should include terms in the variability of those instruments for this reason. Using an *ad hoc* loss function to study the optimal use of QE (as in, for example, Darracq Pariès and Kühl, 2016) may fail to capture the full welfare costs of policy actions, which may in turn determine some of the policy prescriptions.

[36] See Woodford (2003).

The policymaker minimizes the loss function (1.13) subject to (1.8), (1.9) and the relevant constraints on the policy instruments:

$$\hat{R}_t \geq 1 - \beta^{-1} \tag{1.14}$$

$$q_t \geq \underline{q} \tag{1.15}$$

$$q_t \leq \bar{q} \tag{1.16}$$

I assume that there is no commitment technology that allows the policymaker to make credible promises about future policy actions. Examining time-consistent policy is motivated by two considerations. The first is that, for the class of models I consider, the zero lower bound does not pose a substantial problem if the policymaker is able to make commitments about how the short-term policy rate will be set in the future (Eggertsson and Woodford, 2003; Adam and Billi, 2006).[37] The second reason is that many central bankers have expressed doubts over their ability to credibly commit to future policy actions (Nakata, 2015).[38]

In this time consistent setting, the policymaker at date t is treated as a Stackelberg leader with respect to both private agents at date t and policymakers (and private agents) in dates $t + i, i \geq 1$. I seek a Markov perfect policy in which optimal decisions are a function only of the payoff relevant state variables in the model ($\{u_t, r_t^*, q_{t-1}\}$). Under this interpretation, the policymaker understands that future policymakers will choose allocations according to time-invariant Markovian policy functions and therefore that its current policy decisions affect future outcomes through their impact on the endogenous state variable (q).

Appendix 1.F shows that the first order conditions of the policymaker's problem

[37]Levin et al. (2010) point out that if aggregate demand is very sensitive to real interest rates, then the zero bound can be costly, even under commitment. Harrison (2012) studies optimal quantitative easing under commitment in a model with such a calibration.

[38]This evidence is consistent with the observation that, in the United States and United Kingdom, QE was used as a policy tool before explicit forward guidance. Moreover, even when forward guidance was deployed, there was much debate over the extent to which it represented a commitment by policymakers (see, for example, Plosser, 2012).

are given by:

$$
\begin{aligned}
0 =&\ \omega_\pi \hat{\pi}_t - \lambda_t^\pi && (1.17)\\
0 =&\ \omega_x \hat{x}_t + \kappa\lambda_t^\pi - \lambda_t^x && (1.18)\\
0 =&\ \omega_q q_t + \omega_{\Delta q}\left(q_t - q_{t-1}\right) + \beta\frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial q_t} + \beta\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial q_t}\lambda_t^\pi \\
&+ \left[\frac{\partial \mathbb{E}_t x_{t+1}}{\partial q_t} + \sigma\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial q_t} + \sigma\gamma - \beta\sigma\xi\frac{\partial \mathbb{E}_t q_{t+1}}{\partial q_t}\right]\lambda_t^x - \lambda_t^{\bar{q}} - \lambda_t^{\underline{q}} && (1.19)\\
0 =&\ -\sigma\lambda_t^x - \lambda_t^R && (1.20)
\end{aligned}
$$

where $\lambda_t^x, \lambda_t^\pi, \lambda_t^R, \lambda_t^{\underline{q}}, \lambda_t^{\bar{q}}$ are the Lagrange multipliers on the constraints (1.8), (1.9), (1.14), (1.15) and (1.16) respectively. Appendix 1.F reports the required Kuhn-Tucker conditions for the multipliers on the inequality constraints.

The first order condition for quantitative easing (1.19) shows that the policymaker accounts for the fact in which the choice of QE at date t may have effects on welfare and future outcomes because the date $t+1$ policymaker will inherit the stock of QE. In the case that the optimal level of QE is an interior solution (that is, $q_t \in \left(\underline{q}, \bar{q}\right)$), (1.19) can be written as:

$$
\begin{aligned}
q_t =&\ \frac{\omega_{\Delta q}}{\omega_q + \omega_{\Delta q}} q_{t-1} - \frac{\beta}{\omega_q + \omega_{\Delta q}}\frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial q_t} - \frac{\beta}{\omega_q + \omega_{\Delta q}}\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial q_t}\omega_\pi \pi_t \\
&- \frac{1}{\omega_q + \omega_{\Delta q}}\left[\frac{\partial \mathbb{E}_t x_{t+1}}{\partial q_t} + \sigma\frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial q_t} + \sigma\gamma - \beta\sigma\xi\frac{\partial \mathbb{E}_t q_{t+1}}{\partial q_t}\right]\left(\omega_x x_t + \kappa\omega_\pi \pi_t\right)
\end{aligned}
$$

which shows that current QE will be larger if the policymaker inherits a larger initial stock of QE and if additional QE reduces losses in the next period (the first two terms on the right hand side). The policymaker's current choice of QE will also affect current losses via the effects on private agents' expectations and hence current decisions. The third term, for example, captures the effect of QE on current losses via the effect of QE on inflation expectations and hence current inflation choices through the Phillips curve (1.9).

This expression for optimal QE also highlights the importance of 'flow effects' of portfolio changes. If these flow effects are absent, then $\tilde{\xi} = \xi = \omega_{\Delta q} = 0$. Moreover, in that case, q ceases to be a state variable in the model, so that current choices of QE have no effect on expectations or future losses. In this case, for an interior solution for q_t, the first order condition becomes:

$$q_t = -\frac{\sigma\gamma}{\omega_q}\left(\omega_x x_t + \kappa\omega_\pi \pi_t\right)$$

so that the choice of depends only on the current output gap and inflation.

This condition balances the marginal cost of QE ($\omega_q q_t$) with the marginal benefits of improved output gap and inflation stabilization via the effect of QE through the IS equation, (1.8). Crucially, for this equation to hold, the IS equation must be an active constraint on policy choices. For this to be the case, we require that λ_t^x be non-zero which (from (1.20) and the Kuhn-Tucker conditions) requires that the zero bound on the short-term policy rate must be binding. A corollary of this observation is that the policymaker does not use QE when the zero bound is not binding: in this case we have $\omega_x x_t + \kappa\omega_\pi \pi_t = 0$ and hence $q_t = 0$.

The logic of this result is simple. The policymaker has access to two instruments that affect the output gap in the same way, but one is costly to operate (since q appears in the loss function). When the zero bound on the short-term interest rate is not binding, the policymaker is unconstrained in their ability to choose the output gap by the choice of the short term interest rate and hence will not use the costly instrument.

1.4.1 Solution approach

To capture the distortions created by the zero bound on the short-term interest rate, I solve the model using projection methods. The algorithm is an extension of a time iteration algorithm to solve for equilibrium policy functions as in Coleman (1990).[39] I specify a grid for the state vector $\{u_t, r_t^*, q_{t-1}\}$ formed as a tensor product of three linearly spaced vectors. The vector for q_{t-1} is defined on the range $\left[\underline{q}, \bar{q}\right]$ and the grids for u_t and r_t are specified across ± 4 standard deviations. Expectations are computed using Gauss-Hermite quadrature using five nodes for each shock (ε^r and ε^u) and linear interpolation of the policy functions. Appendix 1.F.3 describes the solution algorithm in more detail.

[39] The algorithm is similar to those presented by Adam and Billi (2007) and Nakov (2008), extended appropriately because q is an endogenous state variable.

1.5 Results

This section presents the results from the baseline model. I first examine the model's ability to replicate the effects of QE on asset prices that have been reported in the literature. I then examine the behavior of the model and in particular asset purchases/sales as the economy enters or leaves a recession. Finally, I compare my results with the statements by monetary policymakers about their plans for asset purchases and sales.

1.5.1 Stock and flow effects

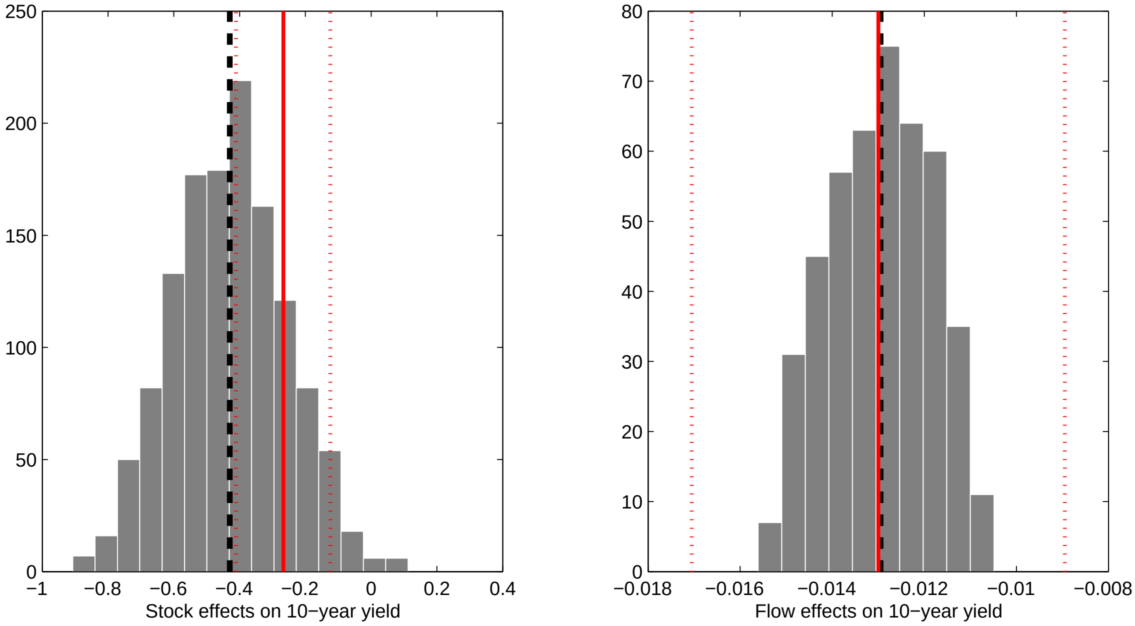
To compare the model's implications for the effects of asset purchases on long-term bond prices with the empirical evidence presented in D'Amico and King (2013), I focus on the estimated 'stock effects' and 'flow effects' of asset purchases reported by those authors.

To uncover the model's implications for stock effects and flow effects, I simulate the model for 100,000 periods and relate surprise movements in QE to surprises in long-term bond yields.[40] By focusing on surprises I mimic the empirical approach of D'Amico and King (2013), which attempts to control for all predictable asset price movements. However, in my model all surprises are generated by optimal policy responses to unforeseen macroeconomic shocks, whereas it is possible that D'Amico and King (2013) estimate the effects of a surprise innovation to a non-optimal QE policy rule (that is, a 'QE policy shock').

Stock effects are computed by isolating the set of QE surprises of a similar magnitude to the FOMC's QE1 programme ($0.10 \leq q_t - \mathbb{E}_{t-1}q_t \leq 0.12$) and recording the corresponding surprise movements in long-term bond rates $\hat{\mathcal{R}}_t - \mathbb{E}_{t-1}\hat{\mathcal{R}}_t$. The mean of the distribution of these surprises (in annualized units), shown as the thick black dashed line in the left panel of Figure 1.2, is -0.43 implying a substantially larger effect than the estimate of -0.27 presented in D'Amico and King (2013) (the solid red line) and just outside the 90% confidence interval (dotted red line).

[40] A simulation of 110,000 periods is used with the first 10,000 periods discarded.

Figure 1.2: Model implied estimates of 'stock effects' and 'flow effects'



Notes: The left panel shows the distribution of the surprise movements in long-term bond yields for QE surprises amounting to approximately 10% of the debt stock. This is calibrated to match the size of from the FOMC's QE1 programme. The dashed black line is the mean of the distribution and the solid red line is the estimated effect on long-term bond yields attributed to QE1 D'Amico and King (2013). The right panel shows the distribution of surprise movements in the yield differential $G_t = \hat{\mathcal{R}}_t - \hat{\mathcal{R}}_t^a$, where $\hat{\mathcal{R}}_t^a$ is defined in equation (1.21), for surprise movements in QE of a similar size to the individual QE1 operations. The dashed black line shows the mean of the distribution and the solid red line shows the estimate of flow effects from D'Amico and King (2013). In both panels, the dotted red lines show 90% confidence intervals around the mean estimate.

To estimate flow effects, I examine the surprise on the difference in yields between the long-term bond and the price of that bond when agents do not face costs of adjusting their portfolio mix ($\tilde{\xi} = \xi = 0$). The yield to maturity of that hypothetical bond is given by:

$$\hat{\mathcal{R}}_t^a = \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1}^a + (1-\chi\beta)\left(\hat{R}_t - (1+\delta)\,\nu q_t\right) \tag{1.21}$$

and the yield differential is defined as $G_t = \hat{\mathcal{R}}_t - \hat{\mathcal{R}}_t^a$. This definition is designed to be the closest match to the effects estimated by D'Amico and King (2013). If estimated accurately, the flow effects in D'Amico and King (2013) reflect reactions to surprises in the maturity composition of asset purchases when the New York Fed enacted the purchases. The model does not incorporate the full maturity structure of government debt, but focusing on marginal effects on yields of *changes* in households' portfolio mix is the closest analogue.

Flow effects are computed by isolating the set of QE surprises of a similar magnitude to individual QE auctions ($0.006 \leq q_t - \mathbb{E}_{t-1}q_t \leq 0.008$) and recording the corresponding surprise movements in the long-term yield differential $G_t - \mathbb{E}_{t-1}G_t$. This distribution is compared to D'Amico & King's estimate of the flow effects of -0.013% shown as the red line in the right hand panel of Figure 1.2, with 90% confidence intervals indicated by the dotted red lines.[41] The model matches the estimated flow effects well.

These results reflect the use of a minimum distance estimator that weights the mean squared deviation from the mean estimated (stock and flow) effects according to the inverse of the confidence intervals. Because the confidence interval for the flow effects is quite narrow (relative to the mean estimate) the flow effect receives a relatively high weight in the estimation process.

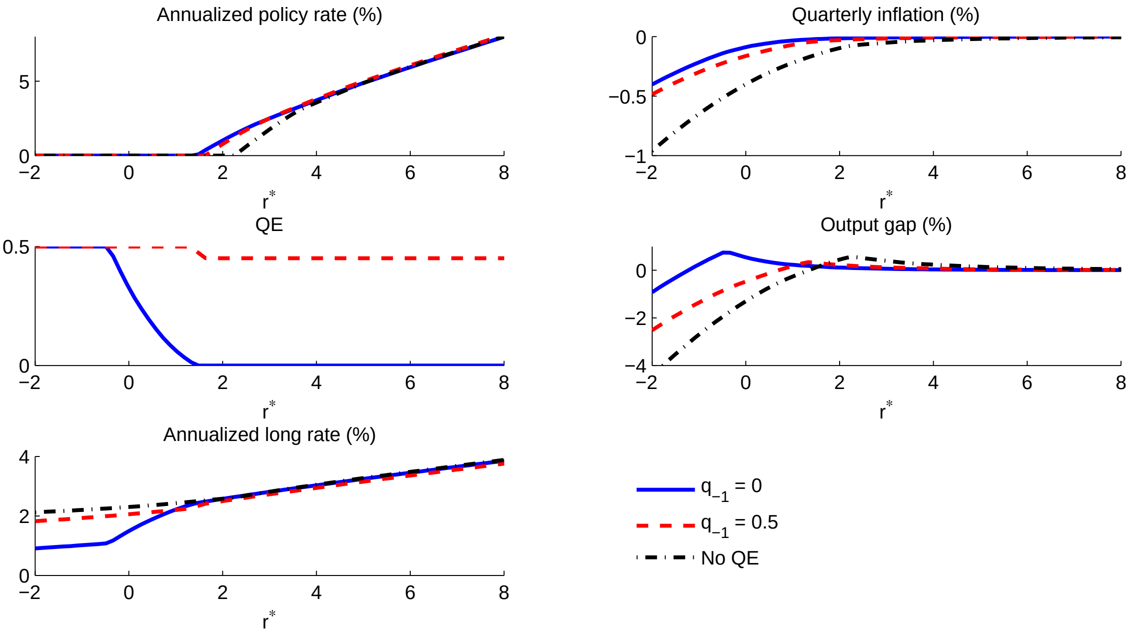
1.5.2 QE entry and exit

In this section, I study the properties of the model and in particular its predictions for asset purchases when the economy enters and leaves a recession. Figure 1.3 plots 'slices' of the policy functions conditioned on particular values for the cost push state, u_t, and the lagged value of the QE instrument, q_{t-1}. In all cases I condition on $u_t = 0$. I then consider policy functions conditional on $q_{t-1} = \underline{q} = 0$ and $q_{t-1} = \bar{q} = 0.5$. By conditioning on the minimum and maximum levels of QE, I can study conditions of 'entry' into and 'exit' from periods in which the central bank holds assets on its balance sheet. Both of these cases are compared to a variant in which the central bank is not allowed to implement QE (that is $q_t = 0, \forall t$).[42] Given the conditioning assumptions, all policy function 'slices' show how optimal outcomes are affected by the natural real interest rate, $r^\star$, holding u_t and q_{t-1} constant.

[41] Appendix 1.D explains how these confidence intervals are constructed from the results in D'Amico and King (2013).

[42] Formally, this case corresponds to a situation in which the policymaker acts to minimize the welfare-based loss function using only the short-term nominal interest rate as the policy instrument. This setup therefore corresponds to the case of discretionary policy subject to the zero lower bound in the canonical New Keynesian model, as studied by Adam and Billi (2007) and Nakov (2008).

Figure 1.3: Policy function comparison



Notes: 'Slices' of policy functions for alternative model variants. The solid blue lines are slices of the policy functions conditional $\{u_t, q_{t-1}\} = \{0, 0\}$. The dashed red lines are slices of the policy functions conditional on $\{u_t, q_{t-1}\} = \{0, 0.5\}$. The dot-dash black lines are policy functions conditional on $u_t = 0$ for a version of the model in which the policymaker does not use QE (so $q_t = 0, \forall t$).

Figure 1.3 demonstrates that when QE is assumed to be unavailable (black dot-dash lines), the policy functions have the same qualitative features as those presented in Adam and Billi (2007).[43] Low values of r^* are associated with the policy rate at the zero bound and negative outcomes for the output gap and inflation. The fact that agents understand that policy will be constrained in this way for low realizations of r^* reduces inflation expectations for values of r^* that are low enough to imply a substantial risk of hitting the zero bound. This effect implies that the policy rate hits the zero bound when the natural rate is positive (around 2%). Moreover, the downward skew in the distribution of future inflation outcomes induces the policymaker to generate a positive output gap for values of r^* are slightly above the value at which the policy rate to hit the zero bound. As described by Adam and Billi (2007), this is the optimal response to the effect of low inflation expectations

[43] The policy functions are quantitatively different because different parameter values are used.

on inflation.[44]

When QE is used, the recessionary consequences of low realizations of r^* are mitigated, since QE can be used to ease monetary conditions when the short-term policy rate is constrained by the zero bound. The policy functions conditioned on $q_{t-1} = 0$ (solid blue lines) show that the policymaker does not make substantial use of QE until the short-term policy rate is constrained by the zero bound. This follows from the first order condition for QE in the case that the policymaker is unconstrained in their instrument settings ($\lambda_t^x = \lambda_t^{\bar{q}} = \lambda_t^q = 0$):

$$q_t = \frac{\omega_{\Delta q}}{\omega_q + \omega_{\Delta q}} q_{t-1} - \frac{\beta}{\omega_q + \omega_{\Delta q}} \frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial q_t} - \frac{\beta}{\omega_q + \omega_{\Delta q}} \frac{\partial \mathbb{E}_t \pi_{t+1}}{\partial q_t} \omega_\pi \pi_t \tag{1.22}$$

When the short-term policy rate is unconstrained, active use of QE does not affect current allocations (since the short-term policy rate can be adjusted to deliver the unconstrained optimal allocations). Moreover, setting $0 \leq q_t \leq \bar{q}$ reduces the scope for subsequent stimulus in the event of bad shocks arriving (such that the short-term policy rate is constrained by the zero bound). So the effects of choosing $0 \leq q_t \leq \bar{q}$ on future losses are positive and the effects on expected future inflation are negative. Taken together, these observations imply that it is not optimal to engage in QE until the short-term policy rate has hit the zero bound.

When the policy rate is constrained by the zero bound, the optimal level of QE rises for lower realizations of r^*. Higher QE reduces the long-term interest rate and provides additional monetary stimulus, hence reducing the recessionary effects of these realizations of r^*. Importantly, the anticipation of additional monetary easing via QE when the short-term policy rate is constrained by the zero bound supports inflation expectations for 'low' values of r^*. As a result, the policy rate becomes constrained at a lower value of r^* (around 1.5% compared with a value of around 2% when QE is unavailable, black dash-dot lines).

Nevertheless, the policy functions in which QE is used as a policy instrument exhibit a similar trade-off between inflation and the output gap for very low values of r^*. This reflects the fact that as r^* reaches very low levels, it becomes optimal

[44] That is, the policymaker pursues the targeting rule $\hat{\pi}_t = -\frac{\omega_x}{\omega_\pi}\hat{x}_t$ when away from the zero bound.

to purchase the maximum possible quantity of assets, $q_t = \bar{q} = 0.5$. In such states, further easing in the event of future recessionary shocks is not possible and the same downward skew in future inflation outturns described above for the standard New Keynesian model once again emerges.

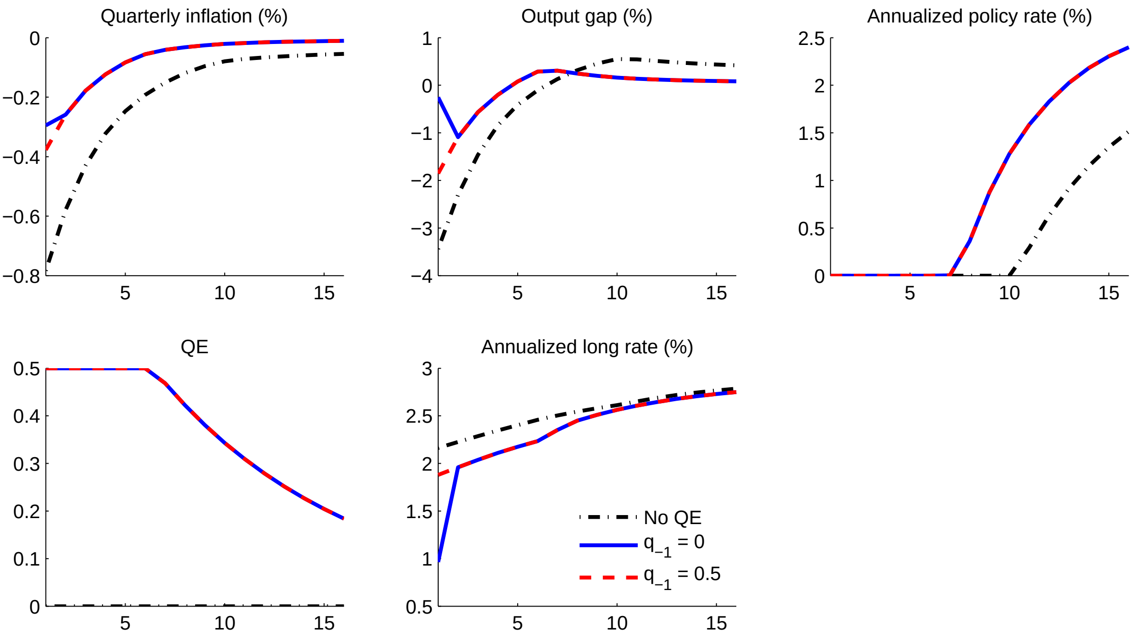
Finally, comparisons of the policy functions for the cases in which $q_{t-1} = 0$ and $q_{t-1} = 0.5$ (dashed red lines) reveal the importance of 'flow effects' in influencing long-term interest rates, monetary conditions and hence output and inflation. When $q_{t-1} = 0$, setting $q_t > 0$ generates both 'stock effects' and 'flow effects' on the long-term rate. For extremely low values of r^*, setting $q_t = 0.5$ reduces the long-term by around 100bp more than setting $q_t = 0.5$ when $q_{t-1} = 0.5$.[45]

To shed further light on how initial conditions effect outcomes as the economy enters or exits a recession, Figures 1.4 and 1.5 present 'modal' simulations for alternative initial conditions. In each case, the simulation traces out the outcomes in the event that the sequence of cost push and natural rate shocks are equal to their most likely value of zero (that is, $\varepsilon_t^u = \varepsilon_t^r = 0, t = 2, \ldots$). The alternative paths represent outcomes for different initial conditions for the exogenous states and QE holdings.

Figure 1.4 shows the case in which the initial condition for the natural rate of interest is extremely low ($r_1^* = -2.25\%$, measured as an annualized rate) and the short-term policy rate is constrained by the zero bound. When the initial condition for QE is zero ($q_0 = 0$, solid blue lines), the policy maker sets QE to its maximum level immediately. The flow effects from this action are sufficient to generate a substantial initial fall in long-term interest rates.

[45] Compare the solid blue and red dashed lines in the bottom left panel of Figure 1.3.

Figure 1.4: Modal simulation of a severe recessionary scenario

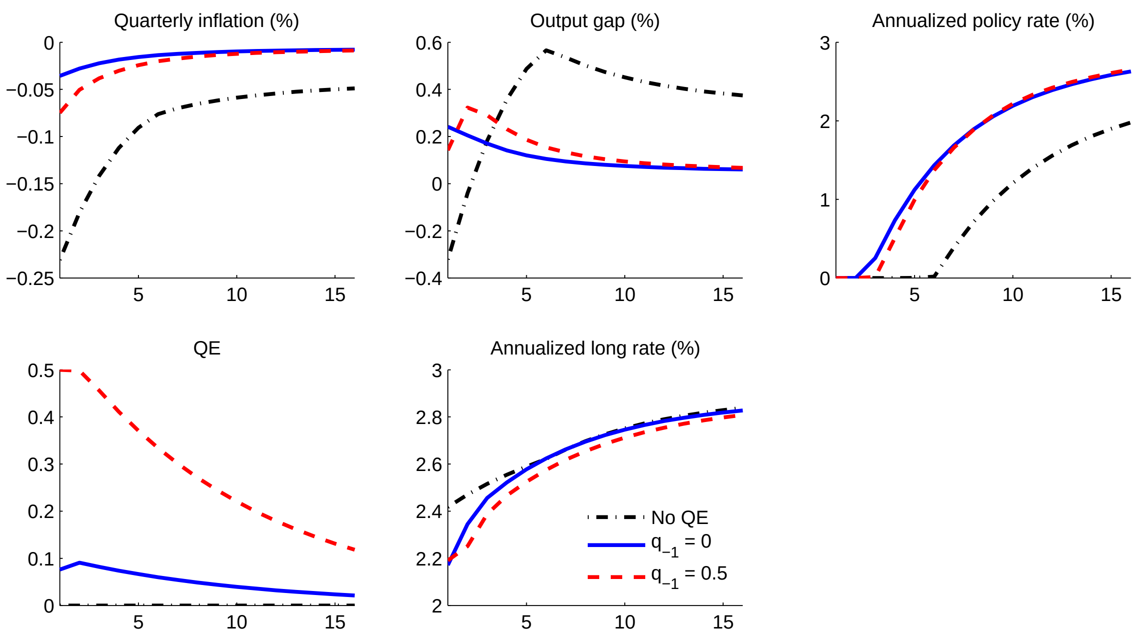


Notes: Each simulated path is computed under the assumptions that the sequence of shocks is equal to the most likely value ($\varepsilon_t^u = \varepsilon_t^r = 0, t = 2, \dots$). The values of the exogenous state variables in period 1 are $u_1 = 0$ and $r_1^* = -2.25\%$ (in annualized units). The solid blue lines correspond to the case in which the initial stock of QE is $q_0 = 0$. The red dashed lines correspond to the case in which the initial stock of QE is $q_0 = 0.5$. The dash-dotted black lines show the case in which the policymaker does not use QE (so $q_t = 0, \forall t$).

In contrast, a policymaker that experiences the same recessionary state ($r_1^* = -2.25\%$), but inherits a maximal stock of QE ($q_0 = \bar{q} = 0.5$, red dashed lines) is unable to loosen policy further. So in this case, inflation and the output gap are more negative in period 1. However, from period 2 onward, the outcomes from these two simulations are identical, because the endogenous state variable is identical in period 2 (that is, $q_1 = 0.5$ in both cases). Compared to the case in which QE is not used (black dash-dotted lines), the additional stimulus from asset purchases allows the short-term policy rate to liftoff from the zero bound several quarters earlier.

Figure 1.5 shows the case of a much milder recessionary state, so that $r_1^* = 0.55\%$ (in annualized units). In the case in which the policymaker does not inherit any assets on its balance sheet ($q_0 = 0$, solid blue lines), it is optimal to engage in a small-scale QE operation that is unwound slowly. When the policymaker inherits

Figure 1.5: Modal simulation of a mild recessionary scenario



Notes: Each simulated path is computed under the assumptions that the sequence of shocks is equal to the most likely value ($\varepsilon_t^u = \varepsilon_t^r = 0, t = 2, \ldots$). The values of the exogenous state variables in period 1 are $u_1 = 0$ and $r_1^* = 0.55\%$ (in annualized units). The solid blue lines correspond to the case in which the initial stock of QE is $q_0 = 0$. The red dashed lines correspond to the case in which the initial stock of QE is $q_0 = 0.5$. The dashed-dotted black lines show the case in which the policymaker does not use QE (so $q_t = 0, \forall t$).

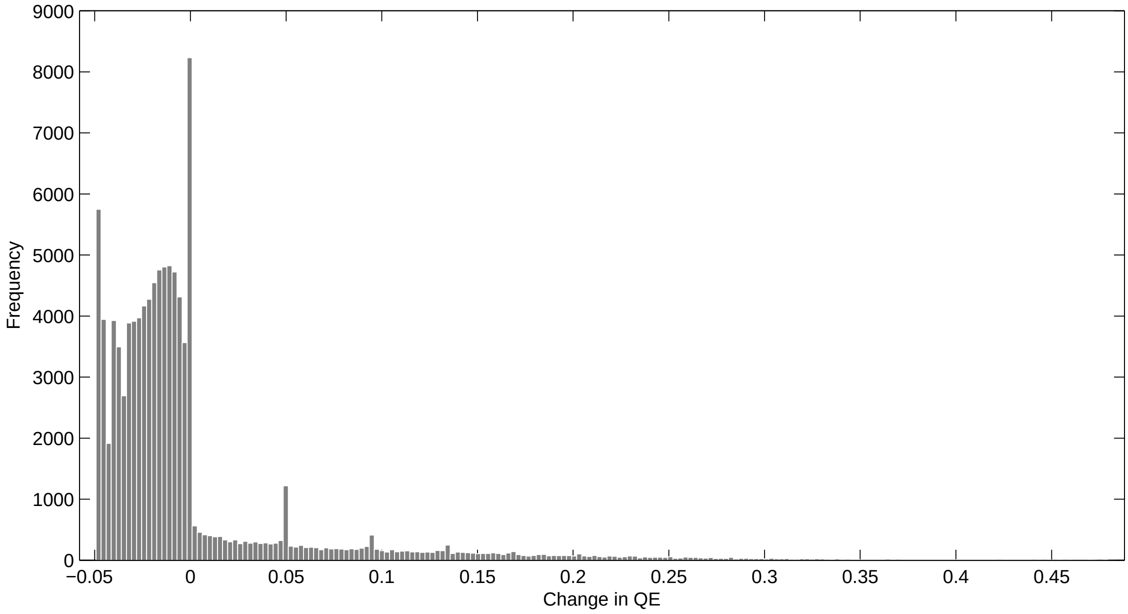
a large stock of assets ($q_0 = \bar{q} = 0.5$, red dashed lines), it is optimal to start unwinding the stock after one period. The unwinding of the large stock of assets tightens monetary conditions, both through the flow effects of asset sales and the stock effect of reducing the size of the balance sheet. So the trajectory of long term interest rates is similar despite the different paths for QE.[46]

In the case of a large inherited balance sheet ($q_0 = 0.5$), exit from the zero bound is delayed by one quarter relative to the case in which the policymaker does not inherit any assets on its balance sheet. Moreover, when the policymaker is unwinding a large initial stock of assets, there is, on average, less capacity to respond to a future shock that constrains QE at its upper bound. As a result, the trade-off

[46] Mechanically, the long rate is calibrated to a duration of around 8 years and the fact that QE is substantially unwound after four years in both simulations implies that the effects on the long rate of the different paths will be relatively small.

between weaker inflation and stronger output is more acute in this case. Compared with the case in which QE is not used (black dashed-dotted lines), however, the trade-off is managed much more effectively and, once again, liftoff from the zero bound occurs earlier.

Figure 1.6: Distribution of changes in QE (Δq_t)



Notes: The histogram records the distribution of outcomes for the change in QE (Δq_t) from a stochastic simulation of 100,000 periods.

The analysis of Figures 1.4 and 1.5 suggests that there is a skew in the distribution of QE policy actions: it is more common to observe large asset purchases than large sales. This is because large scale purchases can be triggered by a large recessionary shock when the policy rate is constrained by the zero bound but exit from QE typically occurs slowly and at least partially during periods in which the short-term policy rate is unconstrained by the zero bound. Figure 1.6 confirms this intuition by plotting the distribution of changes in QE (Δq_t) from a stochastic simulation of the model. The distribution exhibits an upward skew.[47]

[47] One observation from Figure 1.6 is that there are 'spikes' in the distribution of QE changes. This reflects the fact that unwinding existing QE stocks is in many cases a near deterministic process. To see this, recall the first order condition for QE when the policy instruments are unconstrained, (1.22). When the policy rate is unconstrained, the effects of current QE decisions

The policy functions in Figure 1.3 show the trade-off between the output gap and inflation that occurs as the limits of policy accommodation are reached (that is, when the policy rate is at the lower bound and $q \approx \bar{q}$). Figure 1.5 demonstrated that this trade-off may still be present even when policy is relatively unconstrained. Using simple New Keynesian models (without QE), Evans, Fisher, Gourio, and Krane (2016) examine the typical size of the output gap and deviation of inflation from target at the point of liftoff from the zero bound. Appendix 1.A.1 explores this issue in the context of my model.

1.5.3 Discussion

The results from the baseline model show that QE is not actively used until the policy rate hits the zero lower bound and that asset purchases of a very large scale occur fairly frequently. These predictions are consistent with the implementation of QE in early 2009 in the United States and the United Kingdom. In both cases, the initial purchases of long-term government debt were sizable and occurred when (or very soon after) the policy rate hit the effective lower bound.

However, it is less obvious that the model's predictions for exit from QE are consistent with the exit strategies announced and implemented by real-world policymakers. In the United States, the FOMC began to unwind QE only after the short-term policy rate has been increased from the zero bound.[48] In contrast, it is optimal in the model to start reducing the stock of QE at or before the date at which the short-term policy rate lifts off from the zero lower bound.[49] What might

on expectations are likely to be small (because there are many future states of the world in which the short-term policy rate will be unconstrained and current QE decisions will have no impact in those states). In the limiting case where QE has no effect on future outcomes, the first order condition implies $q_t = \frac{\omega_{\Delta q}}{\omega_q + \omega_{\Delta q}} q_{t-1}$. For states in which the effects of current QE on future decisions are small, we will observe $q_t \approx \frac{\omega_{\Delta q}}{\omega_q + \omega_{\Delta q}} q_{t-1}$ and the implied changes in QE will 'bunch' around the values implied by a deterministic unwind of QE.

[48]See Federal Open Market Committee (2011, p3) and Monetary Policy Committee (2015, p34). The Bank of England's MPC has indicated QE unwind will not begin until the policy rate has reached levels that make it possible to respond to negative shocks by reducing the policy rate rather than expanding QE (see Monetary Policy Committee, 2015, p34).

[49]Harrison (2012) and Darracq Pariès and Kühl (2016) reach similar conclusions: optimal policy behavior implies that QE is halted and begins to unwind at or before the date of liftoff. However, both of these studies assume that the policymaker is has access to a commitment technology and adopt a perfect foresight methodology.

explain these differences?

One key difference is that the QE policy variable (q) in the model represents the fraction of the stock of long-term government bonds held by the central bank, rather than the absolute size of the asset stock held by the central bank. Recall that Figure 1.1 plots a crude approximation of q for the United Kingdom. The figure shows that q rises following MPC decisions to increase the stock of assets purchased (dashed vertical lines). But for periods during which the asset stock was held constant and total government debt rose, q was typically falling.[50] This suggests that actual policy behavior has been broadly consistent with the model's predictions: a fixed central bank asset stock when government debt is rising corresponds to a reduction in q. In the model, because government debt is assumed to be fixed, q can only be reduced by active sales of assets.

Another consideration is that the policymaker in the model minimizes a loss function based on the household utility function, whereas the mandates of real-world central banks more closely resemble a so-called 'flexible inflation targeting' loss function which only accounts for the costs of output gap and inflation variability.

Moreover, monetary policymakers have stressed the relative uncertainty over the effect of QE on aggregate demand and inflation relative to the effects of movements in the short-term policy rate.[51] This gives rise to a preference to use the short-term policy rate as the 'primary instrument' to set the overall stance of monetary policy, a result that arises in the simple model analyzed by Williams (2013). This sentiment may be strengthened further by the possibility that asset sales may generate different effects from asset purchases.

In the model, the costs of portfolio misallocation are assumed to be quadratic, suggesting symmetry in the marginal effects of QE tightening and loosening. However, these effects will not in fact be symmetric under optimal policy. Increasing the level of QE reduces the remaining scope for loosening policy via flow effects,

[50] Greenwood et al. (2015) present evidence of a similar effect for the United States.

[51] One source of uncertainty over the model's predictions is that the factors that gave rise to large effects from initial asset purchases may have been related to the particular state of financial stress during the period in which they were implemented. In contrast, the model assumes that the portfolio adjustment costs that give QE traction are structural.

whereas many instances in which QE is reduced will have negligible effects on outcomes because the short-term policy rate is unconstrained.[52] Nevertheless, the model abstracts from any uncertainty over the impact of policy actions on outcomes: the aforementioned asymmetries are perfectly understood by agents in the model.

1.6 Welfare and alternative delegation schemes

The results of Section 1.5 suggest that active use of QE improves welfare by allowing the policymaker to use an additional instrument to offset the effects of shocks on output and inflation. Table 1.2 confirms this by reporting the means of key variables for a simulation of 100,000 periods.[53] The mean of the period loss (that is, $\omega_x \hat{x}_t^2 + \omega_\pi \hat{\pi}_t^2 + \omega_q q_t^2 + \omega_{\Delta q} \left(q_t - q_{t-1}\right)^2$) is also reported.[54]

Table 1.2: Statistics from model simulations

Mean (%)	Baseline	No QE	No ZLB
Qtly inflation	-0.03	-0.10	0.00
Output gap	-0.00	-0.01	-0.00
Policy rate	3.30	2.92	3.31
10-year rate	2.79	2.92	3.31
QE	0.28	0	0
Loss	3.49	7.25	2.52

Table 1.2 shows the results from the baseline version of the model (with active use of QE), a 'no QE' version in which the policymaker sets $q_t = 0, \forall t$ and a 'no ZLB' version in which the zero bound on the short-term policy rate is ignored. Results from this variant represent the best achievable outcomes for the policymaker, conditional on their inability to commit to future policy actions.[55]

[52]See the discussion on page 60.

[53]A simulation of 110,000 periods is produced and the first 10,000 periods are discarded.

[54]Results for welfare losses are often converted into consumption equivalent units (see, for example, Adam and Billi, 2007; Nakov, 2008). As in these papers, applying this conversion to my results generates quantitatively small consumption equivalent losses. However, the *relative* sizes of the losses in consumption equivalent units are very similar to those reported in Table 1.2 since $\sigma = 1$. Moreover, using (1.13) to compare losses ignores the 'terms independent of policy' that generate costly fluctuations in potential output. The scale of the changes in losses therefore represent an upper bound on the changes in welfare.

[55]In a version of the model without QE, the loss under optimal commitment, and in the absence

Comparing the case in which the policymaker does not use QE with the variant in which the zero bound is ignored reveals that the presence of the zero bound reduces mean outcomes for inflation, the output gap and the short-term policy rate. These effects are sufficient to almost triple the average loss.

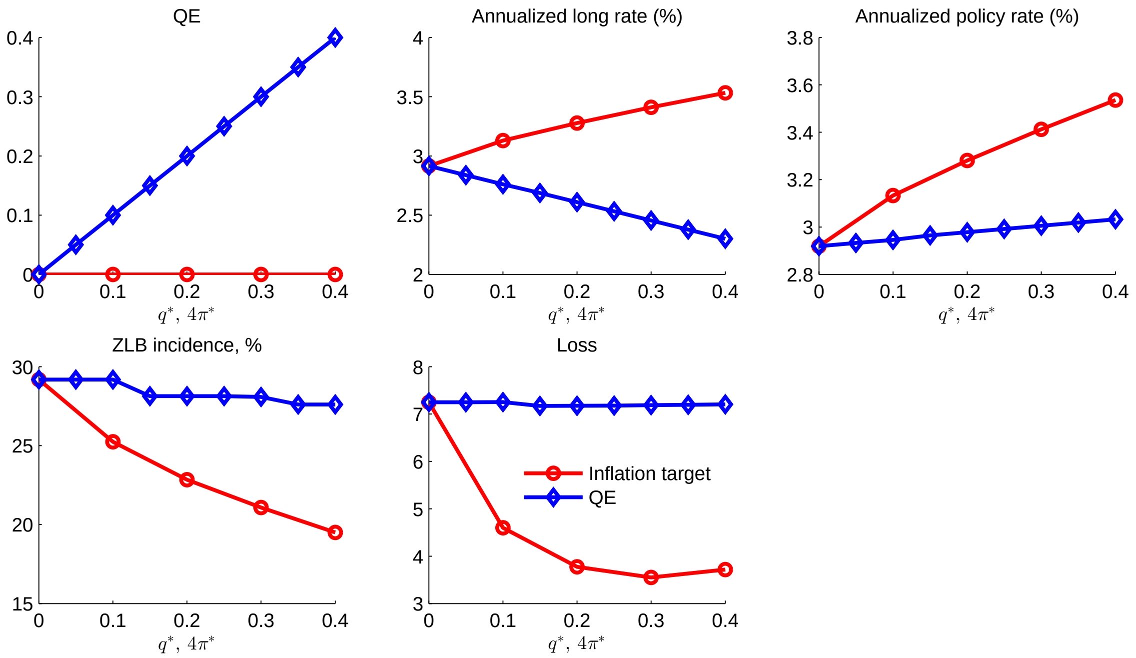
When QE is actively used, the downward skew in the distributions of the output gap and inflation are reduced. Relative to the case in which QE is not used, losses are more than halved. This improved performance is associated with an average level of QE of 0.28, a higher level of the short-term policy rate and a lower average level of the long-term rate.

These observations suggest that it may be possible to mimic the outcomes from active use of QE by mandating that the central bank holds a fixed fraction of the stock of long-term bonds on its balance sheet at all times. This type of 'permanent QE' implies that the central bank sets $q_t = q^*, \forall t$, where $\underline{q} < q^* \leq \bar{q}$. Such a policy might be expected to reduce average long-term nominal interest rates so that a higher short-term nominal interest rate is required, on average, to deliver inflation at target. A higher average short-term nominal interest rate should in turn reduce the frequency with which the short-term policy rate is constrained by the zero lower bound and therefore improve the policymaker's ability to stabilize the economy.

The above logic is similar to the argument for increasing the inflation target: a higher average inflation rate increases the level of the short-term policy rate consistent inflation at target and reduces the frequency with which the short-term policy rate is constrained by the zero bound. So I also compare the results from 'permanent QE' policies with the case in which $q_t = 0, \forall t$, and the short-term policy rate is set to minimize the period loss function $\omega_x \hat{x}_t^2 + \omega_\pi \left(\hat{\pi}_t - \pi^*\right)^2$ where $\pi^* \geq 0$ is the inflation target delegated to the central bank.

of the zero bound, is 1.90. Adam and Billi (2007, Table 2) show that, in that model, the zero bound has a very small impact on losses under commitment. So 1.90 represents a close approximation to the optimal achievable loss in the presence of the zero bound.

Figure 1.7: Mean outcomes under 'permanent QE' and alternative inflation targets



Notes: Each panel reports mean outcomes from a simulation of 100,000 periods for alternative policy specifications. The bottom left panel shows the frequency with which the short-term policy rate is constrained by the zero bound for each of the policy specifications. The blue lines with diamond markers show the outcomes from the case in which the policymaker sets $q_t = q^*, \forall t$ for alternative values of q^* shown on the x-axis. The red lines with circle markers show the case in which the policymaker sets $q_t = 0, \forall t$, but sets the short-term policy rate to minimize the period loss function $\omega_x \hat{x}_t^2 + \omega_\pi (\hat{\pi}_t - \pi^*)^2$. The values of the annualized inflation target ($4\pi^*$) are shown on the x-axis.

Figure 1.7 shows the results of these experiments. Up to a point, increasing the inflation target reduces losses and is associated with a lower frequency of the short-term policy rate being constrained by the zero bound. Beyond this point, while the incidence of the zero bound continues to fall, losses start to increase because the higher level of average inflation is sufficiently costly.[56]

In contrast, 'permanent QE' policies do not improve welfare. As predicted, these

[56]These results are consistent with those of Coibion, Gorodnichenko, and Wieland (2012), who use a much richer model and also account for the implications of positive average inflation on the dynamics of the model (see also, Ascari and Ropele, 2007). My experiments assume that the structure of the economy is described by (1.8) and (1.9), which are linearized around the deterministic steady state. The fact that losses start to increase at very low values of the inflation target (0.3% per year) suggests that my results are robust to this simplification.

policies succeed in 'twisting' the term structure so that the long-term rate falls and the short-term policy rate rises as q^* is increased. However, the strength of these effects is limited and the frequency of zero bound incidents falls only marginally as q^* is increased.

What accounts for these results?

By prohibiting active use of QE in response to shocks, a 'permanent QE' policy influences the term structure only through 'stock effects'. These effects are determined by ν which is small relative to the parameter determining flow effects, ξ. A policy of permanent, but fixed, QE therefore has less traction over the term structure than a policy of adjusting the level of QE in response to shocks.

Moreover, increasing the inflation target has a effect on inflation expectations which is absent for permanent QE policies. Increasing the inflation target raises inflation expectations and nominal yields directly, which mitigates the downward skew in the distribution of inflation outcomes. This improves the policymaker's ability to stabilize outcomes (as measured by the delegated loss function) both at and away from the zero bound. A permanent QE policy does not have this effect on inflation expectations and as a result the only effect on welfare comes through the frequency with which the short-term policy rate is constrained by the zero bound.

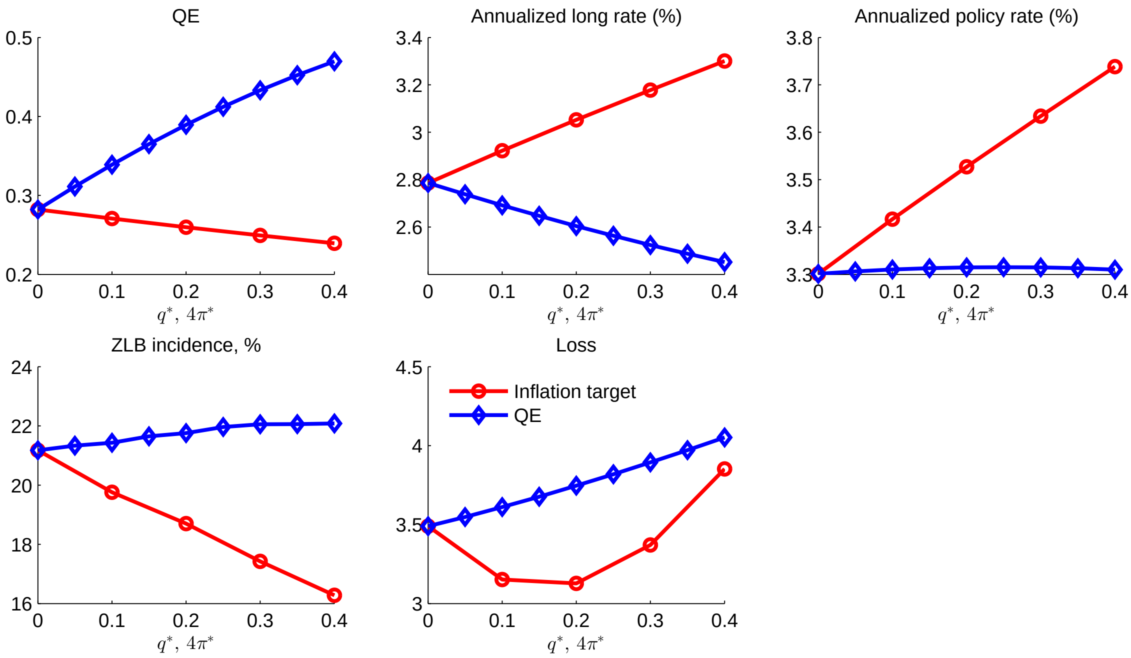
To examine potential gains from more general delegation schemes – when active use of QE is permitted – I assume that the central bank is delegated the following loss function:

$$\widetilde{\mathcal{L}} = \sum_{t=0}^{\infty} \beta^t \left(\omega_x \hat{x}_t^2 + \omega_\pi \left(\hat{\pi}_t - \pi^* \right)^2 + \omega_q \left(q_t - q^* \right)^2 + \omega_{\Delta q} \left(q_t - q_{t-1} \right)^2 \right) \tag{1.23}$$

where π^* and $q_* \in (0, \bar{q})$ (with $\bar{q} \leq 1$) are inflation and QE targets delegated to the policymaker. The loss function coincides with the utility-based benchmark when $\pi^* = q^* = 0$.

Figure 1.8 shows the results of experiments using (1.23). Once again a small increase in the inflation target reduces losses. However, the optimal increase in the inflation target is relatively small, reflecting the fact that active QE is quite effective at offsetting the negative skew in inflation outcomes. As a result, there

Figure 1.8: Mean outcomes with active QE under alternative delegated loss functions



Notes: Each panel reports mean outcomes from a simulation of 100,000 periods for alternative policy specifications. The bottom left panel shows the frequency with which the short-term policy rate is constrained by the zero bound for each of the policy specifications. The blue lines with diamond markers show the outcomes from the case in which the policymaker minimizes the loss function (1.23) for alternative values of q^* shown on the x-axis (with $\pi^* = 0$). The red lines with circle markers show the case in which the policymaker minimizes the loss function (1.23) for values of the annualized inflation target ($4\pi^*$) shown on the x-axis (with $q^* = 0$).

is less benefit from increasing average inflation expectations, so the costs of higher inflation in states of the world when the policymaker is unconstrained offset these benefits quickly as π^* is increased.

Figure 1.8 also illustrates that mandating the central bank to target a higher level of assets on its balance sheet does not improve outcomes. Losses *increase* with q^* because the central bank is forced to hold assets (which imposes costs on households) even when the short-term policy rate is unconstrained.[57]

Despite a mild increase in the short-term policy rate (and a fall in the long-term

[57]Recall that the social welfare function is given by (1.13), which penalizes any $q_t > 0$.

rate) generated by higher average stock effects, the frequency with which the short-term policy rate is constrained by the zero bound actually *rises* as q^* is increased. This reflects the importance of flow effects in determining the effects of QE on output and inflation. To see this, consider the case in which the policymaker inherits a stock of assets equal to the target level ($q_{t-1} = q^* > 0$). Suppose a recessionary shock arrives that constrains the short-term policy rate at the zero bound and necessitates active use of QE. In this case, the maximum 'firepower' that policymaker can deploy through QE is $\bar{q} - q^*$. A higher value of q^* therefore limits the ability of the policymaker to reduce long-term rates via flow effects by increasing QE and so the ability to stabilize the economy at the zero bound is reduced.

1.7 Robustness analysis

This section explores the robustness of the results presented in Section 1.5 to alternative assumptions for key parameter values and to the assumption about the maximal level of QE ($\bar{q}$).

1.7.1 Alternative parameter values

To assess robustness to the choice of parameter values, I focus on those parameters that are most important for the transmission of monetary policy actions.

I consider the case in which the interest elasticity of demand is smaller than in the baseline case by setting $\sigma = 0.5$ following Eggertsson and Woodford (2003). The interest elasticity of demand is a key parameter because it affects the extent to which changes in both short-term and long-term interest rates affect the output gap. A smaller interest elasticity reduces the power of monetary policy, but also reduces the extent to which monetary conditions are tightened when the zero bound binds. In this case, the slope of the Phillips curve (κ) is held fixed to the baseline value of 0.0516 by setting $\alpha = 0.8805$.

I also consider a case in which the Phillips curve is flatter. Setting $\alpha = 0.9$ reduces the Phillips curve slope to $\kappa = 0.024$. This is the value used by Eggertsson and Woodford (2003) and Levin et al. (2010) in their studies of optimal commitment policy at the zero bound. Other things equal, the flatter Phillips curve specification mitigates the downward drag on inflation expectations near the zero bound.

Flattening the IS and Phillips curves is likely to improve the policymaker's ability to stabilize the economy. In a similar vein, I also consider a case in which the standard deviation of the shock to the natural rate of interest is smaller ($100\sigma_r = 0.225$ rather than 0.25 as in the baseline specification). This alternative calibration implies that the zero bound on the short-term policy rate will be less often (and less severely) binding.

Finally, I consider a case in which there are no 'flow effects' ($\xi = 0$). This case is of interest given the uncertainty over the size of flow effects. For example, D'Amico and King (2013) note that flow effects seem to be somewhat short-lived (though are persistent in my model) and Kandrac and Schlusche (2013) find evidence that flow effects from the Fed's LSAP2 and MEP operations were smaller than those for LSAP1.[58]

Table 1.3 presents summary statistics for each of the model variants.

Table 1.3: Model statistics for alternative parameterisations

Variant	Baseline	$\sigma = 0.5$	$\alpha = 0.9$	$100\sigma_r = 0.225$	$\xi = 0$
Mean (%)					
Qtrly inflation	-0.03	-0.06	-0.03	-0.02	-0.06
Output gap	-0.01	-0.01	-0.01	-0.00	-0.01
Policy rate	3.29	3.28	3.33	3.34	3.11
10-year rate	2.78	2.49	2.75	2.86	2.92
QE	0.28	0.43	0.32	0.27	0.10
QE gain (%)	51.89	16.75	12.62	20.74	40.19

Notes: QE gain is the percentage difference in loss when QE is used relative to the case in which $q_t = 0, \forall t$. See footnote 59 for the full definition.

[58]In this case, the policy problem becomes static, as in the simple New Keynesian model with no portfolio balance effects ($\nu = \xi = 0$), because the existing stock of QE does not affect the choices of future policymakers.

As expected, the gain from active use of QE is smaller for all of the variants considered.[59] When the IS curve and Phillips curves are flatter ($\sigma = 0.5$ and $\alpha = 0.9$), the welfare costs of hitting the zero bound in the absence of QE are smaller and hence the gains from using QE are smaller. When the variance of natural rate shocks is smaller ($100\sigma_r = 0.225$) the likelihood of hitting the zero bound is lower, so the benefits of QE are once again reduced. In the absence of flow effects from QE ($\xi = 0$), agents recognize that future QE actions will have a smaller effect on long-term bond yields. This weakens the expectations channel through which QE helps to support inflation expectations and hence reduces the ability of QE to stimulate spending in the event of future negative demand shocks. The policy functions from these alternative parameterizations of the model are consistent with these results, as discussed in Appendix 1.A.2.

1.7.2 The upper bound on QE

As noted in Section 1.3.4, the model makes no distinction between the central bank and government balance sheets (or budget constraints). When the central bank holds long-term bonds on its balance sheet, it faces the risk that the value of those bonds may fall if the long-term interest rate rises. In the model, the government implicitly stands ready to cover any losses incurred on the central bank's portfolio by means of a transfer (or capital injection). Such transfers are funded by levying (lump sum) taxes on households. This fiscal policy is fully credible.

In practice, doubts over whether the government would guarantee such unconditional support to the central bank have been regarded as a limit on the scale of asset purchases by the central bank. For example, Dennis Lockhart, President of the Federal Reserve Bank of Atlanta argues that:

> A second perspective on limits [on monetary policy] might reference statutory or self-imposed limits that central banks observe. These might encompass limits on how far the central bank can or should go in addressing what are fiscal concerns. Monetary policymakers have tried to avoid interventions that put taxpayers at risk of loss. (Lockhart, 2012)

[59] The 'QE gain' is given by $100 \times \left(1 - \bar{\mathcal{L}}/\bar{\mathcal{L}}_{q=0}\right)$ where $\bar{\mathcal{L}}$ is the mean welfare-based loss and $\bar{\mathcal{L}}_{q=0}$ is the mean welfare-based loss computed under a policy in which $q_t = 0, \forall t$.

Such considerations motivate the imposition of the upper bound on the scale of quantitative easing that the central bank may undertake, $\bar{q}$. Appendix 1.A.3 explores the implications of alternative assumptions about $\bar{q}$ for the results presented in Section 1.5. The results show that, unsurprisingly, restricting the maximum scale of asset purchases inhibits the policymaker's ability to stabilize output and inflation for low realizations of the natural real interest rate. In contrast, higher values of $\bar{q}$ deliver better stabilization performance even with relatively low average QE holdings by the central bank because agents recognize that QE *can* be expanded to a substantial level in particularly bad states.

However, a higher value of $\bar{q}$ implies that the central bank will (optimally) hold a larger quantity of long-term government debt in some states. This exposes its balance sheet to greater interest rate risk, compared to a central bank operating under a lower $\bar{q}$ constraint. To explore the extent of the interest rate risk, I calculate the size of the revaluation of the central bank's portfolio as a fraction of steady-state GDP. Appendix 1.E derives the following expression for the revaluation effect:

$$\mathcal{K}_t \approx \frac{\delta\,(b + b_L)}{1+\delta}\left[\hat{R}^1_{L,t} - \hat{R}_{t-1}\right] q_{t-1} \tag{1.24}$$

based on the assumption that purchases of long-term debt are financed by issuing interest-bearing reserves.[60]

Calculating the distribution of $\mathcal{K}_t$ from simulations of the model provides a way to assess the interest rate risk associated with alternative assumptions about $\bar{q}$. Without a fuller treatment of the central bank budget constraint it is not possible to infer from this distribution the likelihood of the central bank paying a negative dividend to the government. However, very large revaluation effects make it more likely that dividends will be negative in some states.[61]

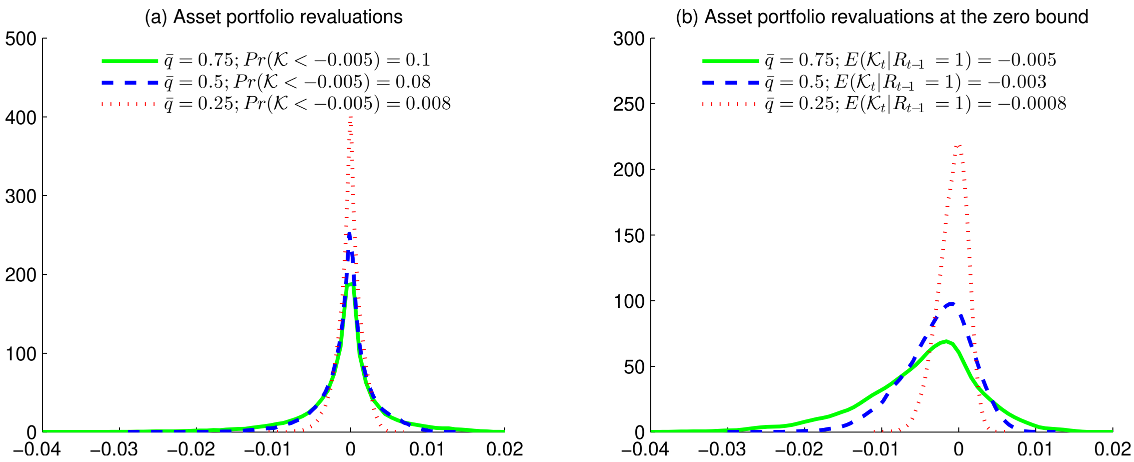
[60] Reserves are assumed to earn the same return as the short-term government bond. A formal model of this type of approach is presented in Bassetto and Messer (2013), which is used to study the feasibility of alternative central dividend policies (see also, Hall and Reis, 2014). Since my model abstracts from the central bank balance sheet and budget constraint, I am restricted to much simpler, indicative, exercises.

[61] As made clear by Reis (2015), negative dividends would only create solvency problems for the central bank under an extremely strict dividend policy (generating what he calls 'period insolvency' in the event that the government insists that all positive net income is transferred to the government as a dividend). Moreover, my calculations represent an upper bound on the extent to which balance

To assess the likelihood of such an event, I calculate the probability of $\mathcal{K} < -0.005$: the probability that capital losses exceed 0.5% of GDP. The choice of critical value is intended to proxy for the fact that, in general, central banks generate seigniorage revenue from the issuance of non-interest bearing currency. A generous estimate for seigniorage revenue as a proportion of GDP is 0.5%, so I interpret the estimated probability as approximating that of a negative dividend payment: the portfolio revaluation exceeds the likely flow of seigniorage revenue.[62]

Panel (a) of Figure 1.9 plots kernel-based estimates of the distribution of $\mathcal{K}_t$ for 100,000 period simulations of the models with $\bar{q} \in \{0.75, 0.5, 0.25\}$. While there is a notable skew in the distributions, the case in which $\bar{q} = 0.75$ has a particularly long left tail.

Figure 1.9: Distributions of portfolio revaluation $\mathcal{K}$ for alternative $\bar{q}$



Notes: Panel (a) shows kernel estimates of the distribution of $\mathcal{K}$ computed using equation (1.24). Panel (b) shows kernel estimates of the the distributions of central bank profits, conditional on being at the ZLB in the previous period ($R_{t-1} = 1$). Results for both panels are constructed from a 100,000 period simulation of the model with $\bar{q} \in \{0.25, 0.5, 0.75\}$.

When $\bar{q} = 0.75$, negative dividends occur with a frequency of almost 10%, and even the baseline assumption of $\bar{q} = 0.5$ implies a frequency of around 8%. Reducing

sheet risks present a problem for central bank solvency because they assume that the central bank's asset portfolio is marked to market.

[62]Reis (2015) argues that the steady-state seigniorage ratio is 0.23% in the model presented by Del Negro and Sims (2015). The data in Aisen and Veiga (2008) give average ratios of 0.3% and 0.4% for the United States and United Kingdom respectively.

$\bar{q}$ to 0.25 reduces the risk to less than 1%. In the absence of a full articulation of the central bank budget constraint and balance sheet, the revaluation effects considered here can only provide an indication of the effects on central bank profitability. However, Benigno (2017, equation (35)) demonstrates that central bank profits will be equal to the revaluation effect computed above in the special case that the nominal interest rate in the previous period is at the zero bound (that is $R_{t-1} = 1$).

Panel (b) of Figure 1.9 plots estimates of the distributions of $\mathcal{K}$, conditional on being constrained by the zero bound in the preceding period. Once again, allowing the central bank to undertake larger asset purchases results in a larger left tail of losses. Conditional on being constrained by the zero bound, losses are 0.5% of steady state GDP when $\bar{q} = 0.75$ compared with less than 0.1% when $\bar{q} = 0.25$.[63] Of course, a higher $\bar{q}$ also allows the central bank to post larger profits on its asset portfolio during a period in which the ZLB is binding, as panel (b) of Figure 1.9 also shows. However, the debate on policy constraints imposed by the central bank's balance sheet has focused on the extent to which the government will stand ready to make transfers to the central bank (or forgo dividend payments).

1.8 Conclusion

I study the optimal use of quantitative easing alongside the short-term policy rate using a textbook New Keynesian model extended to include portfolio adjustment costs. The existence of these costs implies both that QE can influence long-term rates (via a portfolio balance mechanism) and that its use has welfare costs. Reducing long-term rates can increase aggregate demand when the economy is in a recessionary state in which the short-term policy rate is constrained by the zero bound. In such cases, the welfare costs of portfolio distortion are typically outweighed by the benefits of higher aggregate demand. Indeed, relative to the case in which QE is unavailable, use of QE reduces the welfare costs of fluctuations by around 50%.

[63] The frequency of losses is roughly the same for the alternative values of $\bar{q}$: although the distribution of profits is narrower for lower values of $\bar{q}$, the frequency with which the ZLB constraint binds is also higher. These effects roughly cancel out.

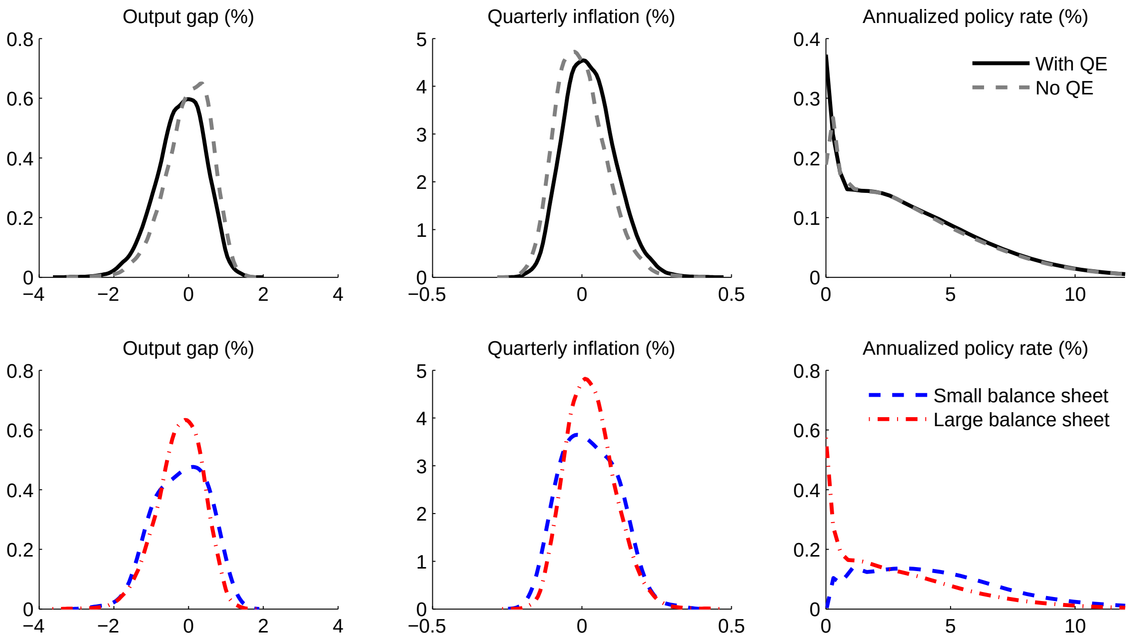
The model predicts that asset purchases can be large and rapid, with purchases commencing as soon as the short-term policy rate hits the zero bound. Exit from QE is gradual. Both of these findings are consistent with the actual implementation of QE in the United States and United Kingdom. However, the model also predicts that 'quantitative tightening' (sales of previously accumulated assets) should start at around the same time that the policy rate lifts off from the zero bound. This contrasts with real-world policy actions, though the comparison is blurred by the fact that QE occurred alongside increases in government debt, which the model abstracts from.

These observations suggest that a comparison of the optimal policy prescriptions from alternative theories of QE, including those that operate via the government budget constraint (e.g., Bhattarai et al., 2015), is an important avenue for further research.

Appendix 1.A Additional results

1.A.1 Macroeconomic conditions at liftoff

Figure 1.10: Distributions of variables at liftoff



Notes: The top row shows kernel based estimates of the distributions of the output gap, inflation and the policy rate for the baseline version of the model in which QE is used (solid black lines) and a case in which QE is not used (dashed grey lines). The distributions are computed using a simulation of 100,000 periods and selecting those periods in which the current policy rate is positive and in which the policy rate in the preceding was at the zero bound. The bottom row shows distributions conditional on the policymaker having a 'large' balance sheet immediately prior to liftoff ($0.4 \leq q_{t-1} \leq 0.5$, dash-dotted red lines) and conditional on a 'small' balance sheet ($0 \leq q_{t-1} \leq 0.1$, dashed blue lines).

The top row of Figure 1.10 shows the distributions of the output gap, inflation and the short-term policy rate in liftoff quarters (defined as those in which the policy rate is positive, but was equal to the lower bound in the previous period). The solid black lines show the distribution in the baseline model, with active use of QE and the dashed grey lines show the distributions when QE is not used ($q_t = 0, \forall t$). These distributions show that, when QE is used as an active policy tool, liftoff of the short-term policy rate will tend to occur with a more negative output gap and smaller inflation overshoot when compared to the case in which QE is not used.[64] The use of QE allows the policymaker to lift off before the output gap has closed (on average).[65]

The bottom row of Figure 1.10 shows the distributions in liftoff quarters for cases in which the central bank's pre-liftoff balance sheet is 'small' ($0 \leq q_{t-1} \leq 0.1$, dashed blue lines) and 'large' ($0.4 \leq q_{t-1} \leq 0.5$, dash-dotted red lines). The distributions for the output gap and inflation in these cases are similar, with little difference in the means.[66] However, it is notable that the variance of the distributions is larger when the policymaker lifts off with a small initial balance sheet. This reflects the fact that such liftoff episodes tend to to occur in relative benign situations (in which previous shocks have not required substantial use of QE). So these liftoff episodes tend to correspond to cases in which policy is less likely to be constrained in the near future. The distribution of the short-term policy rate supports this reasoning: lifting off with a large balance sheet is more likely to be associated with a smaller initial rate rise.

1.A.2 Policy functions for alternative parameterizations

Figure 1.11 plots 'slices' of the policy functions for key variables as functions of the natural real interest rate, r^* (conditional on a zero cost push shock state and zero inherited QE, $u = q_{-1} = 0$) for the alternative parameterizations of the model considered in Section 1.7.1. For the cases in which the IS and Phillips curves are flatter ($\sigma = 0.5$ and $\alpha = 0.9$ respectively) the policy functions for the output gap and inflation are generally closer to zero relative to the baseline parameterization. This reflects the fact that monetary policy is better able to stabilize the economy in light of the smaller downward skews in expected inflation and output gap realizations associated with the zero bound on the short-term policy rate.[67] When $\sigma = 0.5$ the policy functions for the instruments show that more aggressive policy is required to

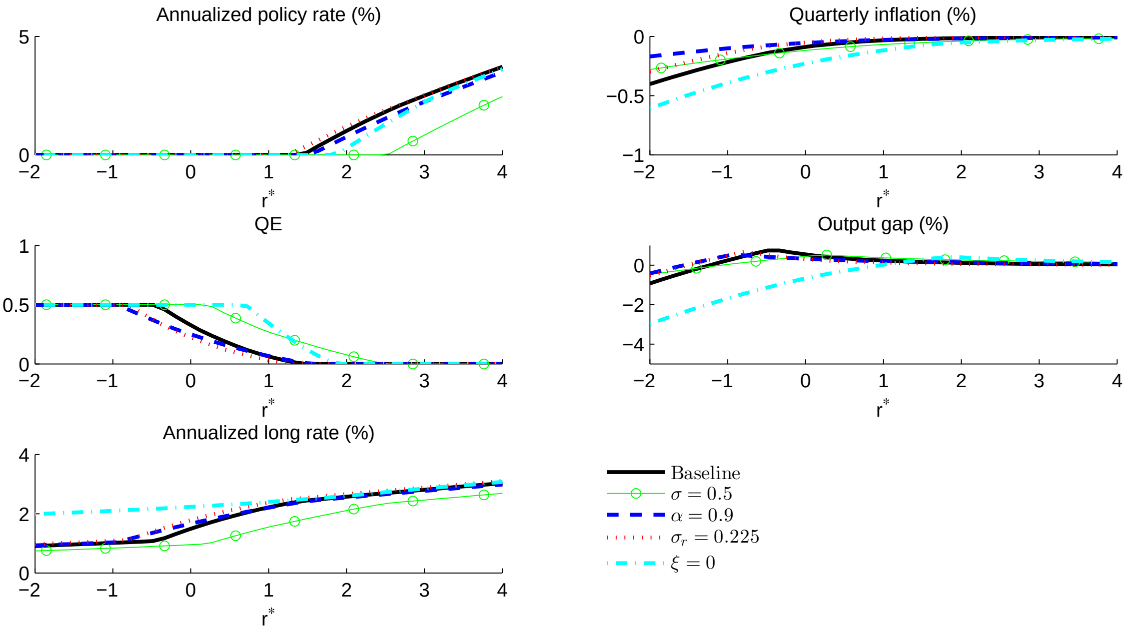
[64]When QE is used, the average output gap is -0.21 and the average inflation rate is 0.02. Without QE the means are both zero to three decimal places.

[65]Inspection of Figure 1.4 indicates that for deep recessionary shocks this is likely to be because liftoff will be somewhat later after the impact of the shock if QE is not used as a policy instrument.

[66]For the 'large' ('small') balance sheet cases the mean output gap is -0.23 (-0.20) and the mean inflation rate is 0.03 (0.03).

[67]Those skews are smaller precisely because the IS or Phillips curves are flatter.

Figure 1.11: Policy functions for alternative parameter values



Notes: 'Slices' of policy functions for alternative model variants. All slices of the policy functions are conditional on $\{u_t, q_{t-1}\} = \{0, 0\}$. See the main text for a full description of the alternative model variants.

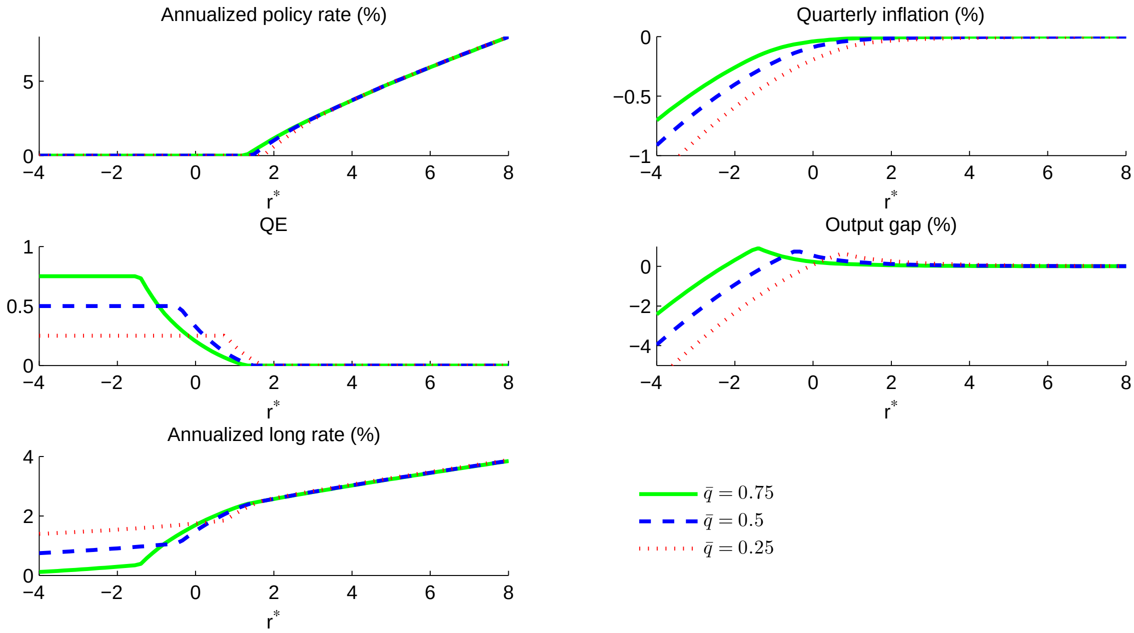
deliver the better stabilization outcomes. That follows from the fact that, when the IS curve is flatter, larger changes in both the short-term and long-term interest rate are required to achieve a given change in the output gap.

Finally, Figure 1.11 shows that the absence of flow effects from QE ($\xi = 0$) generates substantially worse outcomes than the baseline parameterization. The policy function for the long-term interest rate is much flatter because the strong effects of flow effects on depressing long-term yields are absent. As a result, outcomes for both inflation and the output gap are much worse when the economy is at or in the vicinity of the zero bound on the short-term policy rate.

1.A.3 Policy functions for alternative assumptions about maximal QE

Figure 1.12 shows representations of the policy functions for key variables, under alternative assumptions about $\bar{q}$. Each policy function is plotted holding both the

Figure 1.12: Policy functions for alternative $\bar{q}$



Notes: 'Slices' of policy functions for alternative assumptions about the upper bound on asset purchases, $\bar{q}$. Each slice is conditional on $\{u_t, q_{t-1}\} = \{0, 0\}$.

cost push shock and the inherited stock of QE equal to zero.[68] This representation is convenient for assessing the conditions under which a QE regime is entered (that is, the range of values for the natural real interest rate r^* for which asset purchases are initiated) and how the scale of asset purchases is influenced by the natural real interest rate.

The results show that, unsurprisingly, restricting the maximum scale of asset purchases inhibits the policymaker's ability to stabilize output and inflation for low realizations of r^*. When $\bar{q} = 0.25$, the policy rate hits the zero lower bound at a (slightly) higher value of r^*. Moreover, the scale of asset purchases is *larger* over the range $r^* \in (0.5, 1.5)$ when the maximal scale of QE is *smaller*. This means that a higher $\bar{q}$ implies a larger 'bang for buck' of a given scale of asset purchases. The reason for this result is that, as explained in Section 1.6, agents recognize that a policymaker with a larger $\bar{q}$ has more 'firepower' remaining. Agents therefore expect that outcomes will be better stabilized in future bad states. This mitigates

[68] That is, conditioned on $\{u_t, q_{t-1}\} = \{0, 0\}$.

the drag on inflation and output gap expectations generated by the presence of the zero bound on the short-term interest rate and the upper bound on QE.

Table 1.4: Model statistics for alternative $\bar{q}$

Mean (%)	$\bar{q} = 0.75$	$\bar{q} = 0.5$	$\bar{q} = 0.25$
Qtrly inflation	-0.02	-0.03	-0.05
Output gap	-0.00	-0.01	-0.01
Policy rate	3.37	3.29	3.16
10-year rate	2.74	2.78	2.86
QE	0.35	0.28	0.17
Loss	3.08	3.49	4.34

The fact that output and inflation are better stabilized implies that welfare is higher at all points on the slice of the policy function plotted in Figure 1.12. Table 1.4 shows that this is also true on average. Increasing the upper bound on QE reduces welfare losses and keeps inflation and the output gap closer to zero on average. Moreover, comparing the results for $\bar{q} = 0.75$ with those of the baseline specification ($\bar{q} = 0.5$) reveals that improved stabilization is achieved with only slightly higher average QE holdings and very slightly lower long-term bond rates.

Appendix 1.B Model derivation

The model has some similarities to Harrison (2012), but differs in several important respects. The long-term government bond pays a geometrically declining coupon to better approximate the behavior of long-term interest rates. The behavior of fiscal policy is simplified, to focus exclusively on the role of monetary policy. The portfolio friction is in the form of adjustment costs rather than within the utility function and portfolio adjustment costs also depend on changes in households' portfolio mix (between short-term and long-term bonds) as a way to capture 'flow effects' of asset purchases on bond yields. Finally, base money is ignored (a 'cashless limit' following Woodford, 2003), which reduces the scale of the model without affecting the main conclusions.

1.B.1 Households

The optimization problem considered in Section 1.3.2 is

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left\{ \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right\}$$

subject to

$$\begin{aligned} B^h_{L,t} + B^h_t =& R^1_{L,t} B^h_{L,t-1} + R_{t-1} B^h_{t-1} + W_t n_t + T_t + D_t - P_t c_t \\ & - \frac{\tilde{\nu} P_t \left(b^h + b^h_L\right)}{2} \left[\delta \frac{B^h_t}{B^h_{L,t}} - 1\right]^2 \\ & - \frac{\tilde{\xi} P_t \left(b^h + b^h_L\right)}{2} \left[\frac{B^h_t}{B^h_{t-1}} \frac{B^h_{L,t-1}}{B^h_{L,t}} - 1\right]^2 \end{aligned} \tag{1.25}$$

The first-order conditions of the optimization problem are:

$$\phi_t c_t^{-\frac{1}{\sigma}} = \mu_t P_t \tag{1.26}$$

$$\phi_t n_t^{\psi} = W_t \mu_t \tag{1.27}$$

$$\begin{aligned} 0 =& - \mu_t - \mu_t \frac{\tilde{\nu} \delta P_t \left(b^h + b^h_L\right)}{B^h_{L,t}} \left[\delta \frac{B^h_t}{B^h_{L,t}} - 1\right] \\ & - \mu_t \frac{\tilde{\xi} P_t \left(b^h + b^h_L\right)}{B^h_{t-1}} \frac{B^h_{L,t-1}}{B^h_{L,t}} \left[\frac{B^h_t}{B^h_{t-1}} \frac{B^h_{L,t-1}}{B^h_{L,t}} - 1\right] + \beta R_t \mathbb{E}_t \mu_{t+1} \\ & + \beta \mathbb{E}_t \mu_{t+1} \frac{\tilde{\xi} P_{t+1} \left(b^h + b^h_L\right) B^h_{t+1}}{\left(B^h_t\right)^2} \frac{B^h_{L,t}}{B^h_{L,t+1}} \left[\frac{B^h_{t+1}}{B^h_t} \frac{B^h_{L,t}}{B^h_{L,t+1}} - 1\right] \end{aligned} \tag{1.28}$$

$$\begin{aligned} 0 =& - \mu_t + \mu_t \frac{\tilde{\nu} \delta P_t \left(b^h + b^h_L\right) B^h_t}{\left(B^h_{L,t}\right)^2} \left[\delta \frac{B^h_t}{B^h_{L,t}} - 1\right] \\ & + \mu_t \frac{\tilde{\xi} P_t \left(b^h + b^h_L\right) B^h_t}{B^h_{t-1}} \frac{B^h_{L,t-1}}{\left(B^h_{L,t}\right)^2} \left[\frac{B^h_t}{B^h_{t-1}} \frac{B^h_{L,t-1}}{B^h_{L,t}} - 1\right] + \beta \mathbb{E}_t R^1_{L,t+1} \mu_{t+1} \\ & - \beta \mathbb{E}_t \mu_{t+1} \frac{\tilde{\xi} P_{t+1} \left(b^h + b^h_L\right) B^h_{t+1}}{B^h_t} \frac{1}{B^h_{L,t+1}} \left[\frac{B^h_{t+1}}{B^h_t} \frac{B^h_{L,t}}{B^h_{L,t+1}} - 1\right] \end{aligned} \tag{1.29}$$

where μ is the Lagrange multiplier on the nominal budget constraint (1.25).

Define the real Lagrange multiplier as:

$$\Lambda_t \equiv P_t \mu_t$$

and real bond holdings and inflation as

$$\begin{aligned} b_t^h &\equiv \frac{B_t^h}{P_t} \\ b_{L,t}^h &\equiv \frac{B_{L,t}^h}{P_t} \end{aligned}$$

The first order conditions for short-term and long-term bond holdings, (1.28) and (1.29) can be written in terms of real-valued variables as:

$$\begin{aligned} 0 = &- \Lambda_t - \Lambda_t \frac{\tilde{\nu}\delta \left(b^h + b_L^h\right)}{b_{L,t}^h} \left[\delta \frac{b_t^h}{b_{L,t}^h} - 1\right] \\ &- \Lambda_t \frac{\tilde{\xi} \left(b^h + b_L^h\right)}{b_{t-1}^h} \frac{b_{L,t-1}^h}{b_{L,t}^h} \left[\frac{b_t^h}{b_{t-1}^h} \frac{b_{L,t-1}^h}{b_{L,t}^h} - 1\right] + \beta R_t \mathbb{E}_t \Lambda_{t+1} \pi_{t+1}^{-1} \\ &+ \beta \mathbb{E}_t \Lambda_{t+1} \frac{\tilde{\xi} \left(b^h + b_L^h\right) b_{t+1}^h}{\left(b_t^h\right)^2} \frac{b_{L,t}^h}{b_{L,t+1}^h} \left[\frac{b_{t+1}^h}{b_t^h} \frac{b_{L,t}^h}{b_{L,t+1}^h} - 1\right] \end{aligned} \tag{1.30}$$

$$\begin{aligned} 0 = &- \Lambda_t + \Lambda_t \frac{\tilde{\nu}\delta \left(b^h + b_L^h\right) b_t^h}{\left(b_{L,t}^h\right)^2} \left[\delta \frac{b_t^h}{b_{L,t}^h} - 1\right] \\ &+ \Lambda_t \frac{\tilde{\xi} \left(b^h + b_L^h\right) b_t^h}{b_{t-1}^h} \frac{b_{L,t-1}^h}{\left(b_{L,t}^h\right)^2} \left[\frac{b_t^h}{b_{t-1}^h} \frac{b_{L,t-1}^h}{b_{L,t}^h} - 1\right] + \beta \mathbb{E}_t R_{L,t+1}^1 \Lambda_{t+1} \pi_{t+1}^{-1} \\ &- \beta \mathbb{E}_t \Lambda_{t+1} \frac{\tilde{\xi} \left(b^h + b_L^h\right) b_{t+1}^h}{b_t^h} \frac{1}{b_{L,t+1}^h} \left[\frac{b_{t+1}^h}{b_t^h} \frac{b_{L,t}^h}{b_{L,t+1}^h} - 1\right] \end{aligned} \tag{1.31}$$

Several steady-state relationships are useful for log-linearizing the first order conditions:

$$\begin{aligned}
\frac{b_L^h}{b^h} &= \delta \\
\frac{b^h}{b^h + b_L^h} &= (1+\delta)^{-1} \\
\frac{b_L^h}{b^h + b_L^h} &= 1 - (1+\delta)^{-1} = \frac{\delta}{1+\delta}
\end{aligned}$$

Combining (1.26) and (1.30) creates an Euler equation for consumption:

$$\begin{aligned}
\phi_t c_t^{-\frac{1}{\sigma}} =& \beta R_t \mathbb{E}_t \phi_{t+1} c_{t+1}^{-\frac{1}{\sigma}} \pi_{t+1}^{-1} - \phi_t c_t^{-\frac{1}{\sigma}} \frac{\tilde{\nu}\delta\left(b^h + b_L^h\right)}{b_{L,t}^h}\left[\delta \frac{b_t^h}{b_{L,t}^h} - 1\right] \\
& - \phi_t c_t^{-\frac{1}{\sigma}} \frac{\tilde{\xi}\left(b^h + b_L^h\right)}{b_{t-1}^h} \frac{b_{L,t-1}^h}{b_{L,t}^h}\left[\frac{b_t^h}{b_{t-1}^h}\frac{b_{L,t-1}^h}{b_{L,t}^h} - 1\right] \\
& + \beta \mathbb{E}_t \phi_{t+1} c_{t+1}^{-\frac{1}{\sigma}} \pi_{t+1}^{-1} \frac{\tilde{\xi}\left(b^h + b_L^h\right) b_{t+1}^h}{\left(b_t^h\right)^2} \frac{b_{L,t}^h}{b_{L,t+1}^h}\left[\frac{b_{t+1}^h}{b_t^h}\frac{b_{L,t}^h}{b_{L,t+1}^h} - 1\right]
\end{aligned}$$

which can be log-linearized to give:

$$\begin{aligned}
\hat{c}_t =& \mathbb{E}_t \hat{c}_{t+1} - \sigma\left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} + \mathbb{E}_t \Delta\hat{\phi}_{t+1}\right] + \sigma\tilde{\nu}(1+\delta)\left[\hat{b}_t^h - \hat{b}_{L,t}^h\right] \\
& + \sigma\tilde{\xi}(1+\delta)\left[\Delta\left(\hat{b}_t^h - \hat{b}_{L,t}^h\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}_{t+1}^h - \hat{b}_{L,t+1}^h\right)\right] \qquad (1.32)
\end{aligned}$$

The first order conditions for labor supply (1.27) and consumption (1.26) can be combined and log-linearized to give

$$\psi \hat{n}_t = \hat{w}_t - \sigma^{-1}\hat{c}_t \qquad (1.33)$$

Log-linearizing the first order condition for long-term bonds (1.31) gives:

$$\begin{aligned}
\hat{\Lambda}_t =& \tilde{\nu}\delta^{-1}(1+\delta)\left[\hat{b}_t^h - \hat{b}_{L,t}^h\right] + \mathbb{E}_t\left[R_{L,t+1}^1 + \hat{\Lambda}_{t+1} - \hat{\pi}_{t+1}\right] \\
& + \tilde{\xi}\delta^{-1}(1+\delta)\left[\Delta\left(\hat{b}_t^h - \hat{b}_{L,t}^h\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}_{t+1}^h - \hat{b}_{L,t+1}^h\right)\right]
\end{aligned}$$

Log-linearizing the first order condition for short-term bonds (1.30) gives:

$$\begin{aligned}
-\hat{\Lambda}_t =& -\mathbb{E}_t\left[\hat{R}_t + \hat{\Lambda}_{t+1} - \hat{\pi}_{t+1}\right] + \tilde{\nu}(1+\delta)\left[\hat{b}_t^h - \hat{b}_{L,t}^h\right] \\
& + \tilde{\xi}(1+\delta)\left[\Delta\left(\hat{b}_t^h - \hat{b}_{L,t}^h\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}_{t+1}^h - \hat{b}_{L,t+1}^h\right)\right]
\end{aligned}$$

Adding the previous two equations gives:

$$\begin{aligned}0 =& \mathbb{E}_t \left[R^1_{L,t+1} - \hat{R}_t\right] + \tilde{\nu}\delta^{-1}(1+\delta)^2\left[\hat{b}^h_t - \hat{b}^h_{L,t}\right] \\ &+ \tilde{\xi}\delta^{-1}(1+\delta)^2\left[\Delta\left(\hat{b}^h_t - \hat{b}^h_{L,t}\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}^h_{t+1} - \hat{b}^h_{L,t+1}\right)\right]\end{aligned}$$

or

$$\begin{aligned}\mathbb{E}_t R^1_{L,t+1} =& \hat{R}_t - \tilde{\nu}\delta^{-1}(1+\delta)^2\left[\hat{b}^h_t - \hat{b}^h_{L,t}\right] \\ &- \tilde{\xi}\delta^{-1}(1+\delta)^2\left[\Delta\left(\hat{b}^h_t - \hat{b}^h_{L,t}\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}^h_{t+1} - \hat{b}^h_{L,t+1}\right)\right]\end{aligned} \tag{1.34}$$

The final equation can be used to write the Euler equation in terms of returns on short-term and long-term bonds. First note that (1.34) can be rearranged to give:

$$\begin{aligned}\hat{R}_t - \mathbb{E}_t R^1_{L,t+1} =& \tilde{\nu}\delta^{-1}(1+\delta)^2\left[\hat{b}^h_t - \hat{b}^h_{L,t}\right] \\ &+ \tilde{\xi}\delta^{-1}(1+\delta)^2\left[\Delta\left(\hat{b}^h_t - \hat{b}^h_{L,t}\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}^h_{t+1} - \hat{b}^h_{L,t+1}\right)\right]\end{aligned}$$

The right hand side of this expression appears on the right hand side of (1.32), multiplied by $\sigma\delta(1+\delta)^{-1}$. This implies that the Euler equation can be written as:

$$\begin{aligned}\hat{c}_t =& \mathbb{E}_t\hat{c}_{t+1} - \sigma\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} + \mathbb{E}_t\Delta\hat{\phi}_{t+1}\right] + \sigma\delta(1+\delta)^{-1}\left[\hat{R}_t - \mathbb{E}_t R^1_{L,t+1}\right] \\ =& \mathbb{E}_t\hat{c}_{t+1} - \sigma\left[\frac{1}{1+\delta}\hat{R}_t + \frac{\delta}{1+\delta}\mathbb{E}_t R^1_{L,t+1} - \mathbb{E}_t\hat{\pi}_{t+1} + \mathbb{E}_t\Delta\hat{\phi}_{t+1}\right]\end{aligned}$$

1.B.2 Firms

The real profit of producer j is:

$$\frac{(1+s)P_{j,t}}{P_t}y_{j,t} - w_t n_{j,t} = \left((1+s)\frac{P_{j,t}}{P_t} - \frac{w_t}{A}\right)\left(\frac{P_{j,t}}{P_t}\right)^{-\eta_t} y_t$$

where s is a subsidy paid to producers in order to ensure that the steady-state level of output is efficient. This assumption permits the use of a quadratic approximation of the household utility function as an appropriate welfare criterion for policy (see Benigno and Woodford, 2006).

Under a Calvo (1983) pricing scheme the objective function for a producer that is able to reset prices is:

$$\max \mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1+s) \frac{P_{j,t}}{P_k} - \frac{w_k}{A} \right) \left(\frac{P_{j,t}}{P_k} \right)^{-\eta_t} y_k$$

where Λ represents the marginal utility of consumption and $0 \leq \alpha < 1$ is the probability that the producer is *not* allowed to reset its price each period.

The first order condition is

$$\mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1-\eta_t) \frac{(1+s)}{P_k} + \eta_t \frac{w_k}{P_{j,t} A} \right) \left(\frac{P_{j,t}}{P_k} \right)^{-\eta_t} y_k = 0$$

or

$$\mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1-\eta_t) \frac{(1+s)\, p_{j,t}}{\Pi_{t,k}} + \eta_t \frac{w_k}{A} \right) \left(\frac{p_{j,k}}{\Pi_{t,k}} \right)^{-\eta_t} y_k = 0 \tag{1.35}$$

which defined the price set by firm j relative to the aggregate price level as:

$$p_{j,t} \equiv \frac{P_{j,t}}{P_t}$$

and defines the relative inflation factor as

$$\begin{aligned} \Pi_{t,k} &\equiv \frac{P_k}{P_t} = \Pi_k \times \Pi_{k-1} \times ... \times \Pi_{t+1} \text{ for } k \geq t+1 \\ &\equiv 1 \text{ for } k = t \end{aligned}$$

Since all firms are identical in terms of their information and production constraints, all firms that are able to change prices at date t will choose the same price, denoted p_t^*. Thus

$$\mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1-\eta_t) \frac{(1+s)\, p_t^*}{\Pi_{t,k}} + \eta_t \frac{w_k}{A} \right) \left(\frac{p_t^*}{\Pi_{t,k}} \right)^{-\eta_t} y_k = 0$$

The aggregate price is:

$$\begin{aligned} P_t &= \left[\int_0^1 P_{j,t}^{1-\eta_t} dj \right]^{\frac{1}{1-\eta_t}} \\ &= \left[\sum_{k=0}^{\infty} (1-\alpha)\, \alpha^k \left(P_{t-k}^* \right)^{1-\eta_t} \right]^{\frac{1}{1-\eta_t}} \end{aligned}$$

where the equality follows from grouping the firms into cohorts according to the date at which they last reset their price and noting that the mass of firms that have not reset their price since date $t-k$ is $(1-\alpha)\,\alpha^k$. This means that the aggregate price level can be written as

$$P_t = \left[\alpha\left(P_{t-1}\right)^{1-\eta_t} + (1-\alpha)\left(P_t^*\right)^{1-\eta_t}\right]^{\frac{1}{1-\eta_t}}$$

so that

$$1 = \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta_t} + (1-\alpha)\left(p_t^*\right)^{1-\eta_t} \tag{1.36}$$

Log-linearizing the pricing equation gives

$$\mathbb{E}_t\sum_{k=t}^{\infty}(\beta\alpha)^{k-t}\left[\hat{p}_t^* - \hat{\Pi}_{t,k} - \hat{w}_k + \frac{\eta}{\eta-1}\hat{\eta}_k\right] = 0$$

which can be rearranged to give:

$$\hat{p}_t^* = (1-\beta\alpha)\left(\hat{w}_t - \frac{\eta}{\eta-1}\hat{\eta}_t\right) + \beta\alpha\mathbb{E}_t\hat{\pi}_{t+1} + \beta\alpha\mathbb{E}_t\hat{p}_{t+1}^*$$

by using the law of iterated conditional expectations. Log-inearizing the expression for the aggregate price level (1.36) implies that:

$$\hat{p}_t^* = \frac{\alpha}{1-\alpha}\hat{\pi}_t$$

Using this information in the log-linearized pricing equation gives:

$$\hat{\pi}_t = \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\hat{w}_t - \frac{\eta}{\eta-1}\hat{\eta}_t\right) + \beta\mathbb{E}_t\hat{\pi}_{t+1} \tag{1.37}$$

1.B.3 Market clearing and the efficient allocation

Goods market clearing requires:

$$c_t = y_t - \frac{\tilde{\nu}\left(b^h + b_L^h\right)}{2}\left[\delta\frac{b_t^h}{b_{L,t}^h} - 1\right]^2 - \frac{\tilde{\xi}\left(b^h + b_L^h\right)}{2}\left[\frac{b_t^h}{b_{t-1}^h}\frac{b_{L,t-1}^h}{b_{L,t}^h} - 1\right]^2$$

Output for each variety j satisfies:

$$y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\eta_t} y_t$$
$$y_{jt} = An_{jt}$$

Equating the previous expressions and integrating over j gives:

$$\int_0^1 An_{jt}\mathrm{d}j = \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta_t} y_t\mathrm{d}j$$

which implies that:

$$An_t = \mathcal{D}_t y_t$$

where

$$\mathcal{D}_t \equiv \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta_t} \mathrm{d}j \tag{1.38}$$

is a measure of price dispersion.

As noted in the main text, market clearing in government bond markets implies

$$\hat{b}_t^h - \hat{b}_{L,t}^h = -\hat{b}_{L,t}^h = q_t \tag{1.39}$$

It is straightforward to show that in the absence of price-setting and imperfect asset substitutability frictions, the efficient level of output is constant To see this, note that in a flexible price equilibrium with no distortion from monopolistic competition, the real wage will equal the marginal product of labor, which is constant and equal to A. So the efficient allocations, denoted with an asterisk, can be found from the labor supply relation (1.33):

$$\psi\hat{n}_t^* = -\sigma^{-1}\hat{c}_t^*$$

where $\hat{w}_t^* = 0$ because the real wage is constant. Imposing market clearing (with zero portfolio adjustment costs and price dispersion equal to 1) gives $c_t^* = n_t^* = y_t^*$. This in turn implies that $\hat{c}_t^* = \hat{n}_t^* = \hat{y}_t^*$. Since the labor supply equation requires that $\hat{n}_t^* = -\left(\psi\sigma\right)^{-1}\hat{c}_t^*$, we must have $\hat{c}_t^* = \hat{n}_t^* = \hat{y}_t^* = 0$.

1.B.4 The 'gap' representation

The Phillips curve and Euler equation can be written in terms of the output gap, defined as the deviation between output and the efficient level of output ($\hat{y}_t^* = 0$).

Substituting the labor supply equation (1.33) into the Phillips curve (1.37) gives:

$$\begin{aligned}\hat{\pi}_t =& \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi\hat{n}_t + \sigma^{-1}\hat{c}_t\right) + \beta\mathbb{E}_t\hat{\pi}_{t+1} - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\frac{\eta}{\eta-1}\hat{\eta}_t \\ =& \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\left(\psi+\sigma^{-1}\right)\hat{y}_t\right) + \beta\mathbb{E}_t\hat{\pi}_{t+1} - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\frac{\eta}{\eta-1}\hat{\eta}_t \\ =& \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi+\sigma^{-1}\right)\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t\end{aligned}$$

where the second line uses market clearing and the third line uses the definition of the output gap $\hat{y}_t - \hat{y}_t^* \equiv \hat{x}_t$ and defines the cost push shock, u, as:

$$u_t \equiv -\frac{(1-\alpha)(1-\beta\alpha)}{\alpha}\frac{\eta}{\eta-1}\hat{\eta}_t$$

The Phillips curve can therefore be written as:

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t \tag{1.40}$$

where

$$\kappa \equiv \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi+\sigma^{-1}\right)$$

The Euler equation for consumption (1.32) can be written as:

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \sigma\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} + \mathbb{E}_t\Delta\hat{\phi}_{t+1}\right] + \sigma\tilde{\nu}(1+\delta)q_t + \sigma\tilde{\xi}(1+\delta)\left[\Delta q_t - \beta\mathbb{E}_t\Delta q_{t+1}\right]$$

which incorporates the market clearing conditions for output and government bonds.

Collecting terms, this can be written as:

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \sigma\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} + \mathbb{E}_t\Delta\hat{\phi}_{t+1} - (\nu + \xi(1+\beta))q_t + \xi q_{t-1} + \beta\xi\mathbb{E}_t q_{t+1}\right]$$

where

$$\begin{aligned}\nu &\equiv \tilde{\nu}(1+\delta) \\ \xi &\equiv \tilde{\xi}(1+\delta)\end{aligned}$$

In terms of the output gap we have:

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \sigma \left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \gamma q_t + \xi q_{t-1} + \beta \xi \mathbb{E}_t q_{t+1} - r_t^* \right]$$

where

$$\gamma \equiv \nu + \xi \left(1 + \beta\right)$$

and the efficient rate of interest r^* satisfies

$$r_t^* \equiv -\mathbb{E}_t \Delta \hat{\phi}_{t+1} \tag{1.41}$$

Appendix 1.C Utility-based loss function

Ignoring constants, the period utility function is:[69]

$$U_t = \left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

In what follows markup shocks are ignored (by setting $\eta_t = \eta, \forall t$) to simplify notation. Since these shocks are independent of policy this does not affect the derivation.

To derive the loss function, first note that the percentage deviation of any variable z_t from steady state can itself be approximated to second order as:

$$\frac{z_t - z}{z} \approx \hat{z}_t + \frac{1}{2} \hat{z}_t^2$$

where $\hat{z}_t \equiv \ln z_t - \ln z$.

Approximating the utility from consumption to second order gives:

$$\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right) - \frac{1}{2\sigma} c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right)^2$$

[69] I also ignore the preference shock ϕ_t. The utility-based loss function is intended to provide a preference ordering over allocations, rather than measure welfare *per se*. Preference orderings are invariant to a positive monotonic transformation of the utility function. Taking logs of the period utility function implies that ϕ enters in a purely additive manner and hence does not affect the preference ordering.

Ignoring (constant) terms that are independent of policy and denoting as '*t.i.p.*' gives:

$$\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}}\left(\frac{c_t-c}{c}\right)-\frac{1}{2\sigma}c^{1-\frac{1}{\sigma}}\left(\frac{c_t-c}{c}\right)^2+t.i.p.$$

Using the second order approximation for the percentage changes in consumption implies that:

$$\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}}\left(\hat{c}_t+\frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2\right)+h.o.t.$$

where $h.o.t.$ are higher order terms.

Following the same steps for the sub-utility function for labor supply gives:

$$\begin{aligned}\frac{n_t^{1+\psi}}{1+\psi} &\approx n^{1+\psi}\frac{n_t-n}{n}+\frac{\psi n^{1+\psi}}{2}\left(\frac{n_t-n}{n}\right)^2+t.i.p.\\ &\approx n^{1+\psi}\left[\hat{n}_t+\frac{(1+\psi)}{2}\hat{n}_t^2\right]+h.o.t.\end{aligned}$$

The steady-state labor supply relationship is

$$n^{\psi}=wc^{-1/\sigma}=Ac^{-1/\sigma}$$

which follows from the assumption that subsidies to firms are set to eliminate the distortion from monopolistic competition. Steady-state market clearing is

$$c=y=An$$

since steady-state dispersion is $\mathcal{D}=1$.

With $A=1$, this implies that

$$n^{1+\psi}=c^{1-1/\sigma}$$

so that the utility function can be written as

$$U_t \approx c^{1-\frac{1}{\sigma}}\left[\hat{c}_t+\frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2-\hat{n}_t-\frac{(1+\psi)}{2}\hat{n}_t^2\right] \tag{1.42}$$

The goods market clearing condition is:

$$c_t = y_t - \frac{\tilde{\nu}\left(b^h + b^h_L\right)}{2}\left[\delta \frac{b^h_t}{b^h_{L,t}} - 1\right]^2 - \frac{\tilde{\xi}\left(b^h + b^h_L\right)}{2}\left[\frac{b^h_t}{b^h_{t-1}}\frac{b^h_{L,t-1}}{b^h_{L,t}} - 1\right]^2$$

A second order approximation to the goods market clearing condition is:

$$\hat{c}_t + \frac{1}{2}\hat{c}_t^2 = \hat{y}_t + \frac{1}{2}\hat{y}_t^2 - \frac{\tilde{\nu}\left(b^h + b^h_L\right)}{2} q_t^2 - \frac{\tilde{\xi}\left(b^h + b^h_L\right)}{2}\left(\Delta q_t\right)^2$$

where $\hat{b}^h_t - \hat{b}^h_{L,t} = q_t$ has been imposed.

Substituting for $\hat{c}_t$ in (1.42) gives

$$\begin{aligned} U_t &\approx c^{1-\frac{1}{\sigma}}\left[\begin{array}{c} \hat{y}_t + \frac{1}{2}\hat{y}_t^2 - \frac{\tilde{\nu}\left(b^h+b^h_L\right)}{2} q_t^2 - \frac{\tilde{\xi}\left(b^h+b^h_L\right)}{2}\left(\Delta q_t\right)^2 - \frac{\sigma^{-1}}{2}\hat{c}_t^2 \\ -\hat{n}_t - \frac{(1+\psi)}{2}\hat{n}_t^2 \end{array}\right] \\ &\approx c^{1-\frac{1}{\sigma}}\left[\begin{array}{c} \hat{y}_t + \frac{1-\sigma^{-1}}{2}\hat{y}_t^2 - \frac{\tilde{\nu}\left(b^h+b^h_L\right)}{2} q_t^2 - \frac{\tilde{\xi}\left(b^h+b^h_L\right)}{2}\left(\Delta q_t\right)^2 \\ -\hat{n}_t - \frac{(1+\psi)}{2}\hat{n}_t^2 \end{array}\right] + h.o.t \end{aligned}$$

A second order approximation to the aggregate production function is:

$$\hat{y}_t + \frac{1}{2}\hat{y}_t^2 = \hat{n}_t + \frac{1}{2}\hat{n}_t^2 - \hat{\mathcal{D}}_t$$

which uses the fact (shown below) that $\hat{\mathcal{D}}_t$ is a second order term.

Using this result to substitute for $\hat{n}_t$ in the utility function (and ignoring higher order terms) gives:

$$U_t \approx c^{1-\frac{1}{\sigma}}\left[-\hat{\mathcal{D}}_t - \frac{\psi + \sigma^{-1}}{2}\hat{y}_t^2 - \frac{\tilde{\nu}\left(b^h + b^h_L\right)}{2} q_t^2 - \frac{\tilde{\xi}\left(b^h + b^h_L\right)}{2}\left(\Delta q_t\right)^2\right]$$

Defining the discounted loss function to be minimized in terms of the approximated utility function and using the previous results gives:

$$\begin{aligned} \mathcal{L} &= -2c^{\frac{1}{\sigma}-1}\sum_{t=0}^{\infty}\beta^t U_t \\ &= \sum_{t=0}^{\infty}\beta^t\left[2\hat{\mathcal{D}}_t + \left(\psi + \sigma^{-1}\right)\hat{x}_t^2 + \tilde{\nu}\left(b^h + b^h_L\right) q_t^2 + \tilde{\xi}\left(b^h + b^h_L\right)\left(\Delta q_t\right)^2\right] \end{aligned}$$

which makes use of the fact that $\hat{x}_t = \hat{y}_t$.

To eliminate the price dispersion term, recall that:

$$\mathcal{D}_t = \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta} \mathrm{d}j$$

which in equilibrium is given by

$$\mathcal{D}_t = \alpha \mathcal{D}_{t-1} \pi_t^{\eta} + (1-\alpha)\left(p_t^*\right)^{-\eta}$$

Using the price index (1.36), the optimal price can be written as

$$p_t^* = \left[\frac{1-\alpha\pi_t^{\eta-1}}{1-\alpha}\right]^{\frac{1}{1-\eta}}$$

so the price dispersion is

$$\mathcal{D}_t = \alpha \mathcal{D}_{t-1} \pi_t^{\eta} + (1-\alpha)\left[\frac{1-\alpha\pi_t^{\eta-1}}{1-\alpha}\right]^{\frac{\eta}{\eta-1}}$$

Taking a second-order Taylor expansion gives

$$\begin{aligned} \hat{\mathcal{D}}_t &\approx \alpha\left(\hat{\mathcal{D}}_{t-1} + \eta\hat{\pi}_t\right) + (1-\alpha)\left[\frac{-\alpha\eta\hat{\pi}_t}{1-\alpha}\right] \\ &\quad + \frac{\alpha\eta(\eta-1)}{2}\hat{\pi}_t^2 + \frac{1}{2}\left[\frac{\alpha^2\eta}{1-\alpha} - \alpha\eta(\eta-2)\right]\hat{\pi}_t^2 \\ &\approx \alpha\hat{\mathcal{D}}_{t-1} + \frac{\alpha\eta}{2(1-\alpha)}\hat{\pi}_t^2 \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_t &= \alpha\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_{t-1} + \sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2(1-\alpha)}\hat{\pi}_t^2 \\ &= \alpha\hat{\mathcal{D}}_{-1} + \alpha\beta\sum_{t=1}^{\infty}\beta^{t-1}\hat{\mathcal{D}}_{t-1} + \sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2(1-\alpha)}\hat{\pi}_t^2 \\ &= \alpha\hat{\mathcal{D}}_{-1} + \alpha\beta\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_t + \sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2(1-\alpha)}\hat{\pi}_t^2 \end{aligned}$$

reveals that

$$\sum_{t=0}^{\infty} \beta^t \hat{\mathcal{D}}_t = \frac{\alpha}{1-\alpha\beta} \hat{\mathcal{D}}_{-1} + \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \frac{\alpha\eta}{(1-\alpha\beta)(1-\alpha)} \hat{\pi}_t^2$$

Using this information in the definition of the loss function gives

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{\alpha\eta}{(1-\alpha\beta)(1-\alpha)} \hat{\pi}_t^2 + \left(\psi + \sigma^{-1}\right) \hat{x}_t^2 + \tilde{\nu} \left(b^h + b_L^h\right) q_t^2 + \tilde{\xi} \left(b^h + b_L^h\right) (\Delta q_t)^2 \right]$$

because the term in $\hat{\mathcal{D}}_{-1}$ is independent of policy and can be ignored.

Appendix 1.D Rates of return and calibration of stock and flow effects

The following results, shown by Woodford (2001) and Chen et al. (2012) are useful:

$$\text{Yield to maturity} \quad \equiv \mathcal{R}_t = V_t^{-1} + \chi \tag{1.43}$$

$$\text{Duration} \quad \equiv D_t = \frac{\mathcal{R}_t}{\mathcal{R}_t - \chi} \tag{1.44}$$

Log-linearizing the first expression gives:

$$\mathcal{R}\hat{\mathcal{R}}_t = -\frac{1}{V} \hat{V}_t$$

By definition, the one-period return is also linked to the price of the long-term bond. Log-linearizing that relationship gives:

$$R\hat{R}_{L,t}^1 = -\frac{1+\chi V}{V} \hat{V}_{t-1} + \chi \hat{V}_t \implies \hat{R}_{L,t}^1 = -\hat{V}_{t-1} + \frac{\chi}{R_L^1} \hat{V}_t$$

In a zero inflation steady state, with bond issuance in line with household preferences, returns on short-term and long-term bonds are equalized at $R = R_L^1 = \beta^{-1}$. Hence:

$$\hat{R}_{L,t}^1 = -\hat{V}_{t-1} + \chi\beta \hat{V}_t$$

Steady-state one-period returns can be used to pin down steady-state V

$$\beta^{-1} = \frac{1+\chi V}{V} \implies V\left(\beta^{-1}-\chi\right) = 1 \implies V = \frac{1}{\beta^{-1}-\chi} = \frac{\beta}{1-\beta\chi}$$

In steady state, the yield to maturity is:

$$\mathcal{R} = V^{-1} + \chi = \frac{1-\beta\chi}{\beta} + \chi = \beta^{-1}$$

which implies yield to maturity and one period returns are equalized.

So the yield to maturity can be related to the price of the bond by:

$$\hat{\mathcal{R}}_t = -\beta\frac{1-\beta\chi}{\beta}\hat{V}_t = -\left(1-\beta\chi\right)\hat{V}_t \tag{1.45}$$

This expression can also be used to compute the yield to maturity from model outcomes. Note first that the expected one-period return satisfies:

$$\mathbb{E}_t\hat{R}^1_{L,t+1} = -\hat{V}_t + \chi\beta\mathbb{E}_t\hat{V}_{t+1}$$

or

$$\hat{V}_t = -\mathbb{E}_t\hat{R}^1_{L,t+1} + \chi\beta\mathbb{E}_t\hat{V}_{t+1}$$

which can be written in terms of the yield to maturity:

$$\hat{\mathcal{R}}_t = \left(1-\chi\beta\right)\mathbb{E}_t\hat{R}^1_{L,t+1} + \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1}$$

Recall that arbitrage between short-term and long-term bonds implies:

$$\begin{aligned}\mathbb{E}_t R^1_{L,t+1} =& \hat{R}_t - \tilde{\nu}\delta^{-1}\left(1+\delta\right)^2\left[\hat{b}^h_t - \hat{b}^h_{L,t}\right] \\ &- \tilde{\xi}\delta^{-1}\left(1+\delta\right)^2\left[\Delta\left(\hat{b}^h_t - \hat{b}^h_{L,t}\right) - \beta\mathbb{E}_t\Delta\left(\hat{b}^h_{t+1} - \hat{b}^h_{L,t+1}\right)\right]\end{aligned}$$

Imposing bond market clearing and the parameter definitions $\nu \equiv \tilde{\nu}\left(1+\delta\right)$ and $\xi \equiv \tilde{\xi}\left(1+\delta\right)$ gives:

$$\begin{aligned}\mathbb{E}_t R^1_{L,t+1} =& \hat{R}_t - \nu\delta^{-1}\left(1+\delta\right)q_t - \xi\delta^{-1}\left(1+\delta\right)\left[\Delta q_t - \beta\mathbb{E}_t\Delta q_{t+1}\right] \\ =& \hat{R}_t - \nu\delta^{-1}\left(1+\delta\right)q_t - \xi\delta^{-1}\left(1+\delta\right)\left[q_t - q_{t-1} - \beta\mathbb{E}_t q_{t+1} + \beta q_t\right] \\ =& \hat{R}_t - \delta^{-1}\left(1+\delta\right)\gamma q_t + \xi\delta^{-1}\left(1+\delta\right)q_{t-1} + \beta\xi\delta^{-1}\left(1+\delta\right)\mathbb{E}_t q_{t+1}\end{aligned}$$

where $\gamma \equiv \nu + \xi (1 + \beta)$ as before.

This implies that the yield to maturity is given by:

$$\begin{aligned}\hat{\mathcal{R}}_t =& \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1} \\ &+ (1-\chi\beta)\left(\hat{R}_t - \delta^{-1}(1+\delta)\gamma q_t + \xi\delta^{-1}(1+\delta)q_{t-1} + \beta\xi\delta^{-1}(1+\delta)\mathbb{E}_t q_{t+1}\right)\end{aligned}$$

These relationships are used to generate model-consistent measures of the responses of bond yields to QE auctions found by D'Amico and King (2013). Specifically, D'Amico and King (2013, p441) note that \$1bn of asset purchases generates a price increase of around 0.02% for the targeted assets. They argue that this translates into a yield effect of around 0.3 basis points for a representative ten year security.

Equation (1.45) generates a similar result. Given the model calibration, a simple calculation gives:

$$\begin{aligned}\hat{\mathcal{R}}_t - \mathbb{E}_{t-1}\hat{\mathcal{R}}_t &= -(1-\beta\chi)\left(\hat{V}_t - \mathbb{E}_{t-1}\hat{V}_t\right) \\ &= -(1 - 0.9918 \times 0.975)\,0.0002 \\ &= -0.000006599\end{aligned}$$

which when multiplied by 400 to convert into an annualized rate of return gives -0.00264, which is approximately 0.3 basis points.

Average QE auctions were around \$5bn, so the target bond yield change is $-0.00264 \times 5 = -0.013$. Repeating the same calculation for the price changes implied by the point estimate of the elasticity of price to QE purchases plus and minus one standard deviation gives the target range used in Figure 1.2.

Appendix 1.E Profits and losses on the central bank's asset portfolio

This appendix derives approximate expressions for the revaluation of the central bank's asset portfolio under the assumption that asset purchases are financed by issuing interest-bearing reserves. Reserves earn the same (risk free) nominal interest rate as short-term bonds. They are therefore perfect substitutes for short-term bonds and (in equilibrium) households will willingly hold whatever supply of reserves is created by the central bank. Any profits/losses on the central bank's portfolio are transferred to/from the government. At the start of period t the central bank's balance sheet is assumed to have a simple structure. The central bank holds $V_{t-1}\widetilde{Q}_{t-1}$ of previously purchased long-term bonds on the asset side, which is matched by Z_{t-1} of central bank reserves on the liabilities side.

The revaluation effect (or capital gain) on the central bank's existing portfolio is defined as:

$$K_t \equiv [1 + \chi V_t - V_{t-1}] \widetilde{Q}_{t-1} - [R_{t-1} - 1] Z_{t-1}$$

which is the change in the value of the assets minus the change in the cost of the liabilities. The former includes the coupon payment on the long-term bond holdings and the latter includes the risk free interest rate payment on previously issued reserves.

The revaluation effect can be written as:

$$\begin{aligned} K_t &= \left[R^1_{L,t} - 1\right] Q_{t-1} - [R_{t-1} - 1] Z_{t-1} \\ &= \left[R^1_{L,t} - R_{t-1}\right] Q_{t-1} \end{aligned}$$

where the first line uses the definition of the value of assets $\left(Q_{t-1} \equiv V_{t-1}\widetilde{Q}_{t-1}\right)$ and the one-period return on long bonds $\left(R^1_{L,t} \equiv V^{-1}_{t-1} [1 + \chi V_t]\right)$ and the second line uses the fact that the central bank balance sheet satisfies $Z_{t-1} = V_{t-1}\widetilde{Q}_{t-1}$ at the end of period $t-1$.

Since steady-state output is normalized to unity, K_t can be interpreted as a ratio to steady-state output. Given the assumed debt issuance policy, the revaluation effect can be written in real terms as:

$$\mathcal{K}_t \equiv \frac{K_t}{P_t} = \left[R^1_{L,t} - R_{t-1}\right] \delta b q_{t-1} \approx \frac{\delta (b + b_L)}{1 + \delta} \left[\hat{R}^1_{L,t} - \hat{R}_{t-1}\right] q_{t-1}$$

Using the relationships derived in Appendix 1.D, the ex post one-period return on long-term bonds can be written:

$$\hat{R}^1_{L,t} = -\hat{V}_{t-1} + \chi\beta\hat{V}_t = (1 - \chi\beta)^{-1} \hat{\mathcal{R}}_{t-1} - \chi\beta (1 - \chi\beta)^{-1} \hat{\mathcal{R}}_t$$

Appendix 1.F The optimal policy problem

The policymaker sets policy under discretion, with no ability to commit to future policy plans. I seek a Markov perfect policy in which optimal decisions are a function only of the payoff relevant state variables in the model. The policymaker at date t is treated as a Stackelberg leader with respect to both private agents and policymakers in dates $t+i, i \geq 1$.

Under this interpretation, the policymaker understands that future policymakers will choose allocations according to time-invariant Markovian policy functions. I use upper case bold letters to denote these policy functions. For example, inflation at date $t+j$ is given by the function:

$$\hat{\pi}_{t+j} = \mathbf{\Pi}\left(q_{t+j-1}; z_{t+j}\right) \quad , \quad j \geq 1 \tag{1.46}$$

where $z_{t+j} \equiv [u_{t+j} \quad r_{t+j}]'$ are the exogenous state variables. To simplify notation, I present the policy functions as dependent only on q in what follows.

The loss function that the policymaker minimizes is therefore given by:

$$\begin{aligned}
\tilde{\mathcal{L}}_t &= \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \begin{pmatrix} \frac{\omega_x}{2}\hat{x}_{t+i}^2 + \frac{\omega_\pi}{2}\left(\hat{\pi}_{t+i} - \pi^*\right)^2 \\ +\frac{\omega_q}{2}\left(q_{t+i} - q^*\right)^2 + \frac{\omega_{\Delta q}}{2}\left(q_{t+i} - q_{t+i-1}\right)^2 \end{pmatrix} \\
&= \frac{\omega_x}{2}\hat{x}_t^2 + \frac{\omega_\pi}{2}\left(\hat{\pi}_t - \pi^*\right)^2 + \frac{\omega_q}{2}\left(q_t - q^*\right)^2 + \frac{\omega_{\Delta q}}{2}\left(q_t - q_{t-1}\right)^2 + \beta \mathbb{E}_t \tilde{\mathcal{L}}_{t+1}
\end{aligned}$$

where I consider the variant analyzed in Section 1.6 because it nests the loss function derived in Appendix 1.C when $\pi^* = q^* = 0$.

The problem can therefore be expressed as a Lagrangean:

$$\begin{aligned}
\min_{\left\{\hat{\pi}_t, \hat{x}_t, \hat{R}_t, q_t\right\}} & \frac{\omega_x}{2}\hat{x}_t^2 + \frac{\omega_\pi}{2}\left(\hat{\pi}_t - \pi^*\right)^2 + \frac{\omega_q}{2}\left(q_t - q^*\right)^2 + \frac{\omega_{\Delta q}}{2}\left(q_t - q_{t-1}\right)^2 + \beta \mathbb{E}_t \tilde{\mathcal{L}}_{t+1} \\
& - \lambda_t^\pi \left(\hat{\pi}_t - \kappa \hat{x}_t - \beta \mathbb{E}_t \mathbf{\Pi}\left(q_t\right) - u_t\right) \\
& - \lambda_t^x \begin{pmatrix} \hat{x}_t - \mathbb{E}_t \boldsymbol{X}\left(q_t\right) \\ +\sigma\left(\hat{R}_t - \mathbb{E}_t \mathbf{\Pi}\left(q_t\right) - \gamma q_t + \xi q_{t-1} + \beta \xi \mathbb{E}_t \boldsymbol{Q}\left(q_t\right) - r_t^*\right) \end{pmatrix} \\
& - \lambda_t^R \left(\hat{R}_t - \beta^{-1} + 1\right) - \lambda_t^{\bar{q}}\left(q_t - \bar{q}\right) - \lambda_t^{\underline{q}}\left(q_t - \underline{q}\right)
\end{aligned} \tag{1.47}$$

The first order conditions are:

$$0 = \omega_\pi \left(\hat{\pi}_t - \pi^*\right) - \lambda_t^\pi \tag{1.48}$$

$$0 = \omega_x \hat{x}_t + \kappa \lambda_t^\pi - \lambda_t^x \tag{1.49}$$

$$\begin{aligned} 0 = \; & \omega_q \left(q_t - q^*\right) + \omega_{\Delta q}\left(q_t - q_{t-1}\right) + \beta \frac{\partial \mathbb{E}_t \tilde{\boldsymbol{\mathcal{L}}}_{t+1}}{\partial q_t} + \beta \frac{\partial \mathbb{E}_t \boldsymbol{\Pi}\left(q_t\right)}{\partial q_t} \lambda_t^\pi \\ & + \left[\frac{\partial \mathbb{E}_t \boldsymbol{X}\left(q_t\right)}{\partial q_t} + \sigma \frac{\partial \mathbb{E}_t \boldsymbol{\Pi}\left(q_t\right)}{\partial q_t} + \sigma\gamma - \beta\sigma\xi \frac{\partial \mathbb{E}_t \boldsymbol{Q}\left(q_t\right)}{\partial q_t}\right] \lambda_t^x - \lambda_t^{\bar{q}} - \lambda_t^{q} \end{aligned} \tag{1.50}$$

$$0 = -\sigma \lambda_t^x - \lambda_t^R \tag{1.51}$$

The first order condition for quantitative easing, (1.50), indicates that the policymaker accounts for the effects of current QE decisions on the losses incurred by future policymakers.

The solutions for a number of cases corresponding to whether or not the constraints on the instruments are binding are considered in turn. Expectations are taken as given (i.e., known). As described in Appendix 1.F.3, the solution procedure uses the previous guess of the policy functions to compute expectations and then refines the policy function guess conditional on those expectations, iterating in this way until the policy functions converge.

1.F.1 Interior optimum for the policy instruments

Note that we can write the Euler equation as

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - \sigma \left[\tilde{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \tilde{r}_t^*\right] \tag{1.52}$$

where

$$\tilde{R}_t \equiv \hat{R}_t - \gamma q_t \tag{1.53}$$

denotes the 'effective' policy rate and

$$\tilde{r}_t^* \equiv r_t^* - \xi q_{t-1} - \beta \xi \mathbb{E}_t q_{t+1} \tag{1.54}$$

is the 'effective' efficient real interest rate.

This variant of the model is isomorphic to the standard New Keynesian model, conditional on past QE and expected future QE. When the zero bound on the short term interest rate $\hat{R}$ does not bind, the multipliers on the zero bound constraint and IS curve are zero:

$$\lambda_t^R = \lambda_t^x = 0$$

In this case, the optimal effective policy rate can be computed using the following steps.

1. When $\lambda_t^x = 0$ the first order conditions imply a targeting criterion:
$$\hat{x}_t = -\frac{\omega_\pi \kappa}{\omega_x}\left(\hat{\pi}_t - \pi^*\right) \tag{1.55}$$
2. Using (1.55) to eliminate the output gap from the Phillips curve implies a solution for inflation:
$$\hat{\pi}_t = \left(1 + \frac{\omega_\pi \kappa^2}{\omega_x}\right)^{-1}\left[\kappa\frac{\omega_\pi \kappa}{\omega_x}\pi^* + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t\right]$$
3. A solution for the output gap can be computed by plugging the solution for inflation derived in Step 2 into the targeting criterion (1.55).
4. With these solutions in hand, the optimal effective policy rate can be computed from the Euler equation as:
$$\tilde{R}_t = \sigma^{-1}\left(\mathbb{E}_t\hat{x}_{t+1} - \hat{x}_t\right) + \mathbb{E}_t\hat{\pi}_{t+1} + \tilde{r}_t^*$$
5. The next step is to determine whether the optimal effective policy rate can be delivered as an interior optimum for the policy instruments. Under the assumption that the solution for q is an interior solution ($\underline{q}_t \leq q_t \leq \bar{q}_t$), it is the case that $\lambda_t^{\bar{q}} = \lambda_t^{\underline{q}} = 0$ and so the first order condition for q can be solved to give:[70]
$$\begin{aligned} q_t =& \frac{\omega_q}{\omega_q + \omega_{\Delta q}}q^* + \frac{\omega_{\Delta q}}{\omega_q + \omega_{\Delta q}}q_{t-1} \\ &- \frac{\beta}{\omega_q + \omega_{\Delta q}}\left[\frac{\partial \mathbb{E}_t \mathbf{\Pi}\left(q_t\right)}{\partial q_t}\omega_\pi\left(\hat{\pi}_t - \pi^*\right) + \frac{\partial \mathbb{E}_t\tilde{\mathcal{L}}_{t+1}}{\partial q_t}\right] \end{aligned} \tag{1.56}$$

[70]Recall also that at this stage in the solution process, it is assumed that the zero bound on the policy rate does not bind, so that $\lambda_t^x = 0$.

6. If the solution for q_t from equation (1.56) is indeed an interior solution, the optimal policy rate can be computed as $\hat{R}_t = \tilde{R}_t + \gamma q_t$. If this value of $\hat{R}_t$ is greater than the zero bound, the solution computed from these steps represents the equilibrium.

1.F.2 Bounded instruments

The steps presented in Appendix 1.F.1 may fail to deliver a valid equilibrium for two reasons: the implied level of quantitative easing may violate the upper and lower bounds on q; or the implied level of the policy rate required to deliver the desired effective policy rate may violate the zero bound.

Suppose first that Step 5 in Appendix 1.F.1 delivers a solution for q_t which violates the bounds. In this case, the solution for q_t is set to the relevant bound value.[71] If the optimal policy rate $\hat{R}_t = \tilde{R}_t + \gamma q_t$ computed using this value for q_t is greater than the zero bound, then this represents the equilibrium.

In the event that the value of $\hat{R}_t$ computed in Step 6 in Appendix 1.F.1 is below the zero bound, the system is solved as follows. I first assume that, even though the zero bound on the policy instrument is binding, there is an interior solution for QE

[71] If the solution to (1.56) is less than $\underline{q}$, then set $q_t = \underline{q}$. If the solution is greater than $\bar{q}$, set $q_t = \bar{q}$.

(so that $\lambda_t^{\bar{q}} = \lambda_t^{\underline{q}} = 0$).[72] In this case, the equilibrium solves the following system:

$$\begin{bmatrix} \omega_\pi & 0 & 0 & 0 & 0 & -1 \\ 0 & \omega_x & 0 & 0 & -1 & \kappa \\ 0 & 0 & 0 & \omega_q + \omega_{\Delta q} & \mathcal{D}_{\hat{x}} + \sigma\mathcal{D}_{\hat{\pi}} + \sigma\gamma - \beta\sigma\xi\mathcal{D}_q & \beta\mathcal{D}_{\hat{\pi}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \sigma & -\sigma\gamma & 0 & 0 \\ 1 & -\kappa & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \\ \hat{R}_t \\ q_t \\ \lambda_t^x \\ \lambda_t^\pi \end{bmatrix}$$
$$= \begin{bmatrix} \omega_\pi \pi^* \\ 0 \\ \omega_q q^* + \omega_{\Delta q} q_{t-1} - \beta \frac{\partial \mathbb{E}_t \tilde{\mathcal{L}}_{t+1}}{\partial q_t} \\ 1 - \beta^{-1} \\ \mathbb{E}_t \hat{x}_{t+1} + \sigma\left[\mathbb{E}_t \hat{\pi}_{t+1} + \tilde{r}_t^*\right] \\ \beta \mathbb{E}_t \hat{\pi}_{t+1} + u_t \end{bmatrix}$$

where the following notation for derivatives of expectations of variable z_{t+1} with respect to q_t is used to simplify the expressions:

$$\mathcal{D}_z \equiv \frac{\partial \mathbb{E}_t z_{t+1}}{\partial q_t}$$

This system can be solved by matrix inversion. If the solution for q_t is an interior solution, then the equilibrium allocations have been found. Otherwise the solution is found by setting q_t to the relevant bound value and solving the following recursive solutions.

1. The output gap is

$$\hat{x}_t = \begin{cases} \mathbb{E}_t \hat{x}_{t+1} - \sigma\left[1 - \beta^{-1} - \gamma\bar{q} - \mathbb{E}_t \hat{\pi}_{t+1} - \tilde{r}_t^*\right] & \text{if } q_t = \bar{q} \text{ binds} \\ \mathbb{E}_t \hat{x}_{t+1} - \sigma\left[1 - \beta^{-1} - \gamma\underline{q} - \mathbb{E}_t \hat{\pi}_{t+1} - \tilde{r}_t^*\right] & \text{if } q_t = \underline{q} \text{ binds} \end{cases}$$

2. Inflation is:

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t$$

[72] The reasoning is the following. The solution computed in the previous subsection assumed that there was an interior solution. Therefore the solution for q_t was derived conditional on $\lambda_t^x = 0$. If the zero bound on $\hat{R}_t$ is binding, the multiplier λ_t^x is non-zero and hence the solution for q_t implied by the general first order condition (1.50) may admit an interior solution.

1.F.3 Solution algorithm details

The preceding subsections detailed the optimal policy problem and elements of the solution. To solve for policy functions $\boldsymbol{X}, \boldsymbol{\Pi}, \boldsymbol{Q}, \boldsymbol{R}$ a simple policy function iteration scheme is used. The policy functions are defined over a grid for the state vector $s \equiv \{u, r^*, q_{-1}\}$ formed as a tensor product of three linearly spaced vectors. The vector for q_{-1} is defined on the range $\left[\underline{q}, \bar{q}\right]$ with 101 nodes and the grids for u_t and r_t are specified across ± 4 standard deviations with 25 and 101 nodes respectively. The state space is therefore $\mathcal{S} \equiv \mathcal{S}_u \times \mathcal{S}_{r^*} \times \mathcal{S}_{q_{-1}}$ with typical element s.

The iteration scheme to update the estimates of the policy functions is as follows, where the superscript j indexes the iteration number ($j = 1, \dots$):

1. Form $\boldsymbol{P}^j \equiv \{\boldsymbol{X}^j, \boldsymbol{\Pi}^j, \boldsymbol{Q}^j, \boldsymbol{R}^j\}$. To do so:
   a) For each $s \in \mathcal{S}$ compute solutions for $\{\hat{x}, \hat{\pi}, q, \hat{r}\}$ by using the procedure set out in Appendix 1.F.1 and (if necessary) the steps set out in Appendix 1.F.2.
   b) These solutions are conditional on expected outcomes, which are computed using Gauss-Hermite quadrature for the shocks to exogenous states (ε^u and ε^r) using five nodes for each shock.
   c) Expectations are computed by integrating over the estimated policy functions from the previous iteration: $\{\boldsymbol{X}^{j-1}, \boldsymbol{\Pi}^{j-1}, \boldsymbol{Q}^{j-1}, \boldsymbol{R}^{j-1}\}$.
   d) Derivatives are computed using a finite difference method on $\mathcal{S}_{q_{-1}}$. Linear interpolation is used to estimate the value of the derivative conditional on the value of the current stock of QE (i.e., q, which is the relevant state variable for expected outcomes) using the *previous* estimate of the policy function $\boldsymbol{Q}^{j-1}$.
2. Update the estimate of the loss function using
$$\tilde{\mathcal{L}}_t = \frac{\omega_x}{2}\hat{x}_t^2 + \frac{\omega_\pi}{2}\left(\hat{\pi}_t - \pi^*\right)^2 + \frac{\omega_q}{2}\left(q_t - q^*\right)^2 + \frac{\omega_{\Delta q}}{2}\left(q_t - q_{t-1}\right)^2 + \beta \mathbb{E}_t \tilde{\mathcal{L}}_{t+1}$$
   a) The loss function is computed for each $s \in \mathcal{S}$. Expectations are computed using the quadrature scheme described above.
   b) Convergence of the solution for the loss function is very slow, so the solution algorithm performs policy function iteration a number of times

(up to 10 depending on whether successive iterations are within solution tolerance) for each iteration on the policy functions.

3. If $\min|\text{vec}\,(\boldsymbol{P}^j - \boldsymbol{P}^{j-1})| < \epsilon$ then stop, otherwise return to Step 1. I set $\epsilon = 10^{-6}$.

One thing to note about this solution approach is that it does not involve solving a fixed point problem in step 1(a). For example, the first order condition for q for an interior solution, (1.56), implies that q is a function of the derivative of expected inflation with respect to q. So the right hand side of the equation for q is itself a function of q. One approach would be to solve for q as the fixed point of the equation for each point in $\mathcal{S}$. However, such an approach is computationally intensive, so instead I use derivative of expected inflation with respect to q evaluated using the previous iteration of the q policy function (that is, $\boldsymbol{Q}^{j-1}$). As the policy functions converge, $\boldsymbol{Q}^{j-1} \to \boldsymbol{Q}^j$ and the derivative is computed at the fixed point value of q. Approximations for the policy functions for the yield to maturity are computed using a simple variant of the policy function iteration approach.

Appendix 1.G Equilibrium without the zero bound

In the case in which there is no zero bound on the short term interest rate, the model is linear and it is possible to derive analytical expressions for the endogenous variables in terms of the state variables r^* and u.

The model is solved for the general case, in which the policymaker may be minimizing a delegated loss function so that q^* and π^* may be non-zero. The initial stock of QE inherited by the monetary policymaker is assumed to be $q_{-1} = q^*$. In this case, the subsequent choices of QE satisfy $q_t = q^*, \forall t$. This follows from inspection of the first order condition (1.50). First note that an interior solution for QE implies $\lambda_t^{\bar{q}} = \lambda_t^{q} = 0$ and the absence of a zero bound on $\hat{R}_t$ implies that $\lambda_t^R = 0$. The remaining condition for the conjectured policy $q_t = q^*$ to be optimal is that $\frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial q_t} = 0$. This is indeed the case if $\hat{R}_t$ can always be freely chosen. In that case, equilibrium outcomes for the output gap and inflation are uniquely pinned down by the effective policy rate $\tilde{R}_t$ which in turn can be set to any required value by an appropriate choice of $\hat{R}_t$. This implies that q_{t-1} ceases to be a meaningful state variable in the model, because monetary conditions can be chosen independently of the level of quantitative easing.

If $q_t = q^*$, then the optimal allocations for inflation and the output gap satisfy:

$$\hat{x}_t = -\frac{\kappa\omega_\pi}{\omega_x}\left(\pi_t - \pi^*\right)$$

which can be substituted into the Phillips curve (1.9) to give:

$$\pi_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} - \frac{\kappa^2\omega_\pi}{\omega_x}\left(\pi_t - \pi^*\right) + u_t$$

or

$$\begin{aligned}
\pi_t &= \frac{\beta\omega_x}{\omega_x + \kappa^2\omega_\pi}\mathbb{E}_t\hat{\pi}_{t+1} + \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi}u_t \\
&= \sum_{i=0}^{\infty}\left(\frac{\beta\omega_x}{\omega_x + \kappa^2\omega_\pi}\right)^i\left(\frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi}u_{t+i}\right) \\
&= \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi}\sum_{i=0}^{\infty}\left(\frac{\beta\omega_x\rho_u}{\omega_x + \kappa^2\omega_\pi}\right)^i u_t \\
&= \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t
\end{aligned}$$

where we make use of the fact that:

$$\sum_{i=0}^{\infty}\left(\frac{\beta\omega_x}{\omega_x + \kappa^2\omega_\pi}\right)^i = \frac{1}{1 - \frac{\beta\omega_x}{\omega_x+\kappa^2\omega_\pi}} = \frac{\omega_x + \kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}$$

and similarly

$$\sum_{i=0}^{\infty}\left(\frac{\beta\rho_u\omega_x}{\omega_x + \kappa^2\omega_\pi}\right)^i = \frac{1}{1 - \frac{\beta\rho_u\omega_x}{\omega_x+\kappa^2\omega_\pi}} = \frac{\omega_x + \kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}$$

The targeting criterion implies that the output gap is:

$$\begin{aligned}
\hat{x}_t &= -\frac{\kappa\omega_\pi}{\omega_x}\left(\frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t - \pi^*\right) \\
&= -\frac{\kappa\omega_\pi}{\omega_x}\left(\frac{\kappa^2\omega_\pi - \omega_x - \kappa^2\omega_\pi + \beta\omega_x}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \frac{\omega_x}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t\right) \\
&= \frac{\left(1-\beta\right)\kappa\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* - \frac{\kappa\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t
\end{aligned}$$

To solve for the nominal interest rate, note that:

$$\mathbb{E}_t \hat{x}_{t+1} = \frac{(1-\beta)\,\kappa\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* - \frac{\kappa\omega_\pi\rho_u}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t$$
$$\mathbb{E}_t \pi_{t+1} = \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \frac{\omega_x\rho_u}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t$$

Using this in the IS curve (1.8) gives:

$$-\frac{\kappa\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t = -\frac{\kappa\omega_\pi\rho_u}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t - \sigma\begin{pmatrix} \hat{R}_t - \nu\delta q^* - \frac{\kappa^2\omega_\pi}{\omega_x+\kappa^2\omega_\pi-\beta\omega_x}\pi^* \\ -\frac{\omega_x\rho_u}{\omega_x+\kappa^2\omega_\pi-\beta\omega_x\rho_u}u_t - r^* \end{pmatrix}$$

So

$$\sigma\left(\hat{R}_t - \nu q^* - \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* - \frac{\omega_x\rho_u}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t - r^*\right) = \frac{\kappa\omega_\pi\,(1-\rho_u)}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t$$

Or

$$\sigma\left(\hat{R}_t - \nu q^* - \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* - r^*\right) = \frac{\sigma\omega_x\rho_u + \kappa\omega_\pi\,(1-\rho_u)}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u}u_t$$

Which implies that the nominal interest rate satisfies:

$$\hat{R}_t = r_t^* + \frac{\kappa^2\omega_\pi}{\omega_x + \kappa^2\omega_\pi - \beta\omega_x}\pi^* + \nu q^* + \frac{\sigma\omega_x\rho_u + \kappa\omega_\pi\,(1-\rho_u)}{\sigma\,(\omega_x + \kappa^2\omega_\pi - \beta\omega_x\rho_u)}u_t \tag{1.57}$$

With $q_t = q^*, \forall t$, the yield to maturity is given by:

$$\begin{aligned}
\hat{\mathcal{R}}_t =& (1-\chi\beta)\left(\hat{R}_t - \nu\delta^{-1}(1+\delta)\,q^*\right) + \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1} \\
=& (1-\chi\beta)\,\mathbb{E}_t\sum_{j=0}^{\infty}(\chi\beta)^j\left(\hat{R}_t - \nu\delta^{-1}(1+\delta)\,q^*\right) \\
=& -\nu\delta^{-1}(1+\delta)\,q^* \\
&+ (1-\chi\beta)\,\mathbb{E}_t\sum_{j=0}^{\infty}(\chi\beta)^j\left[\begin{array}{l} r^*_{t+j} + \frac{\kappa^2\omega_\pi}{\omega_x+\kappa^2\omega_\pi-\beta\omega_x}\pi^* + \nu\delta q^* \\ +\frac{\sigma\omega_x\rho_u+\kappa\omega_\pi(1-\rho_u)}{\sigma(\omega_x+\kappa^2\omega_\pi-\beta\omega_x\rho_u)}u_{t+j} \end{array}\right] \\
=& -\nu\delta^{-1}(1+\delta)\,q^* + \frac{1-\chi\beta}{1-\chi\beta\rho_r}r^*_t + \frac{\kappa^2\omega_\pi}{\omega_x+\kappa^2\omega_\pi-\beta\omega_x}\pi^* + \nu\delta q^* \\
&+ \frac{1-\chi\beta}{1-\chi\beta\rho_u}\frac{\sigma\omega_x\rho_u+\kappa\omega_\pi(1-\rho_u)}{\sigma(\omega_x+\kappa^2\omega_\pi-\beta\omega_x\rho_u)}u_t
\end{aligned}$$

Collecting terms gives:

$$\begin{aligned}
\hat{\mathcal{R}}_t =& \frac{1-\chi\beta}{1-\chi\beta\rho_r}r^*_t + \frac{\kappa^2\omega_\pi}{\omega_x+\kappa^2\omega_\pi-\beta\omega_x}\pi^* - \nu\delta^{-1}q^* \\
&+ \frac{1-\chi\beta}{1-\chi\beta\rho_u}\frac{\sigma\omega_x\rho_u+\kappa\omega_\pi(1-\rho_u)}{\sigma(\omega_x+\kappa^2\omega_\pi-\beta\omega_x\rho_u)}u_t
\end{aligned}$$

Chapter 2

Monetary financing with interest-bearing money*

Abstract

Recent results suggesting that monetary financing is more expansionary than bond financing in standard New Keynesian models rely on a duality between policies specified in terms of money growth and the short-term bond rate, rather than a special role for money. Two features are added to a simple sticky-price model to generalize these results. First, that money may earn a strictly positive rate of return, motivated by recent debates on central bank digital currencies and interest-bearing reserves. This allows money-financed transfers to be used as a policy instrument at the effective lower bound, without giving up the ability to use the short-term bond rate to stabilize the economy in normal times. Second, a simple financial friction generates a wealth effect on household spending from government liabilities. Though temporary money-financed transfers to households can stimulate spending and inflation when the short-term bond rate is at the lower bound, similar effects could be achieved by bond-financed tax cuts.

2.1 Introduction

The global financial crisis prompted macroeconomic policymakers to employ a range of unconventional policies to stabilize economic activity and inflation, as short-term policy rates hit the effective lower bound. Despite these dramatic policy actions, the

*This chapter is co-authored with Ryland Thomas.

protracted effects of the crisis and the potential limits to unconventional measures have prompted proposals for fiscal stimulus as a policy tool. In particular, many have argued that this stimulus should be financed by money creation, as money-financed deficits are argued to have some direct effect on spending and lead to a smaller crowding out effect via higher interest rates.

Recent research has analyzed the effects of money-financed fiscal policies using conventional New Keynesian models (Galí, 2014a; English, Erceg, and López-Salido, 2017), under the standard assumption that money earns no interest. In such models there is no wealth effect from a money-financed deficit even if money earns no interest and does not require higher future taxes to meet the servicing costs of conventional government debt (Weil, 1991).

In this chapter, a small sticky price model is used to study the possible advantages of money-financed transfers to households, relative to conventional fiscal policy, when the economy is in a temporary liquidity trap. The model is standard in most respects. Firms maximize profits, subject to price adjustment costs. Households maximize utility. Money alleviates transactions frictions faced by households.

However, the model includes two important features that permit previous results to be extended. First, money may earn a strictly positive rate of return. This is motivated by the recent debates on the introduction of central bank digital currencies (which in some forms could be remunerated) and the introduction of interest-bearing central bank reserves in many economies following the financial crisis. Second, a simple financial friction implies that households regard government liabilities as net wealth (Weil, 1991; Ireland, 2005).

The first feature allows policymakers to control the stock of money independently of the nominal interest rate on short-term government bonds. In contrast, conventional monetary models assume that the rate of interest on money is zero. Monetary policy in these models is implemented *either* by a policy rule for the evolution of the money stock or by a policy rule for the rate of interest on short-term nominal bonds. The latter can be implemented by supplying whatever quantity of money is required (via open market operations in short-term bonds) to deliver the rate of interest implied by the rule. This conventional approach implies a duality between the rate of money growth and the short-term bond rate.[1] This approach therefore allows a comparison of the effects of monetary and bond-financed transfers *for a given monetary policy rule.*

The second feature of the model implies that expansions of government liabilities (money and one-period government bonds) do have the potential to increase aggregate demand through a wealth effect. To incorporate this effect it is assumed that each period households face a constant probability of a default-like event that restricts participation in asset markets (Castelnuovo and Nisticò, 2010; Nisticò, 2012; Del Negro, Giannoni, and Patterson, 2015a). The presence of this friction implies that, in equilibrium, households 'over-discount' future income streams. One implication is that changes in real money balances can stimulate spending by increasing households' net wealth. This implementation implies that the wealth effect also applies to bond-financed stimulus, which permits an assessment of whether there are specific benefits to money-financed fiscal actions.

Simulations using the model show that recent proposals for money-financed fiscal stimulus (in particular, Galí, 2014a) rely on the conventional duality between the rate of money growth and the short-term bond rate.[2] The results confirm the findings of English et al. (2017), who show that the stimulative effects of such policies are determined by the fact that the money supply rule implies that the short-term bond rate responds weakly to inflation, rather than because of any special property of money itself. These results are extended to demonstrate that the macroeconomic responses to non-fiscal shocks may be undesirable when this type of monetary policy rule is used.

With interest-bearing money, the policymaker can control the money stock independently of the short-term bond rate. This allows monetary transfers to be used as a policy instrument at the effective lower bound, without giving up the ability to use the short-term bond rate to stabilize the economy in response to shocks in normal times.

[1] This duality is a well-known feature of conventional monetary models. A textbook analysis can be found in Woodford (2003, Chapter 2).

[2] Since this chapter was completed, a revised version of Galí (2014a) has been published (as Galí, 2019). The published version of the paper acknowledges that the money financing rule requires abandonment of control of the short-term nominal interest rate.

The potential quantitative effects of such policies are explored by simulating the effects of a temporary increase in the rate of money growth. Monetary policy continues to be governed by a rule for the short-term bond rate. So for households to willingly hold the additional money balances, the rate of interest on money must be increased. Under these conditions, the simulations show that money-financed transfers to households can increase output and inflation when the economy is in a temporary liquidity trap. Moreover, such transfers increase household wealth and hence spending and inflation, even if they are implemented in the form of a *temporary* increase in the stock of money.

However, these results reveal three reasons to be cautious about the use of money-financed transfers to stimulate the economy at the lower bound. First, the scale of the monetary transfers required to deliver a meaningful increase in aggregate demand and inflation is likely to be extremely large. Second, the frictions in the model suggest that equivalent effects could be achieved by an increase in government debt, without requiring interest-bearing money. Finally, the stimulative effect of money-financed transfers is likely to be sensitive to the precise nature of the frictions that give rise to a meaningful role for money and the policy rule used to set the short-term bond rate.

The simulation results represent a quantitative contribution to the recent debate on the potential efficacy of money-financed policy measures in a liquidity trap. Many recent contributions refer to the idea of 'helicopter drops', named for Milton Friedman's famous thought experiment in which "a helicopter [...] drops an additional $1,000 in bills from the sky".[3] These contributions follow Bernanke (2002) and interpret Friedman's thought experiment as "essentially equivalent" to a money-financed tax cut. Prominent recent proponents of such policies include Bernanke (2016), Buiter (2014) and Turner (2015).[4]

Other recent work has studied the implications of interest-bearing money or reserves (see, for example, Ireland, 2014). The primary interest for this chapter is the

[3] Friedman (1969).

[4] Many economists have used blog posts to set out the arguments for and against helicopter drops. See, for example, Bossone, Fazi, and Wood (2014), Bossone (2013), Cabellero (2010), Galí (2014b), Grenville (2013), Reichlin, Turner, and Woodford (2013), Wren-Lewis (2014) and Yates (2014).

extent to which an additional policy instrument expands the set of possible outcomes achievable by monetary and fiscal policies, under the conventional New Keynesian assumption that the government adjusts taxes to stabilize the real present value of its nominal liabilities, for all possible paths of the price level.[5] This approach abstracts from potential implications of interest-bearing money operating via the government's present value budget constraint as studied by Buiter (2014) and Cochrane (2014), among others.

The rest of the chapter is organized as follows. Section 2.2 sets out the model. Section 2.3 analyzes money-financed government spending shocks, following Galí (2014a) and English et al. (2017). Section 2.4 examines the efficacy of money-financed transfers to households when the return on money is adjusted to ensure that households willingly hold the additional money. Section 2.5 investigates the sensitivity of the results to alternative assumptions about the specification of money demand. Section 2.6 relates the results to several issues raised in recent discussions of the likely efficacy of monetary-financed fiscal stimulus, including the importance of allowing the rate of return on money to be adjusted. Section 2.7 concludes.

2.2 The model

This section provides a description of the baseline model, focusing on the innovations. A full derivation is presented in Appendix 2.B.

The infinite-horizon model is cast in discrete time, with time periods indexed by $t = 1, \ldots, \infty$. Agents in the model have perfect foresight.

A simple financial friction causes households to regard government liabilities as net wealth. Similarly to Castelnuovo and Nisticò (2010), Nisticò (2012) and Del Negro et al. (2015a), among others, households are assumed to face a risk of a default-like event that restricts participation in asset markets. Specifically, each household is subject to a fixed per-period probability of experiencing an 'asset reset' event that causes the household's previously accumulated assets to be lost. After

[5] That is, fiscal policy is 'passive' in the sense of Leeper (1991).

a household experiences an 'asset reset' they must reformulate a new consumption plan starting from a zero asset position.[6]

The asset reset friction effectively causes households to discount the future more heavily than otherwise. The fact that previously accumulated assets may be lost creates a disincentive to save for future consumption and a stronger incentive to use assets to finance current consumption, given the risk that those assets may be lost in future periods. These properties of the model imply that even temporary increases in holdings of government liabilities can have net wealth effects that stimulate consumption.

2.2.1 Individual households

The population consists of a continuum of households of measure one. A household that last experienced an asset reset at date j faces a budget constraint in period $t \geq j$ given by:

$$\frac{M^p_{j,t}}{P_t} + \frac{B^p_{j,t}}{P_t} = \gamma^{-1}\left[\frac{R^M_t M^p_{j,t-1}}{P_t} + \frac{R_t B^p_{j,t-1}}{P_t}\right] + \tilde{w}_{j,t} - \left(1 + \varphi\left(\frac{c_{j,t}}{M^p_{j,t}/P_t}\right)\right) c_{j,t} \tag{2.1}$$

The household invests in money (M) and short-term government bonds (B) and receives interest income from its portfolio of money and bonds (at gross nominal rates R^M and R respectively) and real net labor income ($\tilde{w}$, defined below). Net labor income and net proceeds from portfolio changes are used to finance expenditure on consumption, c, which is measured inclusive of transactions costs (φ, discussed below). The price of consumption is denoted by P.

Throughout, nominal quantities and prices are denoted using upper case letters, real valued quantities and relative prices (relative to P) are denoted using lower case letters. Similarly to the notation of Buiter (2005), the p superscript denotes private sector demand for assets. Interest rates are defined so that R_{t+1} is the (gross) rate of

[6]This approach is similar to the perpetual youth model of Yaari (1965) and Blanchard (1985), in which a randomly selected fraction of the population dies each period and is replaced by a cohort of newborn households. The 'asset reset' interpretation is preferred because it allows the calibration of the asset reset probability to be tied to factors (other than mortality) that are likely to cause households to discount the future more heavily. Section 2.2.8 discusses this in more detail.

return on a bond held between periods t and $t+1$. So R_{t+1} and R^M_{t+1} are determined at date t. The net rate of return on money may be positive (so that $R^M > 1$) and vary over time.

The total returns on assets reflect the presence of the asset reset mechanism. Returns on bonds and money are adjusted by $0 < \gamma \leq 1$, which is the (constant) probability that the household reaches the following period without losing their accumulated assets. This reflects the presence of an actuarially fair insurance market that pools the risk of asset resets across households.[7]

By definition, an asset reset means that the household has no previously accumulated assets. So $M_{t-1,t} = B_{t-1,t} = 0$, for a household that experienced an asset reset in the previous period ($j = t-1$).

The real net income of the household is defined as:

$$\tilde{w}_{j,t} \equiv w_t n_{j,t} + d_{j,t} - \tau_{j,t}$$

where the household receives labor income (real wage w times labor supply, n) and lump sum transfers (either positive or negative) in the form of real dividends (d) and tax/transfer payments from the government, τ. All elements are expressed in real terms (i.e., nominal income/expenditures deflated by the price level, P).

Money provides transactions services (Schmitt-Grohé and Uribe, 2010; Del Negro and Sims, 2015). The transaction cost, φ, associated with each unit of consumption is a declining function of the household's holdings of money relative to their consumption. That is, $\varphi = \varphi(v_{j,t})$, with $\varphi'(v_{j,t}) > 0$ and $v_{j,t} \equiv P_t c_{j,t}/M^p_{j,t}$.

Following Del Negro and Sims (2015), the transactions cost function is given by:

$$\varphi(v_{j,t}) = Z \exp\left[-\frac{\zeta}{v_{j,t}}\right]$$

where $Z, \zeta > 0$.

[7]Assets taken from those households who are randomly selected for an asset reset (with probability $1-\gamma$) are redistributed among those households who do not experience an asset reset in proportion to their asset holdings.

The household maximizes a time-separable and additively separable lifetime utility function specified over consumption and hours worked:

$$\max \sum_{t=0}^{\infty} (\gamma\beta)^t \vartheta_t \left[\ln c_{j,t} - \frac{\chi_{j,t}}{1+\psi} n_{j,t}^{1+\psi}\right] \tag{2.2}$$

where ϑ_t and $\chi_{j,t}$ are exogenous shocks to utility. The first order conditions are derived in Appendix 2.B.1.

The effective discount factor for the household has two components. The factor $0 < \beta < 1$ captures the standard assumption that households discount future utility relative to current utility. The additional factor $0 < \gamma \leq 1$ reflects the fact that previously accumulated assets are reset to zero with a probability of $1 - \gamma$ each period. Thus γ is the probability that a current consumption plan is still in effect next period.[8]

Variations in ϑ_t generate fluctuations in output and inflation that the monetary policymaker will seek to stabilize in the simulations studied in Section 2.4. The exogenous process for ϑ_t, common to all households, is:

$$\Delta \ln \vartheta_{t+1} = \rho_\vartheta \Delta \ln \vartheta_t + \varepsilon_t^\vartheta \tag{2.3}$$

where $\rho_\vartheta \in [0, 1)$ governs the persistence of ϑ and ε^ϑ is an exogenous disturbance.

The disutility of labor supply is also subject to a preference shifter, $\chi_{j,t}$. Although $\chi_{j,t}$ depends on the households' last reset date j as well as t, each individual household treats $\chi_{j,t}$ parametrically as it is a function of cohort-j aggregates rather than an individual household's decisions.

The first order conditions for labor and consumption derived in Appendix 2.B.1 can be combined to give a labor supply relationship:

$$\chi_{j,t} n_{j,t}^{\psi} = \frac{w_t}{c_{j,t}\left[1 + \varphi(v_t)\left(1 + \zeta v_t^{-1}\right)\right]} \tag{2.4}$$

[8]The only state variables in the household's problem are asset stocks. Since these are are not carried forward in the event that the household's assets are reset, utility flows when the household experiences a reset are independent of the expected utility flow enjoyed until the point that a reset does occur. This means that the relevant maximand for the household is the latter.

where v denotes the aggregate velocity of circulation.[9]

While all households that last experienced an asset reset on the same date make identical decisions, the consumption levels of households that experienced resets at different dates will vary depending on the assets that have been accumulated in the meantime. If χ was fixed then, in principle, some households could accumulate a stock of assets to support a level of consumption sufficiently large that the solution to (2.4) implies an arbitrarily small number of hours worked ($n_{j,t} \to 0$), which complicates aggregation.[10]

This choice of preferences is intended to be simple and conventional, particularly for the case in which $\gamma = 1$. However, when $\gamma < 1$, each household's consumption will increase over time, even in the steady state. If labor supply decisions depend on consumption, then a household's expected labor income and human wealth will depend on the date of their last asset reset, which complicates aggregation. In order to simplify, it is therefore assumed that $\chi_{j,t}$ evolves over time in a way that offsets the effects of cohort-specific consumption on labor supply decisions.

Specifically, the preference shifter is given by:

$$\chi_{j,t} = \chi \frac{c_t}{\bar{c}_{j,t}} \tag{2.5}$$

where $\bar{c}_{j,t}$ denotes the average consumption level of households that last experienced an asset reset in period j and c_t is aggregate consumption. This specification of preferences therefore includes a "consumption externality in labor supply" similar to the method used by Galí, Smets, and Wouters (2011) and Jaimovich and Rebelo (2009) to reduce wealth effects on labor supply in business cycle models. However, equation (2.5) is designed not to mitigate wealth effects, but rather to remove *distributional* effects generated by differences in the date on which households last experienced an asset reset.

[9] Appendix 2.B.1 shows that the chosen form of the transactions cost function generates a money demand function with the property that velocity is identical for all households.

[10] Ascari and Rankin (2007) note that for some utility functions this effect may even imply that some households wish to supply a negative quantity of hours. The authors demonstrate how a generalization of the preferences proposed by Greenwood, Hercowitz, and Huffman (1988) can be used to avoid the problem.

In equilibrium, all households that experienced a reset in the same period will make identical decisions, so that $c_{j,t} = \bar{c}_{j,t}, \forall j$. This implies that the labor supply relationship becomes

$$\chi n_{j,t}^{\psi} = \frac{w_t}{c_t \left[1 + \varphi\left(v_t\right)\left(1 + \zeta v_t^{-1}\right)\right]} \tag{2.6}$$

so that all households will supply the same labor, regardless of when they last experienced an asset reset. This means that the human wealth of all households is the same, facilitating aggregation.[11]

2.2.2 Firms

A set of monopolistically competitive producers indexed by $j \in (0,1)$ produce differentiated products that form a Dixit-Stiglitz bundle that is purchased by households. Preferences over differentiated products are given by

$$y_t = \left[\int_0^1 y_{j,t}^{1-\eta^{-1}} dj\right]^{\frac{1}{1-\eta^{-1}}}$$

where y_j is firm j's output.

Firms produce using a constant returns production function in the single input (labor):

$$y_{j,t} = A n_{j,t}$$

where A is a productivity parameter.

Firms are subject to Rotemberg (1982) price adjustment costs, so that the real profit of producer j is:

$$\frac{P_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} - \frac{\Phi}{2}\left(\frac{P_{j,t}}{\pi^* P_{j,t-1}} - 1\right)^2 = \left(\frac{P_{j,t}}{P_t} - \frac{w_t}{A}\right)\left(\frac{P_{j,t}}{P_t}\right)^{-\eta} y_t - \frac{\Phi}{2}\left(\frac{P_{j,t}}{\pi^* P_{j,t-1}} - 1\right)^2$$

where $\Phi \geq 0$ is the parameter governing the strength of price adjustment costs, which are indexed to the steady-state inflation rate (the inflation target, π^*). Profits are distributed lump sum as dividends to households with each household receiving an equal share, regardless of the date at which they last experienced an asset reset.

[11] In a version of the model in which there are no asset resets ($\gamma = 1$), there is a single representative household and there is no distinction between individual consumption, average cohort consumption and aggregate consumption. That is, $c_t = \bar{c}_{j,t} = c_{j,t}$. In that case $\chi_{j,t} = \chi$ and so equations (2.6) and (2.4) are equivalent.

2.2.3 Monetary policy

The short-term bond rate is adjusted according to a simple rule, similar to that examined by Taylor (1993), subject to an effective lower bound:

$$R_{t+1} = \max\left\{ R\frac{\vartheta_t}{\vartheta_{t+1}}\left(\frac{\pi_t}{\pi^*}\right)^{\theta_\pi}\left(\frac{y_t}{y_t^f}\right)^{\theta_y}, \underline{R} \right\} \tag{2.7}$$

where R denotes the steady-state bond rate and $\underline{R} \geq 1$ is the effective lower bound.

Away from the effective lower bound, the policy rate is adjusted in response to deviations of inflation from the target and the output gap. The output gap is computed relative to the level of output that would prevail under flexible prices, y_t^f.[12] The coefficients $\theta_\pi > 1$ and $\theta_y > 0$ determine the strength of the policy response. The inclusion of the term $\frac{\vartheta_t}{\vartheta_{t+1}}$ in the rule incorporates an approximation to exogenous variations in the natural rate of interest that the policymaker seeks to offset.[13]

2.2.4 Fiscal policy

The period government budget constraint is:

$$M_t^g + B_t^g = R_t^M M_{t-1}^g + R_t B_{t-1}^g + P_t\left(g_t - \tau_t\right) \tag{2.8}$$

where the g superscript indicates that the quantities refer to government choices of liabilities. The flow budget constraint says that the government issues money and bonds to finance its interest payments on existing liabilities and the primary deficit. The government budget constraint is written in terms of economy-wide aggregates.

[12] As is common in models with transactions frictions, flexible price allocations are not independent of the levels of the nominal interest rates on bonds and money. Following Kim and Subramanian (2006) and Ravenna and Walsh (2006), a 'supply side' flexible price equilibrium concept, conditional on steady-state nominal returns on money and bonds is used. See Appendix 2.B.5 for a full derivation.

[13] In the absence of transactions frictions and the asset reset mechanism, the aggregate Euler equation would be $c_t = \frac{\pi_{t+1}}{\beta}\frac{\vartheta_t}{\vartheta_{t+1}}\frac{1}{R_{t+1}}c_{t+1}$. Under flexible prices with inflation at target, the rate of interest that keeps consumption stable is: $R_{t+1} = \frac{\pi^*}{\beta}\frac{\vartheta_t}{\vartheta_{t+1}}$. As noted previously, the presence of transactions frictions complications the definition of the flexible price equilibrium. See Appendix 2.B.5.

We assume that the pattern of government spending is determined exogenously by:

$$\ln g_t = \rho_g \ln g_{t-1} + (1 - \rho_g) \ln g^* + \varepsilon_t^g \tag{2.9}$$

where $\rho_g \in [0, 1)$ controls the persistence of the process, $g^* > 0$ is the steady state level of government spending and ε_t^g is an exogenous disturbance.

As the focus of this chapter is the effects of money-financed fiscal policies, for the simulations in Section 2.4 the government is assumed to hold the real debt stock constant:

$$b_t^g = b^*$$

where, $b_t^g = B_t^g / P_t$, though other experiments relax this assumption.

2.2.5 Money-financed transfers

Two specifications for the determination of equilibrium money holdings are considered.

In the first specification, no interest is paid on money: $R_t^M = 1, \forall t$. The pattern of money holdings is determined by the demand for money, given the short-term bond rate R_t set according to the policy rule (2.7). This is the conventional approach.

The second specification is to allow the monetary authority to directly control the stock of money M_t. This permits an analysis of the effects of a money-financed transfer to households. In this case, the stock of money is determined by a rule and the central bank adjusts R_t^M to ensure that households are willing to hold that stock. In Section 2.4 this specification is used to analyze the effects of money-financed transfers when the short-term bond rate is constrained by the effective lower bound.

The baseline assumptions for fiscal policy in those experiments are that government spending and debt are held fixed in real terms. Inspection of the government budget constraint (2.8) indicates that, if the short-term bond rate is fixed at $\underline{R}$, an increase in M_t^g requires a reduction in nominal lump sum taxes $P_t\tau_t$. This observation shows that expansions in the money stock can be interpreted as money-financed net transfers, such that the overall level of lump sum taxation falls.[14]

These observations show that a money-financed transfer is a fiscal policy action. However, since central banks generally have operational responsibility for the creation of base money, there is a debate over the feasibility of such policies under traditional institutional relationships between the central bank and fiscal authority. For example, Benigno and Nistico (2015) and Del Negro and Sims (2015) study cases in which the composition of public sector liabilities might have an effect on equilibrium allocations. These authors focus on cases in which the central bank and government have separate intertemporal budget constraints. While these issues are indeed likely to be of practical importance, the model sidesteps such concerns by assuming that there is a single consolidated (government and central bank) budget constraint for two reasons.

First, the model is configured so that there is as much chance as possible for money-financed fiscal policies to be effective. Even advocates of monetary financing in principle acknowledge the potential institutional difficulties with implementation.[15] These concerns are set aside here to focus on the potential efficacy of monetary-financed fiscal policy under the assumption that such institutional difficulties can be solved.

Second, there are real-world examples of mechanisms to ensure that capital injections from the government to cover potential losses on the central bank's balance sheet arising from unconventional policies are guaranteed *ex ante*.[16] So the assumption of a single consolidated government budget constraint is not necessarily unrealistic.

[14]For a sufficiently large expansion in M_t^g, τ_t may become negative so that the government makes gross transfers to households.

[15]See, for example, Turner (2015, Chapter 14).

[16]One example of such an arrangement is the indemnity provided by the UK government on any losses sustained by the Asset Purchase Facility used by the Bank of England to conduct quantitative easing.

2.2.6 Market clearing

Asset market clearing requires equality between government supply of assets and private sector demand. Superscripts are removed for market clearing equilibrium asset stocks:

$$b_t^p = b_t^g = b_t \tag{2.10}$$

$$m_t^p = m_t^g = m_t \tag{2.11}$$

Goods market clearing requires that output, net of adjustment costs, is purchased by the government or consumed by households:

$$y_t = c_t + g_t + \frac{\Phi}{2}\left(\frac{\pi_t}{\pi^*} - 1\right)^2 \tag{2.12}$$

which implies that the dividend paid by firms to each household is:

$$d_t = y_t - \frac{\Phi}{2}\left(\frac{\pi_t}{\pi^*} - 1\right)^2 - w_t n_t \tag{2.13}$$

2.2.7 Aggregation

The heterogeneity across households' asset reset dates requires aggregation across these cohorts to obtain aggregate quantities. Each variable x is aggregated as follows:

$$x_t \equiv \sum_{j=-\infty}^{t} \gamma^{t-j}\left(1-\gamma\right) x_{j,t}$$

where x_t is the aggregate quantity, $x_{j,t}$ is the quantity chosen by each household that last experienced an asset reset at date $j \leq t$ and $\gamma^{t-j}\left(1-\gamma\right)$ is the share of that cohort in the population.

Appendix 2.B.2 demonstrates that the aggregate money demand equation is:

$$m_t = \zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+1} - R_{t+1}^M}{R_{t+1}}\right] c_t \tag{2.14}$$

and that aggregate consumption satisfies:

$$\widetilde{c}_t = \frac{\pi_{t+1}}{\beta R_{t+1}} \frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}} \left[\widetilde{c}_{t+1} + \left(1-\gamma\right)\gamma^{-1}\mu_{t+1}\pi_{t+1}^{-1}\left(R_{t+1}^M m_t + R_{t+1} b_t\right)\right]$$

where μ is the marginal propensity to consume from wealth, $\widetilde{c}_t = (1 + \varphi(v_t))\, c_t$ denotes consumption inclusive of transactions costs and

$$\widetilde{\vartheta}_t \equiv \frac{\vartheta_t (1 + \varphi(v_t))}{1 + \varphi(v_t)\left(1 + \zeta v_t^{-1}\right)}$$

2.2.8 Parameter values

Table 2.1 summarizes the parameter values for the baseline version of the model. Each time period is interpreted as one quarter of a year. The parameters A and χ are used to normalize steady-state output and labor supply to 1.[17] Most parameter values are set with reference to those in other studies, or to deliver the same steady-state allocations as those studies. Appendix 2.B.6 provides details of the required calculations. The discussion focuses on the parameters of most relevance to the present inquiry.

Table 2.1: Model parameters

	Value	Source/motivation
π^*	1.005	Annual inflation target of 2%
β	0.99917	Steady-state annual real interest rate $\approx$ 1.5%
γ	0.97	Del Negro et al. (2015a)
g^*	0.2	Sims and Wolff (2013)
b^*	2	Reinhart et al. (2012) (advanced economies, pre-crisis)
Z	20.33	$\frac{m}{c} = 0.428$ (Del Negro and Sims, 2015)
ζ	25.75	Del Negro and Sims (2015)
η	7.88	Rotemberg and Woodford (1997)
Φ	80.83	Calvo (1983) price adjustment probability $\approx$ 0.25
ψ	0.55	Smets and Wouters (2007)
θ_π	1.5	Taylor (1993)
θ_y	0.125	Taylor (1993)
$\underline{R}$	1.0006	Effective lower bound of 25bp (annualized)

The central bank's inflation target is 2% per year, consistent with the inflation targeting regimes in many economies. The discount rate, β, is chosen to be consistent with a steady-state risk free real interest rate of 1.5% per year. This is somewhat lower than assumptions often used in analysis before the financial crisis. This reflects

[17] Appendix 2.B.6 derives the required values to deliver these normalizations.

the notion that risk-free real interest rates may be somewhat lower, relative to the pre-crisis period (see, among others, King and Low, 2014; Bean, 2017; Fischer, 2016, 2017; Williams, 2017). Given the chosen value for γ (discussed below), this parameterization requires a discount factor (β) very close to unity.

The parameters governing money demand (Z and ζ) are chosen to deliver the same steady-state velocity and elasticity of real money balances with respect to the nominal interest rate estimated by Del Negro and Sims (2015) using US data.[18]

The most important parameter for generating net wealth effects is $1 - \gamma$, the probability that a household transitions to a state in which it has no assets. Following Del Negro et al. (2015a), the probability of an asset reset is set at 3% per quarter so that $\gamma = 0.97$. Del Negro et al. (2015a) reach this calibration for γ by studying the frequency of events that lead households to transition to a state of default or in which there are other constraints on using assets to finance spending.

There is ample empirical evidence that households discount the future even more heavily than implied by the baseline calibration. Experimental evidence generates a wide range of estimates. The averages of the lower and upper bound estimates surveyed by Frederick, Loewenstein, and O'Donoghue (2002) suggests values for γ of 0.86 and 0.95 respectively. The posterior mean estimated by Castelnuovo and Nisticò (2010) using macroeconomic data implies $\gamma = 0.87$.

2.2.9 Simulation approach

The simulations of the model considered in later sections are consistent with the perfect foresight assumption under which the model is derived. The perfect foresight assumption captures non-linear effects (in particular of money demand when the return on money approaches that on bonds) without requiring the use of projection methods to solve the model.

In the simulations, it is assumed that in period $t = 0$ the economy is at its deterministic steady state. At the beginning of period $t = 1$ information about the exogenous disturbances and the behavior of policy is revealed. In particular,

[18] The value for ζ is one quarter of the value reported by Del Negro and Sims (2015) because their estimation uses annualized interest rates.

announced temporary policies (such as an expansion of the monetary base) are regarded as fully credible by private agents. Perfect foresight implies that equilibrium outcomes are consistent with the information revealed at the beginning of period $t = 1$. The TROLL modeling software is used to compute the equilibrium outcomes.

2.3 Pitfalls of money-financed government spending

This section explores experiments in which monetary policy is configured so that changes in government spending are financed by money creation.

2.3.1 Stimulus from money-financed government spending

The first experiment considers the effect of financing a (temporary, exogenous) government spending increase by money creation (rather than debt issuance) in a similar manner to Galí (2014a). Government spending is determined by (2.9) and we set $\varepsilon_1^g = 0.05$ and $\varepsilon_t^g = 0, t = 2, \ldots$. The persistence parameter is set to $\rho_g = 0.9$ which corresponds to the "high persistence" calibration used by Galí (2014a). This calibration is chosen because it is associated with large effects of money-financed government spending increases in Galí's model.

The experiments in this section adopt the conventional assumption that money earns no interest (so that $R_t^M = 1, \forall t$). The macroeconomic effects of a government spending increase are considered for two alternative assumptions about the conduct of monetary and fiscal policy.

In the first case ('debt financing'), higher government spending is financed by issuing short-term debt. A fiscal rule adjusts the lump sum tax to ensure that real government debt returns to b^* in the long run. However, this adjustment is not immediate so that, in the short run, government debt rises as the government borrows to finance the additional spending. A simple fiscal rule for the lump sum tax acts to stabilize the value of government liabilities in the long run:

$$\tau_t = \tau^* + \theta_b \left(b_{t-1} - b^*\right) \tag{2.15}$$

where $\theta_b > 0$ determines the strength of the fiscal feedback and τ^* is the steady-state value of the tax. However, this rule is only activated $K + 1$ periods ($K \geq 0$) after the initial increase in government expenditure, so that $\tau_t = \tau^*, t = 1, \ldots, K$. For the simulations, it is assumed that $K = 12$, which mimics Galí's analysis with fixed tax rates.[19] The short-term interest rate, R_t, is determined by the Taylor rule, (2.7).

The second case ('money financing') assumes that government debt is held constant: $b_t = b^*, \forall t$. In this case it is assumes that lump sum taxes are held constant indefinitely: $\tau_t = \tau^*, \forall t$. For the government's budget constraint to hold, money balances must be adjusted according to:

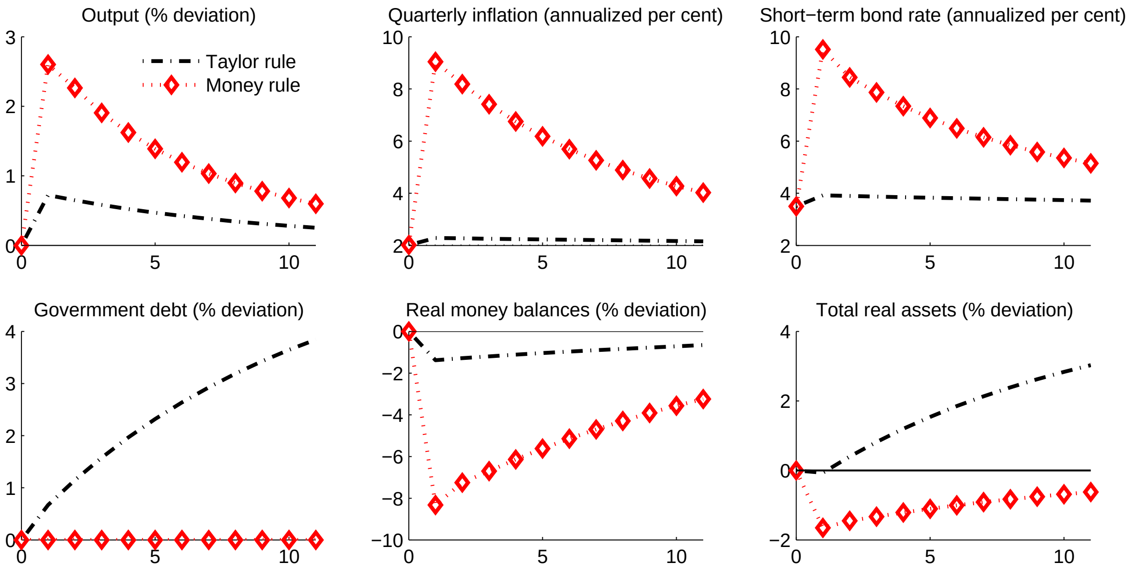
$$m_t = R_t^M \pi_t^{-1} m_{t-1} + \left(R_t \pi_t^{-1} - 1\right) b^* + g_t - \tau^* \tag{2.16}$$

Because both taxes and government debt are held fixed in this case, real money balances must be adjusted to satisfy the government budget constraint. For households to willingly hold the required level of real money balances, the interest rate on short-term government debt must adjust. So the money creation rule (2.16) is used *in place of* the Taylor rule (2.7). Thus Galí's experiment amounts to considering a government spending shock under alternative policy rules that determine the short-term bond rate, R_t, as well as those that determine debt and taxes.

Figure 2.1 shows the results. An expansionary government spending shock generates much more stimulus under 'money financing' than under 'debt financing', replicating Galí's results. However, Figure 2.1 also reveals that the additional stimulus generated by money financing is not associated with stronger wealth effects (via higher real asset values). Indeed, real assets fall in this case (bottom right panel). Instead, money financing is associated with a lower path for real interest rates which stimulates spending through the Euler equation for consumption.

[19] Specifically, the fiscal rule is assumed to be inactive for K periods (during which time $\tau_t = \tau^*$) and from period $K + 1$ taxes are set according to (2.15) with $\theta_b = 0.1$. Galí (2014a) abstracts from changes in tax rates. Since he uses a model with $\gamma = 1$, it is legitimate to assume that taxes are held fixed for an arbitrary period, as long as there is an eventual adjustment in taxes to ensure that the government's solvency condition is satisfied. The nature of Galí's model means that equilibrium allocations are invariant to the pattern of taxes as long as government liabilities are eventually stabilized. The model used for the experiments in this section features net wealth effects, so in principle the horizon K over which taxes are held fixed will matter. Quantitatively, however, results are almost identical to those shown here for choices of $K > 12$. Appendix 2.A.1 shows results for the case in which $\gamma = 1$, which replicates Galí's set up.

Figure 2.1: A government spending increase under debt financing and money financing



Notes: The model starts in period 0 at the steady state. Government spending is given by (2.9). The panels show the responses to $\varepsilon_1^g = 0.05$ (with $\varepsilon_t^g = 0, t \geq 2$). In the Taylor rule case, the tax rate fiscal rule is initially inactive so that $\tau_t = \tau^*$ for $t = 1, \ldots, K$, where $K = 12$. From period $t = K + 1$ taxes are set according to (2.15) with $\theta_b = 0.1$. The short-term interest rate is set according to (2.7). In the money rule case, the tax rate and debt stock are held constant and the government spending increase is financed by money creation according to equation (2.16). There is no interest paid on money ($R_t^M = 1, \forall t$) so the short-term bond rate R_t adjusts to ensure that (2.16) holds.

To see this, recall that consumption satisfies:

$$\widetilde{c}_t = \frac{\pi_{t+1}}{\beta R_{t+1}} \frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}} \left[\widetilde{c}_{t+1} + (1-\gamma)\,\gamma^{-1}\mu_{t+1}\pi_{t+1}^{-1}a_{t+1}\right]$$

where $a_{t+1} \equiv R_{t+1}^M m_t + R_{t+1} b_t$ represents real financial assets.

This means that current consumption can be increased by reductions in the real interest rate $\frac{R_{t+1}}{\pi_{t+1}}$ and increases in the real value of assets a_{t+1}. As noted above, real asset values fall in response to the government spending increase under money financing. However, the real interest rate also falls: inflation rises materially, but the short-term bond rate increases by (slightly) less.[20]

[20] In Figure 2.1 inflation rises by around 7 percentage points and the short-term bond rate by around 6 percentage points (both measured as annualized percentage point deviations from steady state).

To confirm this intuition Appendix 2.A.1 repeats this experiment for a version of the model with $\gamma = 1$, so there are no net wealth effects. The results in Figure 2.9 are virtually indistinguishable from those in Figure 2.1. This underscores the fact that the key mechanism at work is the real interest rate channel, since the consumption equation collapses to $\widetilde{c}_t = \frac{\pi_{t+1}}{\beta R_{t+1}} \frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}} \widetilde{c}_{t+1}$ when $\gamma = 1$.

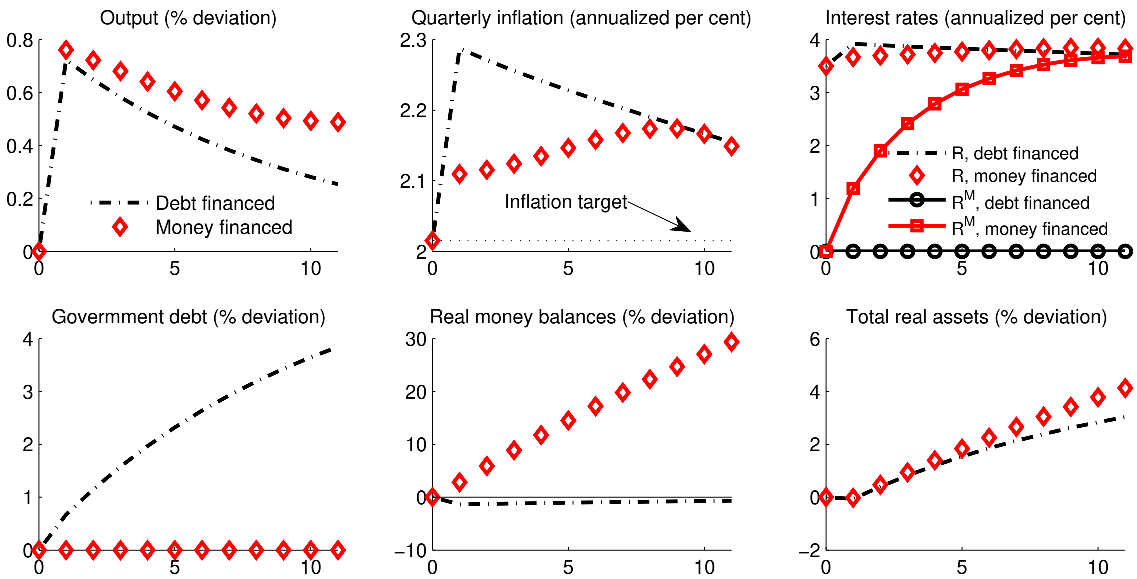
In a recent paper, English et al. (2017) also replicate Galí's result and note the importance of the extent to which monetary policy accommodates the inflationary impetus of the government spending increase. Importantly, they demonstrate that the money-financing rule (a variant of (2.16)) can be represented as a rule for the short-term bond rate (R) that responds to the deviation of the price level from a target path. The target path is determined by the level of government spending such that small increases in spending generate a large rise in the target path. As a result, monetary policy accommodates a temporary but substantial rise in inflation so that the price level moves up to the new target path.

The analysis of English et al. (2017) uses the well-known result that in standard models (with no interest on money) there is an equivalence between a policy rule written in terms of the money stock (such as (2.16)) and a policy rule written in terms of the short-term bond rate, R. Indeed, Eggertsson and Woodford (2003, p147) use this result to develop "an irrelevance proposition for open-market operations in a variety of types of assets that the central bank might acquire, under the assumption that the open-market operations do not change the expected future conduct of monetary or fiscal policy". Their result implies that any macroeconomic effects that can be generated by a particular type of policy rule specified in terms of the money stock can also be achieved by an appropriately specified interest rate rule. From this perspective, Galí's policy prescription can be viewed as advocacy of a particular rule for the *short-term bond rate*, the form of which is uncovered and analyzed by English et al. (2017).

2.3.2 Financing government spending with interest-bearing money

To demonstrate the importance of policy behavior in determining the effects of money-financed government spending, this subsection examines a case in which the short-term bond rate continues to be determined by the Taylor rule (2.7). To implement this variant, real money balances must therefore satisfy (2.16) while the short-term bond rate satisfies (2.7). To achieve this, the interest rate on money, R^M, is adjusted to ensure that the additional money created to finance the additional government spending increase is willingly held.

Figure 2.2: Government spending increase financed via interest-bearing money vs debt



Notes: The model starts in period 0 at the steady state. Government spending is given by (2.9). The panels show the responses to $\varepsilon_1^g = 0.05$ (with $\varepsilon_t^g = 0, t = 2, \dots$). In the debt financed case, the tax rate fiscal rule is initially inactive so that $\tau_t = \tau^*$ for $t = 1, \dots, K$, where $K = 12$. From period $t = K + 1$ taxes are set according to (2.15) with $\theta_b = 0.1$. In the money financed case, the tax rate and debt stock are held constant and the government spending increase is financed by money creation according to equation (2.16). The return on money is adjusted to ensure that households willingly hold the additional money balances.

Figure 2.2 demonstrates that, when the return on money (R^M) adjusts, the effects of the government spending increase are much more similar, regardless of whether it is financed through debt or money creation. The bottom row reveals that the two financing arrangements have markedly different implications for the paths of government debt and real money balances. However, the path for total real assets (a_t), which is the relevant determinant of consumption expenditure in the model, is very similar for the two financing approaches.

A key reason for the (relatively small) difference between the responses for debt-financed and money-financed government spending increases in Figure 2.2 is that the consumption Euler equation is 'tilted' by the change in the transactions costs associated with holding money.[21] Appendix 2.A.2 demonstrates that in a variant of the model in which the demand for money is additively separable, the responses from money and debt financed government spending are almost identical.

These results show that the effects of a money-financed government spending increase depend on the precise monetary policy arrangements in place. If the short-term bond rate R is set using a rule with a strong response to inflation (as in (2.7)), then the inflationary effects of the government spending increase are contained (as in Figure 2.2). This result is only achievable if the rate of return on money is allowed to rise, so that the additional money created to finance the spending increase is willingly held. If no interest is payable on money, then the Taylor rule (2.7) must be abandoned to allow the short-term bond rate R to adjust to deliver the monetary financing rule (2.16). In that case, the implicit interest rate feedback rule has a weak response to inflation (as in Figure 2.1).

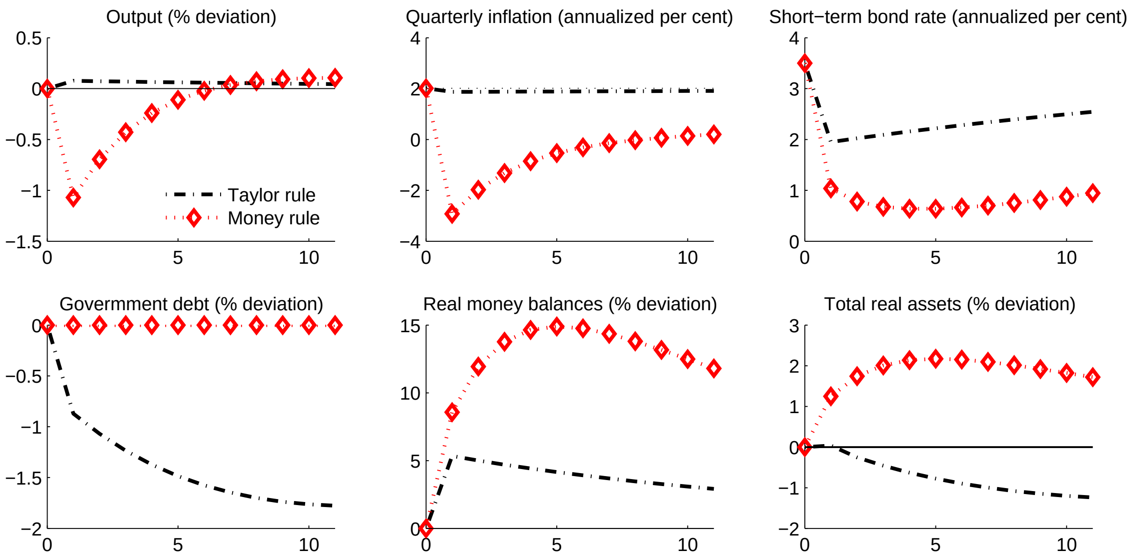
2.3.3 Broader implications of a weak policy response to inflation

The preceding results indicate that the ability of money-financed government spending increases to stimulate demand and inflation stems from the fact that such policies imply a weak response of the short-term bond rate to inflationary developments. Indeed, part of the rationale for such policies is that such a response may be beneficial in some circumstances: if the economy is stuck at the effective lower bound, then pursuing a policy that increases inflation expectations (and ultimately inflation) may be very desirable. However, replacing the Taylor rule (2.7) with the money financing

[21] Since $\widetilde{\vartheta}$ depends on velocity, v.

rule (2.16) will affect the equilibrium responses to *all* shocks, not just those that provide stimulus such as an increase in government spending.

Figure 2.3: Responses to a preference shock under alternative monetary policy rules



Notes: The panels show responses to an unanticipated preference shock ε^{ϑ} in period 1. In the 'Taylor rule' case, taxes are set according to (2.15) with $\theta_b = 0.1$. The short-term bond rate is set according to (2.7). In the 'money rule' case, the tax rate and debt stock are held constant and the government budget constraint is enforced by money creation according to equation (2.16). There is no interest paid on money ($R_t^M = 1, \forall t$) so the short-term bond rate R_t adjusts to ensure that (2.16) holds.

For example, Figure 2.3 compares the responses to a preference shock, ε^{ϑ}, when the short-term rate R is determined by the money rule (2.16) (red lines with diamond markers) and the Taylor rule (2.7) (black dash-dot lines).[22] Under the Taylor rule, the short-term policy rate is cut sharply to offset the variation in the natural real interest rate generated by the preference shock. Inflation stays close to target and output is barely changed.[23] In contrast, under the money rule, the short-term bond rate rate is determined by (2.16), which ensures that the debt stock is held fixed by adjusting the rate of money growth. In equilibrium, this generates a a rise in the real interest rate, leading to a recession and prolonged undershoot of inflation.

[22] In both cases, the model starts in steady state in period 0. The preference shock is determined by equation (2.3), with $\rho_{\vartheta} = 0.85$. In period 1, the preference shock is $\varepsilon_1^{\vartheta} = 0.005$ (with $\varepsilon_t^{\vartheta} = 0, t = 2, \ldots$).

[23] Full stabilization is not achieved because the measure of flexible price output in the policy rule is computed conditional on the assumption that the short-term bond rate is fixed.

This example illustrates the far-reaching implications of *permanently* replacing a standard Taylor-type monetary policy rule with a monetary financing rule.

2.4 Money-financed transfers at the effective lower bound

While many recent policy proposals have focused on money-financed government spending increases, similar to those investigated in the previous section, Friedman's original thought experiment was cast as a direct monetary transfer to households. Whether or not such a transfer stimulates spending depends on households' reaction to an increase in nominal income and the extent to which the transfer is permanent. The original 'helicopter drop' experiment assumed a one-off, permanent, increase in the supply of money. This section explores the consequences of experiments in which the money stock is increased with varying degrees of permanence.

Importantly, we assume that the rate of return on money is adjusted to ensure that the monetary injection is willingly held by households and that the short-term bond rate is determined by the Taylor rule (2.7). The objective is to explore whether it is possible to achieve a stimulative effect from expanding the stock of money without requiring a *permanently* weak response of the short-term bond rate to inflation. As demonstrated in section 2.3.3, such a weak response to inflation may be beneficial in some circumstances, but not others.

2.4.1 A recessionary scenario

A baseline simulation in which the short-term bond rate hits the zero bound is used to explore the potential for money-financed transfers to stimulate the economy. As in previous sections a shock arrives in period $t = 1$, with the model at steady state in period $t = 0$. Specifically, a large preference shock (ϑ) generates a recession large enough to constrain the monetary policy rule (2.7) at the effective lower bound $\underline{R}$. The shock sequence is $\varepsilon_1^{\vartheta} = 0.0115$ and $\varepsilon_t^{\vartheta} = 0, t = 2, \ldots$. The preference disturbance evolves according to equation (2.3), with ρ_{ϑ} set to 0.95 to generate a persistent spell at the effective lower bound.

The effective lower bound on the nominal interest rate is set at 0.25%.[24] This is broadly consistent with the experience of some advanced economies (e.g., the United Kingdom and United States) but not for those economies that have implemented negative policy rates (including the Euro area and Japan). The effective lower bound must be strictly positive, otherwise the demand for money (in the baseline simulation) would be infinite. From a practical perspective, the fact that large economies have successfully implemented negative policy interest rates without explosive increases in the demand for money suggests that the true rate of return on money is likely to be slightly negative (for example, reflecting storage and security costs that do not appear in the model).[25]

In the baseline recessionary scenario, the return on money is fixed at $R_t^M = 1, \forall t$ and that the short-term bond rate R evolves according to the monetary policy rule (2.7). The subsequent experiments explore the extent to which money-financed transfers could stimulate spending and inflation.

2.4.2 Money-financed transfers with interest-bearing money

This section considers the effects of a temporary money-financed transfer to households announced in period $t = 1$. The temporary transfer is determined by the following process for the aggregate nominal money stock:

$$\frac{M_t}{M_{t-1}} = \left(\frac{M_{t-1}}{M_{t-2}}\right)^{\rho_m} (\pi^*)^{(1-\rho_m)} \exp\left(\varepsilon_t^m\right) \tag{2.17}$$

for periods $t = 1, \ldots, K$. For the duration of the monetary expansion, money growth follows an autoregressive process, which is assumed to be weakly persistent by setting $\rho_m = 0.33$. The process is driven by a disturbance term ε^m, with $\varepsilon_1^m = 1.01$ and $\varepsilon_t^m = 0$ for $t = 2, \ldots, K$. The value of ε_1^m is chosen to deliver an increase in the money stock similar to that observed following the global financial crisis. The calibration is discussed further below. A two-year monetary expansion ($K = 8$) is considered:

[24] For the annualized *net* interest rate.

[25] Alternatively, there may be a zero bound on deposit rates (again abstracted from in our model) that does not apply to the overnight policy rate.

for periods $t = 1, \dots, K$, the policymaker adjusts the rate of return on money (R_t^M) to ensure that the additional money stock is willingly held by households.

The assumption of a temporary monetary expansion is intended to facilitate comparisons with variants of the model that contain alternative monetary frictions (analyzed in Section 2.5). Moreover, a temporary monetary expansion during a period in which the short-term bond rate is constrained by the effective lower bound may be sufficient to stimulate spending and inflation by ensuring that the policy response to higher inflation expectations is *temporarily* weaker than usual. A more permanent monetary expansion is analyzed in Section 2.4.3.

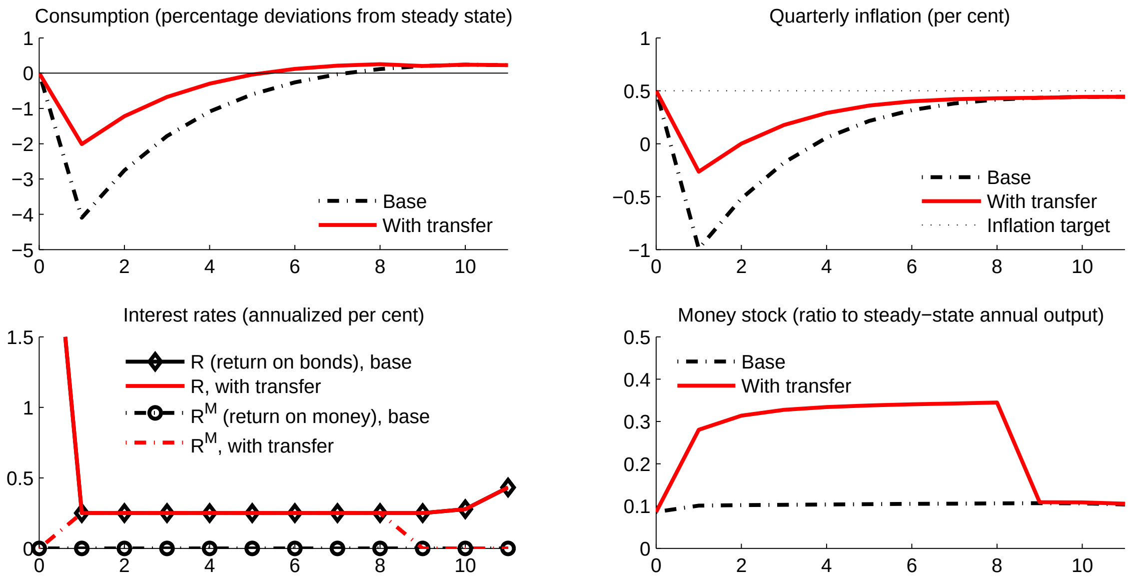
The size of the monetary injection is calibrated with reference to the quantitative easing experiences of the United Kingdom and United States following the financial crisis. While this experiment does *not* analyze the effects of quantitative easing, calibrating the transfer with reference to the scale of money expansion associated with those policies is intended to deliver a policy intervention that is 'large', but not unprecedented in recent economic history.

Reis (2016) documents the evolution of the balance sheets of major central banks between 2007 and 2015. If 'money' in the model is interpreted as a composite of currency and interest bearing reserves, then Reis (2016, Figure 1) suggests that the stock of money increased by around 25 percentage points of (annual) GDP between 2007 and 2015 in both the United States and United Kingdom. In the simulation, the monetary injection is measured relative to *steady-state* GDP and calibrated to be approximately 25pp.[26]

Figure 2.4 shows the results. The black (dash-dot) lines show the baseline simulation, without a monetary transfer. The short-term bond rate is immediately cut from its steady-state level of 3.5% to the lower bound of 0.25% and remains there for nine quarters. Given the scale of the shock, however, this policy response is insufficient to stabilize spending and inflation. Consumption falls by 4% in period 1 and *quarterly* inflation undershoots the target by 1.5 percentage points.

[26] Reis reports stocks of assets and liabilities as a proportion of actual GDP. The change in GDP over the period in question is small relative to the observed changes in reserves and currency, so little is lost by calibrating the monetary injection with reference to steady-state output.

Figure 2.4: A money-financed transfer at the effective lower bound



Notes: The model starts in period 0 at the steady state and a recessionary shock $\varepsilon_1^{\vartheta} = 0.0115$ arrives in period 1. The black dot-dashed lines show the effects of the recessionary shock. The solid red lines show the case in which a temporary money-financed transfer is used to combat the effects of the recessionary shock. For periods $t = 1, \ldots, 8$, the money stock is determined by (2.17). The value of ε_1^m is chosen to deliver the desired total increase in the money stock and $\varepsilon_t^m = 0$ for $t = 2, \ldots, 8$. For the duration of the money-financed transfer, the rate of return on money R^M is endogenously determined. From period 9 onwards, the rate of return on money is fixed at unity and the quantity of money is determined by households' demand for money.

The solid red lines in Figure 2.4 show the effect of the shock when a money-financed transfer is also announced at the start of period 1. The transfer ends in period 8, so that from period 9 onwards no interest is paid on money and the level of real money balances is determined by household demand. This would be akin to central banks that did not pay interest on reserves prior to the crisis temporarily paying interest on reserves, before reverting to the pre-crisis policy as the economy recovers.

The expansion in the money stock requires an increase in the return on money (bottom left panel, dashed red line) so that households are willing to hold the additional money balances. The monetary transfer increases consumption by around 2 percentage points in period 1. Quarterly inflation is around 0.8 percentage points higher in period 1. The monetary transfer stimulates the economy via a direct wealth effect as the real money balances held by households increase. Because there is no conventional monetary policy response to the stimulus (the short-term bond rate does not change relative to the baseline simulation), inflation expectations increase and the expected real interest rate falls. This provides a further boost to consumption.

As discussed in Section 2.2.5, an expansion of M at the lower bound leads to a reduction in taxes $P\tau$, given the assumed behavior of government spending and debt. Because the expansion in M is temporary, there is a sharp fall in M when the transfer ends, corresponding to a *rise* in $P\tau$. In the absence of the asset reset mechanism, households would save the additional income from the temporary transfer, to pay the subsequent increase in taxes. However, in the presence of the asset reset mechanism there is a risk that households will experience an asset reset before the tax rise occurs. In that case, they will experience the increase in taxes without having additional assets from which to finance it. This creates an incentive to spend some of the additional income generated by the money-financed transfer.

These results show that when the short-term bond rate is temporarily constrained at the lower bound, it does not rise to counteract the stimulative effects of the money-financed transfer. As in the experiments of Section 2.3.1, a policy intervention that increases inflation has more effect when the short-term bond rate responds weakly. However, in this case the lack of a short-term bond rate response is a *temporary* consequence of the lower bound constraint rather than a permanent change in the monetary policy rule. As discussed above, one of the main reasons for advocating money-financed policies is to provide stimulus when other monetary policy instruments are constrained.

2.4.3 A 'permanent' money-financed transfer

It is generally argued that monetary transfers that are permanent (as in Friedman's original thought experiment) are more effective than temporary transfers. Indeed, some authors argue that achieving *any* stimulus via monetary transfers at the zero bound requires those transfers to be permanent (Krugman, 1998; Eggertsson and Woodford, 2003).

The monetary transfer in Section 2.4.2 lasts for just eight quarters. After the policy intervention, the interest rate on money returns to zero and the monetary policymaker supplies whatever quantity of money is demanded by households at this rate. As described in Section 2.4.2, there is a *withdrawal* of money from households (via higher taxes) when the policy ends.

The fact that a temporary transfer stimulates spending reflects the fact that it operates via a wealth channel and can be implemented without changing the path of the short-term bond rate (because the interest rate on money is adjusted appropriately).[27] Nevertheless, it is instructive to consider whether a more permanent money-financed transfer would be more powerful.

In this section, the experiment is one in which the policymaker never withdraws the stock of money initially transferred to households. To implement this experiment the nominal money stock is determined by:

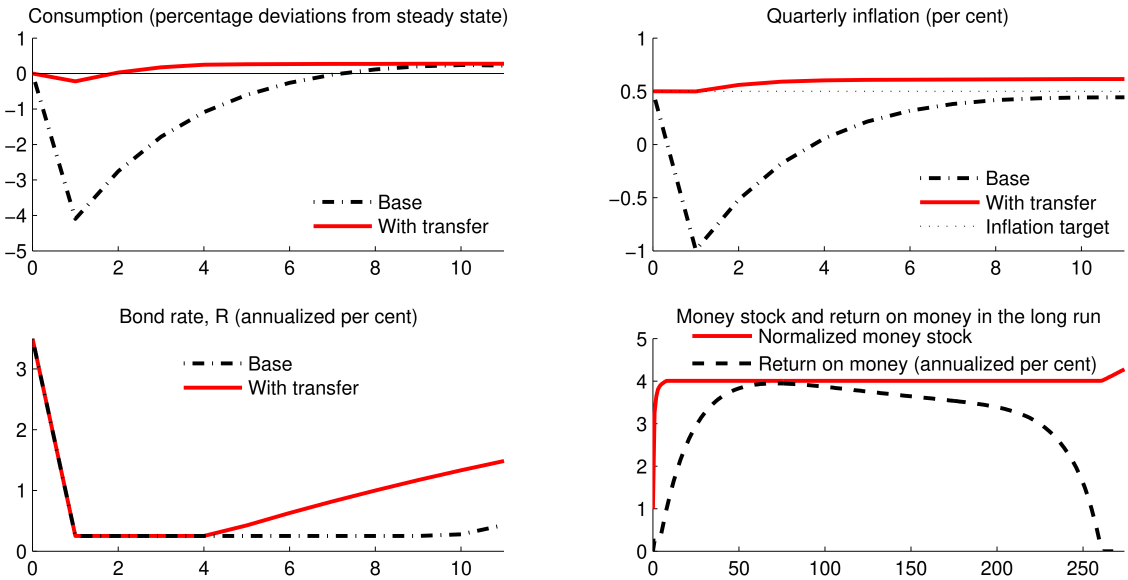
$$M_t = \begin{cases} M_{t-1}\left(\frac{M_{t-1}}{M_{t-2}}\right)^{\rho_m}(\pi^*)^{(1-\rho_m)}\exp\left(\varepsilon_t^m\right) & t = 1,\ldots,K \\ \max\left\{M_{t-1}\ ,\ \zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+1}-1}{R_{t+1}}\right]P_t c_t\right\} & t = K+1,\ldots \end{cases} \tag{2.18}$$

Again, $K = 8$ and the same value of ε_1^m as in Section 2.4.2 is used. These assumptions mean that the monetary expansion follows the same path as the experiment in Section 2.4.2 for $t = 1,\ldots,K$. Equation (2.18) specifies that the money stock is held constant for periods $t = K+1,\ldots$ unless that value exceeds $\zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+1}-1}{R_{t+1}}\right]P_t c_t$. Inspection of (2.14) reveals that this quantity corresponds to the demand for *nominal* money when the net interest rate on money is zero ($R_{t+1}^M = 1$). So this specification requires that the initial increase in the money stock is maintained as long as the rate of return required for the money stock to be willingly held is non-negative.

Figure 2.5 shows the results of this experiment. Comparing the top row with the top row of Figure 2.4 reveals that the permanent money-financed transfer has a

[27]The results of Krugman (1998) and Eggertsson and Woodford (2003) rely on the duality between the operation of a policy in terms of a path for the short-term bond rate and the quantity of money. A policy that delivers a permanently higher stock of money must also deliver a permanently higher price level. In forward-looking models that is achieved by a temporarily higher inflation rate generated by a path for the short-term bond rate that responds weakly (if at all) to higher inflation.

Figure 2.5: A permanent money-financed transfer at the effective lower bound



Notes: The model starts in period 0 at the steady state and a recessionary shock $\varepsilon_1^{\vartheta} = 0.0115$ arrives in period 1. The black dot-dashed lines show the effects of the recessionary shock. The solid red lines show the case in which a permanent money-financed transfer is used to combat the effects of the recessionary shock. In this case the money stock is determined by (2.18). Responses in the bottom right panel are plotted for 275 quarters. The nominal money stock is normalized to 1 in period 0.

much larger macroeconomic effect than a temporary transfer. Other things equal, the permanent transfer has a larger net wealth effect and hence a more expansionary effect on aggregate demand and inflation. Indeed, the permanent transfer is sufficiently stimulative that it brings forward the date at which the short-term bond rate lifts off from the zero bound.

The bottom right panel of Figure 2.5 (showing the *long-run* implications of the transfer) shows that the permanent monetary transfer requires a strictly positive rate of return on money for more than 65 years. The ‘hump shaped’ response on the return on money reflects the strength of the wealth effects in the near term. This drives up the short-term bond rate (which rises to offset the wealth effects on consumption and hence output) and hence the return on money required to induce households to hold the additional money balances. In the longer term, the price level rises in line with the inflation target and the return on money required for households to hold a given nominal stock of money falls.

Much of the power of the permanent money-financed transfer is in fact driven by the nature of the reaction function for the short-term bond rate, R. As described in Section 2.2.3, the reaction function responds to a measure of the output gap based on a 'supply side' concept of flexible price equilibrium. As a result, the policy rule (2.7) does not fully stabilize spending and inflation away from the lower bound. Indeed, the money-financed transfer is sufficient in this case to generate a small, but persistent, overshooting of the inflation target. Section 2.5.2 considers a variant of the model in which the reaction function for the short-term bond rate *does* completely stabilize spending and inflation away from the lower bound. In that case there is a limit to the extent to which a permanent money-financed transfer can provide additional stimulus.

Finally, it is worth noting that a bond-financed transfer of a similar magnitude could generate similar stimulative effects on consumption and inflation at the zero bound. The primary friction through which the stimulus operates is the asset reset friction, which applies to all government liabilities.

2.5 Robustness to alternative monetary frictions

This section examines the robustness of the results to alternative assumptions about the underlying frictions that give rise to the demand for money.

2.5.1 A cash in advance friction

Appendix 2.C develops a variant of the model in which the demand for money arises from a cash in advance constraint. This approach greatly reduces the sensitivity of money demand to the interest differential between money and bonds. On the other hand, a simple cash in advance assumption typically implies much larger average money holdings than observed in the data.[28]

The cash in advance variant assumes that consumption spending must be financed by existing money holdings brought forward from the previous period, income

[28] One modification to address this is to assume that only a subset of consumption goods are subject to the cash in advance constraint with the remainder being 'credit goods'.

from maturing bonds (net of new bond purchases) and a transfer from the government. Money earns no interest.

The cash in advance constraint will bind if the rate of return on bonds is strictly positive: households will hold only the money balances required to finance consumption, allocating the remainder of their portfolio to bonds. However, when the return on bonds is zero, the cash in advance constraint will not bind and households are indifferent between allocating their portfolios between money and bonds. This means that an expansion in the stock of money beyond the level required to finance consumption expenditures is willingly held.

These assumptions give rise to a model with almost identical behavioral equations to the baseline variant. The key differences are in the consumption equation and the Phillips curve, which Appendix 2.C shows to be:

$$c_t = (1-\mu_t)^{-1}\left[\frac{\gamma\pi_{t+1}}{R_{t+1}}\frac{\mu_t}{\mu_{t+1}}c_{t+1} + (1-\gamma)\,R_{t+1}^{-1}\mu_t\left(R_{t+1}b^* + m_t\right)\right]$$

$$\frac{\Phi\pi_t}{y_t\pi^*}\left(\frac{\pi_t}{\pi^*}-1\right) = 1-\eta+\eta\chi y_t^{\psi}c_tR_{t+1} + \frac{\Phi}{y_t}\frac{\pi_{t+1}}{R_{t+1}}\frac{\pi_{t+1}}{\pi^*}\left(\frac{\pi_{t+1}}{\pi^*}-1\right)$$

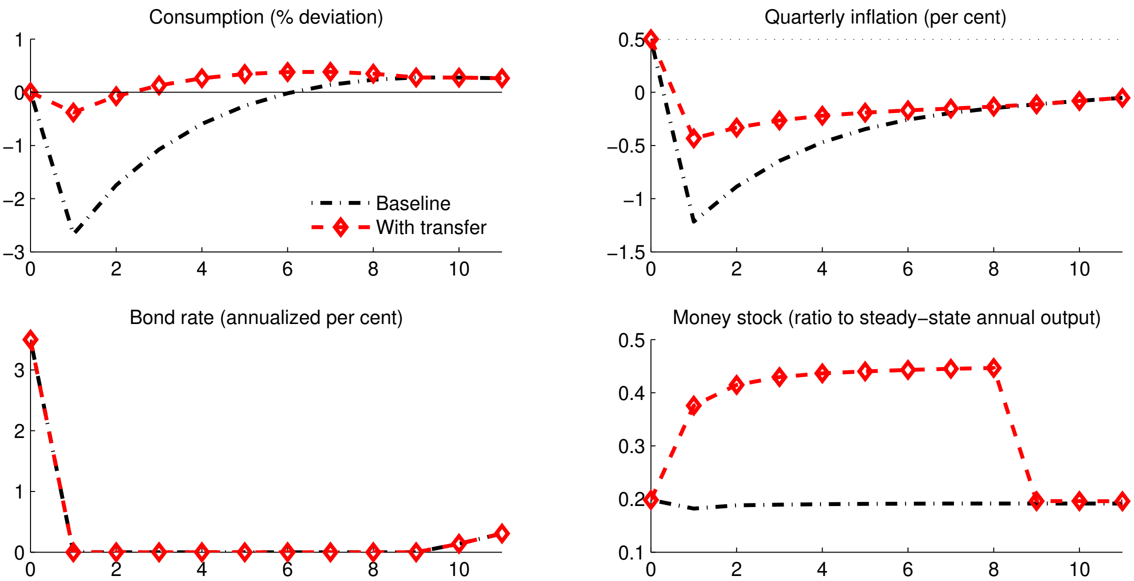
The consumption equation, while very similar to the baseline model, contains a slightly different term that gives rise to the wealth effect from government liabilities.[29] However, it is also clear that the wealth effects disappear when $\gamma = 1$, as in the baseline version of the model. The Phillips curve equation has the feature that marginal cost is an increasing function of the interest rate on short-term bonds: there is a 'cost channel' (Barth and Ramey, 2002). This arises because the marginal rate of substitution between consumption and leisure depends on the Lagrange multiplier on the cash in advance constraint and (hence) the nominal interest rate.[30]

[29]This difference implies that a slightly different calibration for β is required to deliver the same steady state return on bonds as the baseline model. All other parameter values in the simulations below are unchanged from the baseline model.

[30]The baseline model includes a similar effect since the labor supply relationship depends on velocity, v.

Figure 2.6 shows the results of an experiment similar to that considered in Figure 2.4 using the cash in advance variant. Specifically, the model begins in period $t = 0$ in steady state. In period $t = 1$, a shock to ϑ generates a fall in demand, prompting a cut in the interest rate on short-term bonds via the Taylor rule. The shock is sufficiently large that the short-term interest rate is constrained by the effective lower bound. In this variant of the model, the lower bound on the net short-term bond rate is set to zero ($\underline{R} = 1$).

Figure 2.6: A money-financed transfer at the effective lower bound: cash in advance model



Notes: The model starts in period 0 at the steady state and a recessionary shock $\varepsilon_1^{\vartheta} = 0.005$ arrives in period 1, which drives the economy to the zero lower bound in the baseline simulation (black dash-dot lines) for 9 quarters. In the baseline simulation, while the economy is at the zero bound the money stock is adjusted to the level at which the cash in advance constraint *would* (just) bind if operative. In the case of a money-financed transfer (red dashed lines with diamond markers), for periods $t = 1, \ldots, 8$ the path of the money stock delivers a 25pp increase in nominal money balances relative to steady-state annual GDP. From period 9 onwards, the money stock is determined by the cash in advance constraint, which binds.

When the lower bound is binding, the cash in advance constraint will be slack and households will be willing to hold money balances in excess of the minimum quantity required for consumption purposes. The baseline assumption in Figure 2.6 is that the government monetary transfer is adjusted to ensure that households hold the minimum quantity of money required to financed consumption expenditure.

This assumption is intended to be a neutral benchmark against which to assess alternative policy options.

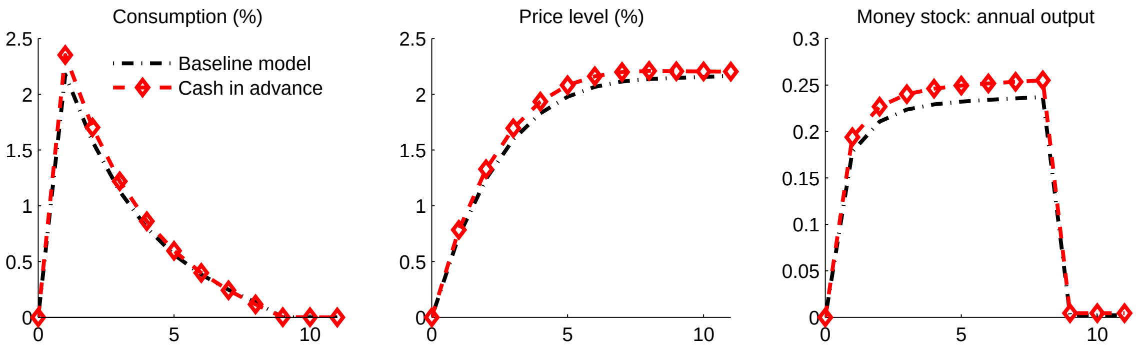
The baseline recessionary scenario is plotted as black dash-dotted lines in Figure 2.6. As in the experiment in Section 2.4.2, the shock generates a substantial fall in consumption and inflation and the interest rate on short-term bonds remains at the zero bound for ten quarters before rising very gradually. The recession is associated with a small reduction in the money stock. That reflects the fact that the level of money balances required to satisfy the cash in advance constraint responds more strongly to the level of economic activity than to the nominal return on short-term bonds.

The dashed red lines with diamond markers in Figure 2.6 depict the effect of a temporary monetary-financed transfer to households. This monetary expansion is calibrated to deliver the same increase in nominal money balances as the experiment in Section 2.4.2, which implies a smaller proportionate increase in money balances given the larger steady state money stock in this variant of the model. The results indicate that large monetary expansions can be effective at the zero lower bound even without interest-bearing money, though in this case the duration of the monetary transfer is limited by the length of the liquidity trap.

While the effects appear to be somewhat larger than the baseline specification of the model (Figure 2.4), this mainly reflects the differences in behavior in response to the underlying recessionary shock. In particular, there are three key differences between the two model variants. First, the baseline model exhibits a small increase in money demand as the short-term bond rate is reduced in response to the recessionary shock. Other things equal, higher real money balances support consumption spending via a wealth effect. This means that a somewhat larger shock is required to drive the baseline model to the zero bound. Second, the cash in advance variant implies that steady-state money balances are larger than in the baseline model, so that a given proportionate change in money generates a larger wealth effect on consumption. Third, the cash in advance variant features a cost channel which suppresses inflation when nominal interest rates are low.

To aid the comparison between the two variants, Figure 2.7 plots the marginal effects of the policy experiments simulated in Figures 2.4 and 2.6, normalizing each variable in a way that facilitates the comparison of the effects. Figure 2.7 shows that the stimulatory effects of a money-financed transfer are slightly larger in the cash-in-advance variant of the model, but the pattern and order of magnitude of the effects are very similar.

Figure 2.7: Marginal effects of money-financed transfers in different model variants



Notes: Each panel plots the marginal effect of the experiments shown in Figures 2.4 and 2.6. The dot-dashed black lines depict the responses from Figure 2.4, using the baseline model. The red dashed lines with diamond markers show the results from 2.6, using the cash in advance variant. In each case, the marginal effects are computed as the differences between the baseline simulation and the simulation in which a temporary money-financed transfer is undertaken. The right panel shows the marginal effect on the money stock as a fraction of steady-state annual output.

2.5.2 Additively separable money demand

To illustrate the importance of the monetary frictions, we consider a variant of the model with additively separable money demand. This assumption delivers a tractable specification for money demand, consistent with much of the recent literature (notably Buiter (2005, 2014)). Importantly, it implies that flexible price allocations are independent of monetary developments and hence the monetary policy rule can be parameterized to deliver complete stabilization of the output gap and inflation away from the zero bound. This in turn is important in isolating the importance of the policy rule for the short-term bond rate in determining the effects of monetary transfers. Of course, this variant also has some weaknesses. In particular, the elasticity of money demand to changes in the relative returns on money and bonds is implausibly large.

Appendix 2.D sets out a variant of the model in which (log) real money balances enter the utility function in an additively separable manner. The consumption equation, Phillips curve and monetary policy rule in this case are:

$$\begin{aligned} c_t &= (1-\mu_t)^{-1}\frac{\gamma\pi_{t+1}}{R_{t+1}}\left[\frac{\mu_t}{\mu_{t+1}}c_{t+1} + (1-\alpha)\,\mu_t\,(1-\gamma)\,\gamma^{-1}a_{t+1}\right] \\ \frac{\Phi\pi_t}{\pi^* y_t}\left(\frac{\pi_t}{\pi^*}-1\right) &= 1-\eta+\eta\frac{\chi c_t y_t^{\psi}}{1-\alpha} + \Phi\frac{\pi_{t+1}}{R_{t+1}}\frac{\pi_{t+1}}{\pi^* y_t}\left(\frac{\pi_{t+1}}{\pi^*}-1\right) \\ R_{t+1} &= \max\left\{R^f_{t+1}\left(\frac{\pi_t}{\pi^*}\right)^{\theta_\pi}\left(\frac{y_t}{y_t^f}\right)^{\theta_y}, \underline{R}\right\} \end{aligned} \tag{2.19}$$

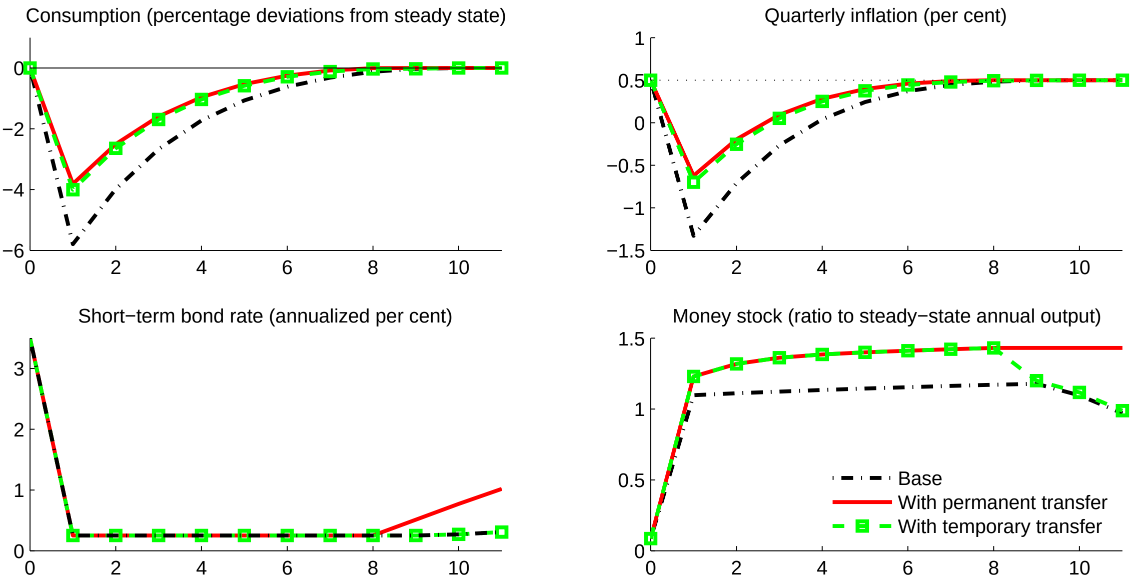
where R^f is the nominal interest rate prevailing under flexible prices (derived in Appendix 2.D) and the marginal propensity to consume satisfies:

$$\mu_t^{-1} = 1 + \gamma\beta\frac{\vartheta_{t+1}}{\vartheta_t}\mu_{t+1}^{-1}$$

The main differences between the baseline model and this variant are that there is no direct impact of money demand (or transactions frictions) in either the Phillips curve or the intertemporal substitution components of the consumption equation.[31] This means that the flexible price allocations can be derived independently of monetary frictions. As a result, it is possible to specify the monetary policy rule so that, away from the lower bound, fluctuations in inflation around target and output around flex-price output are completely stabilized. This is achieved in the rule used in this model variant via its response to the flexible price interest rate R^f_{t+1}.

[31] That is, while assets appear on the right-hand side of the consumption equation, reflecting the net wealth effect, the slope of the consumption equation (in particular the marginal propensity to consume) does not depend on real money balances.

Figure 2.8: Money-financed transfers at the lower bound: additively separable money



Notes: The model starts in period 0 at the steady state and a recessionary shock $\varepsilon_1^{\vartheta} = 0.0185$ arrives in period 1, which drives the economy to the effective lower bound in the baseline simulation (black dash-dot lines) for 9 quarters. Solid red lines show the effect of a permanent monetary transfer and dashed green lines with square markers show the effect of a temporary monetary transfer.

Figure 2.8 demonstrates the importance of the specification of the Taylor rule by comparing the effects of a temporary and permanent monetary transfer at the lower bound. The baseline (black dash-dot lines) is a recessionary scenario that drives the short-term bond rate to the effective lower bound (which is set at 25bp on an annualized basis). Two money-financed transfers are considered. The solid red lines show the case of a permanent transfer, calibrated in the same way as the experiment in Section 2.4.3.[32] The dashed green lines with square markers show an 8 quarter temporary transfer of the same size. In each case, for the duration of the transfer, the interest rate on money is adjusted to ensure that the additional money balances are willingly held.

The results show that the additional stimulus from the permanent transfer is extremely small. The reason is that the monetary policy rule (2.19) delivers complete stabilization of inflation at target when not constrained by the zero bound. The

[32]The transfer is calibrated to deliver the same increase in the nominal money stock (measured as a fraction of steady-state GDP) relative to the baseline response of money balances.

additional stimulus from the permanent money transfer causes the short-term bond rate to lift off from the lower bound earlier. However, inflation is fully stabilized after the liftoff date, limiting the extent to which the real interest rate can be reduced by higher inflation expectations. In contrast, for the baseline model, the results in Figure 2.5 generated a small but extremely persistent inflation overshoot, depressing real interest rates and providing additional stimulus to spending. That result was driven by the fact that the monetary policy rule in the baseline model does not deliver full stabilization of preference shocks away from the lower bound.

The results in Figure 2.8 suggest that there may be a limit to the degree of stimulus that can be provided by a money-financed transfer, when the money stock and short-term bond rate are both used as policy instruments. For an appropriately specified rule for the short-term bond rate, a money-financed transfer may not generate a *sustained* reduction in real interest rates via a prolonged increase in inflation expectations. In the variant of the model with additively separable money demand, a permanent money-financed transfer cannot achieve this when the short-term bond rate is determined by (2.19) because this rule will offset the inflationary effects of the transfer when not constrained by the zero bound.

2.6 Discussion

This section relates the results from preceding sections to the recent debate on the use of monetary-financing to stimulate spending and inflation.

2.6.1 Interest-bearing money and bank deposits

A key assumption underpinning many of the results is that money-financed transfers are implemented by varying the rate of interest paid on money. In this way, policymakers can influence the stock of real money balances in the economy while retaining control over the interest rate on short-term bonds because these balances are willingly held by households at the prevailing interest rates on short-term bonds and money.

While this approach differs from the textbook assumption of non-interest bearing money, it seems appropriate in the context of recent discussions of monetary financing. Such discussions often focus on the effects of the increased supply of money generated by such a policy. However, the effects of increased money holdings on spending depend on the extent to which households are willing to hold the additional money balances. For a given rate of return on short-term bonds, ensuring that households willingly hold a particular level of real money balances requires the interest rate on money to adjust.

If money is interpreted as banknotes and coins, then the concept of a variable, non-zero rate of return on money is unrealistic. However, money in the model is interpreted in the context of the post-crisis practice of remunerating commercial bank reserves at the policy rate and the fact that the vast majority of the money stock is held in the form of bank deposits (see, McLeay, Radia, and Thomas, 2014). Many central banks began paying interest on reserves as policy rates approached a (positive) effective lower bound on rates and quantitative easing policies allowed a marked increase in reserves without forcing short-term market rates below that level. Moreover, any monetary transfer from government to households, even if made in the form of notes, would quickly result in an increase in reserves.[33] However, unlike Ireland (2014), the model does not explicitly include the demand for bank deposits by households independently of the (derived) demand for reserves by the banking system.

A further implication of paying interest on money is that as that rate approaches the short-term bond rate, money becomes virtually indistinguishable from short-term government debt as a means of financing a deficit. However, the quantitative results hinge on a particular friction that generates net wealth effects – the asset-reset mechanism – which applies equally to bonds and interest-bearing money. Features of real-world economies that are likely to make money-financed transfers effective are also likely to make other policies effective, for example a debt-financed tax cut. This implies that the focus on money financing options as a 'special' policy tool may be misguided, though there are other frictions and attributes of central bank money

[33]Households would be expected to deposit their notes in the banking system. Those notes in turn would be exchanged by the banking system for interest-bearing reserves at the central bank.

that could generate a greater efficacy of money financing, as discussed below.[34]

2.6.2 Wealth effects via 'irredeemable' money

Several recent papers have explored the notion that central bank money is special because it is viewed as "irredeemable" by the issuing government (Buiter, 2005, 2014; Buiter and Sibert, 2007). In this approach, households still view interest-bearing money as an asset but, in equilibrium, they do not expect future taxes to be raised to ensure the that the principal of the reserve liability is 'paid off' in present value terms (unlike bonds).[35]

This setup implies an asymmetry in which the present value of the terminal stock of irredeemable reserves will be positive whereas that of redeemable government bonds will be zero. A permanent helicopter drop of money in such an economy could stimulate spending through a type of wealth effect. However for that irredeemability belief to be credible requires a commitment to permanently expand the stock of money at a rate equal to the the nominal interest rate (Buiter, 2014), while expanding the stock of government bonds at a rate below the rate of interest. This setup has several interesting practical implications.

First, a helicopter drop policy of this type would require a shift to a new equilibrium and institutional mode of operation for central banks. Current institutional arrangements imply that central banks and governments do *not* regard money as irredeemable. In the model, the public sector budget constraint is consolidated, so there is no distinction between the balance sheet of the central bank and wider fiscal authority. In practice, however, the payment of interest on reserve liabilities are typically financed by the interest from central bank holdings of government

[34]One reason why monetary financing may be interpreted as 'special' is that, under some circumstances, it operates through mechanisms that could be regarded as unique to monetary policy. For example, monetary financing as modeled by Galí (2014a) implies that government spending is financed using the inflation tax and the resulting high inflation leads to a reduction in real interest rates.

[35]This result holds *in equilibrium* by combining the household sector intertemporal budget constraint (which treats both money and bonds as redeemable) and the government intertemporal budget constraint (in which money is treated as irredeemable).

bonds, which in turn are financed by taxes and government borrowing. This implies that money is regarded as redeemable from the perspective of the government.[36]

Second, even if such a policy could be implemented, it is not clear that moving to an equilibrium in which the nominal money stock grows at the nominal interest rate is achievable. For example, Buiter and Sibert (2007) present a general equilibrium treatment which shows that the existence of irredeemable money can rule out deflationary bubbles.[37] However, that result also implies that the limiting value of the present discounted value of real money balances is zero *in equilibrium*. This rules out the use of policies that involve expanding the money supply at the rate of interest as advocated by Buiter (2014) in equilibrium.

2.6.3 Debt versus money finance

One argument in favor of money-financed rather than debt-financed fiscal expansions is that increases in the government debt stock may increase the real interest rates at which the government can finance that debt (see, for example, Smets and Trabandt, 2012).[38] Modeling such effects typically involves a friction whereby some agents have a preference or requirement for holding longer term government liabilities rather than short-term liabilities (such as one-period government bonds or money). A long-term bond-financed deficit will increase the term or risk premium on such debt and crowd out private sector spending. Such frictions are similar to those that are often used to motivate a role for quantitative easing (as in Chapter 1, for example). In these cases, a money-financed fiscal expansion could be regarded as the equivalent of a bond-financed expansion plus a quantitative easing operation.

These observations raise the possibility that there may be some circumstances in which a combined bond-financed expansion plus a quantitative easing operation

[36]One way to make money irredeemable would be for the government to cancel the debt used to 'back' reserves. In that case, to implement a helicopter drop, the central bank would need to finance interest payments on reserves by the creation of more reserves. This raises questions of central bank solvency and whether such a concept has any practical relevance (Cumming, 2015; Reis, 2015).

[37]Transversality conditions on the total value of government liabilities alone are not sufficient to rule out equilibria in which the real values of money and bonds diverge in opposite directions.

[38]In the limit, it may become impossible for a government to borrow more if its current debt level is sufficiently high.

would be more effective than either policy alone.

Finally, the quantitative results hinge on the precise friction that generates net wealth effects: the asset-reset mechanism. As explained in Section 2.2, this device is used for analytical convenience rather than realism. In practice, there are distortions in the economy that are likely to lead to more substantial departures from stark Ricardian equivalence results. For example, in a model without interest on money, Auerbach and Obstfeld (2005) show that monetary expansions at the zero bound can be effective when taxes are distortionary rather than lump sum.

The extent to which different frictions apply to debt versus money financing is likely to have implications for the optimal mix of these financing methods and is a topic that deserves further research.

2.6.4 Welfare implications

Much of the debate on money-financed fiscal actions takes it for granted that increasing spending and inflation in response to a recessionary shock is welfare improving. However, as Ireland (2005) shows in a similar model, if net wealth effects from real money balances are generated through a redistributional channel then policies that rely heavily on that channel may reduce welfare for many households. Similarly, a money-financed fiscal stimulus implemented in the same way as studied by Galí (2014a) generates a large gap between output and its flexible price counterpart. As Galí (2014a) notes, a more appropriate metric of welfare is the efficient level of output.[39] He shows that, under some conditions, a policy rule in which government spending is financed by money creation can improve welfare. More broadly, the transactions friction in the baseline model would (other things equal) suggest that the returns on money and bonds be equalized, so that transactions costs vanish. A full welfare analysis of money-financed transfers is beyond the scope of this chapter, but is an interesting topic for future research.

[39] That is, the level of output that would prevail when distortions from monopolistic competition are eliminated.

2.6.5 Permanent liquidity trap versus temporary lower bound episode

The focus of this chapter is the potential efficacy of money-financed fiscal policy during a period in which the short-term interest rate is temporarily constrained by the effective lower bound. This is in line with most of the literature studying policy options at the zero lower bound, much of it inspired by the seminal contributions of Krugman (1998) and Eggertsson and Woodford (2003). It is also consistent with much of the recent commentary on the use of helicopter drops as a temporary measure to provide stimulus.[40]

The extent to which monetary-financing may be useful in a permanent liquidity trap, in which the short-term bond rate remains at the zero bound forever is analyzed by Buiter (2014) in a partial equilibrium context. Ireland (2005) studies the property of a very similar model in a liquidity trap environment, but in his case the liquidity trap is a policy choice.[41] Again, an assessment of the efficacy of money-financed fiscal policy in a permanent liquidity trap as an interesting avenue for future research.

2.7 Conclusion

This chapter assesses recent proposals for the use of money-financed transfers to stimulate economic activity and inflation using a simple sticky-price model. The efficacy of these policies is studied in the context of a recessionary shock that temporarily forces the short-term policy rate to the effective lower bound. The model allows for net wealth effects and, in this case, money-financed transfers can stimulate spending and inflation. However, the scale of the transfers required to generate meaningful effects is very large and could also be achieved by a bond-financed deficit.

In the baseline model, money may earn a non-zero rate of return and may therefore be interpreted as a digital currency rather than cash. Exploring this interpretation of the framework used here is an interesting topic for future research.

[40] Though Turner (2015) argues that monetary financing may become a conventional monetary policy tool in a world of secular stagnation.

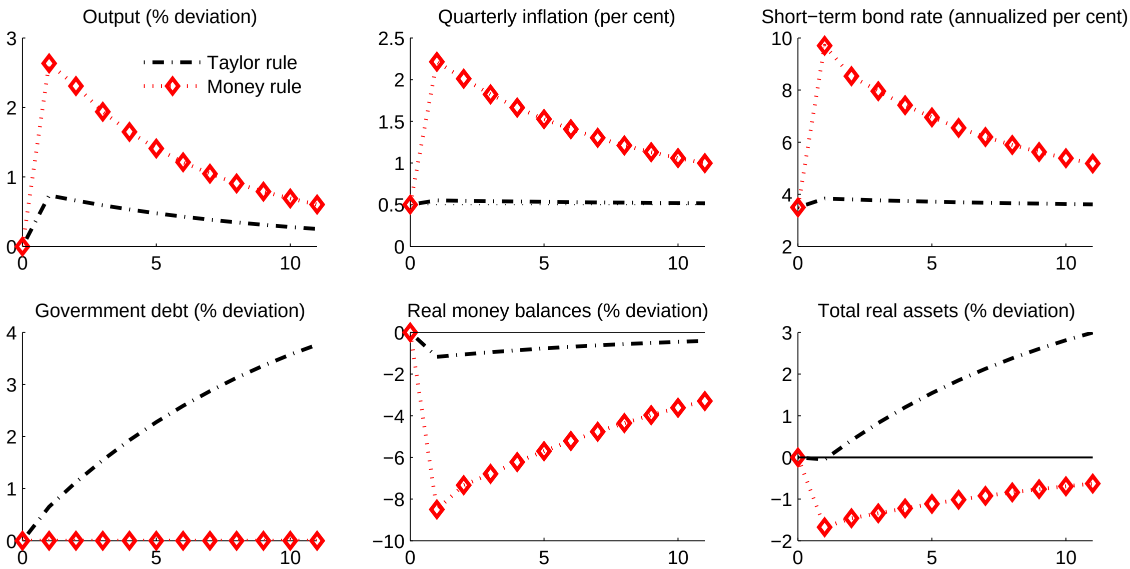
[41] The short-term nominal interest rate remains permanently at zero if the government chooses to expand the money stock at a sufficiently low rate.

Appendix 2.A Additional results

2.A.1 Money vs debt financing when $\gamma = 1$

Figure 2.9 shows the results of replicating the experiment in Figure 2.1 when there are no real balance effects (i.e., setting $\gamma = 1$). The results are virtually identical to those shown in Figure 2.1.

Figure 2.9: Money-financed and debt-financed government spending with $\gamma = 1$

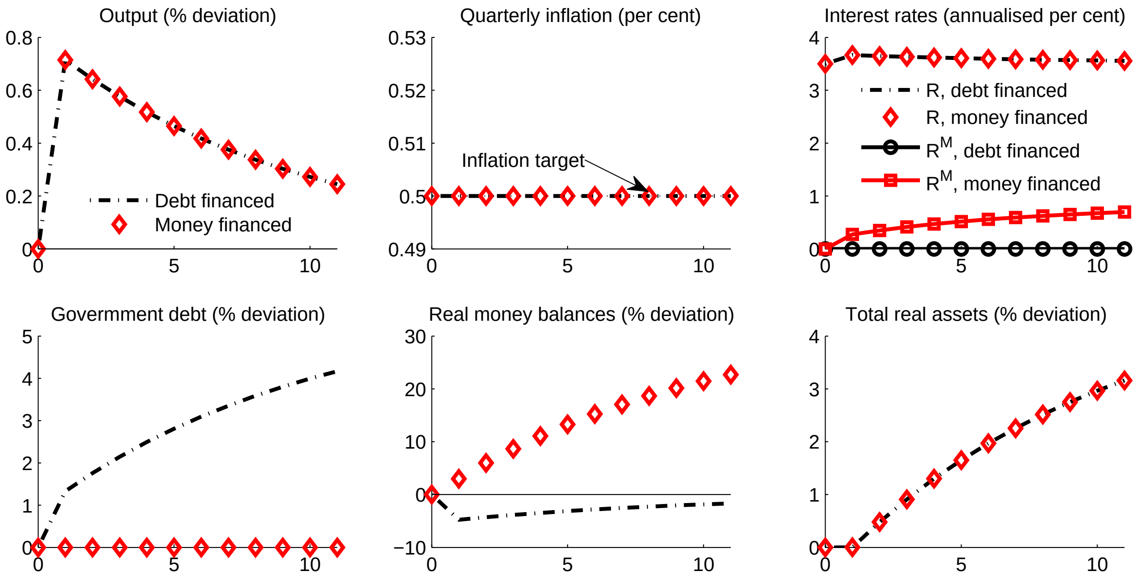


Notes: The model starts in period 0 at the steady state. Government spending is given by (2.9). The panels show the responses to $\varepsilon_1^g = 0.05$ (with $\varepsilon_t^g = 0, t \geq 2$). In the Taylor rule case, the tax rate fiscal rule is initially inactive so that $\tau_t = \tau^*$ for $t = 1, \ldots, K$, where $K = 12$. From period $t = K + 1$ taxes are set according to (2.15) with $\theta_b = 0.1$. The short-term interest rate is set according to (2.7). In the money rule case, the tax rate and debt stock are held constant and the government spending increase is financed by money creation according to equation (2.16). There is no interest paid on money ($R_t^M = 1, \forall t$) so the short term interest rate R_t adjusts to ensure that (2.16) holds. The experiment is conducted in a version of the model without asset resets (i.e., $\gamma = 1$).

2.A.2 Additively separable money demand

This Appendix reports the results of the experiment shown in Figure 2.2 in a version of the model with additively separable money demand (derived in Appendix 2.D) and without asset resets ($\gamma = 1$). This variant is closest to the one analyzed by Galí (2014a), with the key difference being that money may earn interest.

Figure 2.10: Financing government spending with interest-bearing money: additively separable case



Notes: The model starts in period 0 at the steady state. Government spending is given by (2.9). The panels show the responses to $\varepsilon_1^g = 0.05$ (with $\varepsilon_t^g = 0, t = 2, \ldots$). In the debt financed case, the tax rate fiscal rule is initially inactive so that $\tau_t = \tau^*$ for $t = 1, \ldots, K$, where $K = 12$. From period $t = K + 1$ taxes are set according to (2.15) with $\theta_b = 0.1$. In the money financed case, the tax rate and debt stock are held constant and the government spending increase is financed by money creation according to equation (2.16). The return on money is adjusted to ensure that households willingly hold the additional money balances. The experiment is conducted in a version of the model with (additively separable) money in the utility function and without asset resets.

Figure 2.10 repeats the experiment shown in Figure 2.2. As in Figure 2.2, the government spending increase is financed either using short-term debt (with the short-term nominal interest rate adjusting according to the Taylor rule) or by interest-bearing money (with the interest rate on money adjusting to ensure that households willingly hold the additional real money balances). In the variant with additively separable money demand, the outcomes for output, inflation and total real assets are identical. Inflation does not respond to this shock because the policy rule in this variant of the model responds to a measure of the flexible price output gap, which is fully stabilized in response to the government spending shock.

Appendix 2.B Derivation of the baseline model

2.B.1 Households

The maximization problem is:

$$\max \sum_{t=0}^{\infty} (\gamma\beta)^t \vartheta_t \left[\ln c_{j,t} - \frac{\chi_{j,t}}{1+\psi} n_{j,t}^{1+\psi}\right]$$

subject to

$$m_{j,t}^p + b_{j,t}^p = (\gamma\pi_t)^{-1}\left[R_t^M m_{j,t-1}^p + R_t b_{j,t-1}^p\right] + w_t n_{j,t} + d_t - \tau_t - \left(1 + \varphi\left(\frac{c_{j,t}}{m_{j,t}^p}\right)\right) c_{j,t}$$

where the budget constraint is written in real terms and

$$\varphi\left(v_{j,t}\right) = Z\exp\left[-\frac{\zeta}{v_{j,t}}\right]$$

$Z, \zeta > 0$.

The first order conditions are:

$$\begin{aligned}
\lambda_{j,t} - \beta\pi_{t+1}^{-1} R_{t+1}\lambda_{j,t+1} &= 0\\
\vartheta_t c_{j,t}^{-1} - \lambda_{j,t}\left(1 + \varphi\left(v_{j,t}\right) + \varphi'\left(v_{j,t}\right) v_{j,t}\right) &= 0\\
\lambda_{j,t}\left(1 - \varphi'\left(v_{j,t}\right) v_{j,t}^2\right) - \beta\pi_{t+1}^{-1} R_{t+1}^M \lambda_{j,t+1} &= 0\\
\chi_{j,t}\vartheta_t n_{j,t}^{\psi} - \lambda_{j,t} w_t &= 0
\end{aligned}$$

This functional form of φ implies that

$$\varphi'\left(v_{j,t}\right) = \varphi\left(v_{j,t}\right)\zeta v_{j,t}^{-2}$$

This implies that the set of first order conditions can be written as:

$$\lambda_{j,t} - \beta\pi_{t+1}^{-1} R_{t+1}\lambda_{j,t+1} = 0 \tag{2.20}$$

$$\vartheta_{j,t} c_{j,t}^{-1} - \lambda_{j,t}\left[1 + \varphi\left(v_{j,t}\right)\left(1 + \zeta v_{j,t}^{-1}\right)\right] = 0 \tag{2.21}$$

$$\lambda_{j,t}\left(1 - \zeta\varphi\left(v_{j,t}\right)\right) - \beta\pi_{t+1}^{-1} R_{t+1}^M \lambda_{j,t+1} = 0 \tag{2.22}$$

$$\chi_{j,t}\vartheta_t n_{j,t}^{\psi} - \lambda_{j,t} w_t = 0 \tag{2.23}$$

Combining the first order conditions for bonds and money gives:

$$1 - \zeta\varphi\left(v_{j,t}\right) = \frac{R^M_{t+1}}{R_{t+1}}$$

which shows that velocity is determined entirely by the difference between the rates of return on money and bonds. Since these rates of return are the same for all households, velocity is the same for each household:

$$v_{j,t} = v_t \quad \forall j, t$$

where v_t denotes aggregate velocity (total consumption divided by total real money balances). This result is imposed for the rest of the derivation.

The previous result means that

$$\varphi\left(v_t\right) = \zeta^{-1}\frac{R_{t+1} - R^M_{t+1}}{R_{t+1}}$$

which in turn implies that

$$\ln Z - \frac{\zeta}{v_t} = -\ln\zeta + \ln\frac{R_{t+1} - R^M_{t+1}}{R_{t+1}}$$

so that the demand for real money balances is given by:

$$m^p_{j,t} = \zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+1} - R^M_{t+1}}{R_{t+1}}\right]c_{j,t}$$

which implies that aggregate money demand satisfies

$$m_t = \zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+1} - R^M_{t+1}}{R_{t+1}}\right]c_t$$

Rearranging the first order condition for consumption gives:

$$\lambda_{j,t} = \frac{\vartheta_t}{c_{j,t}\left[1 + \varphi\left(v_t\right)\left(1 + \zeta v_t^{-1}\right)\right]}$$

which can be combined with the first order condition for bonds to give an Euler equation

$$\widetilde{c}_{j,t+1} = \beta\pi^{-1}_{t+1}R_{t+1}\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\widetilde{c}_{j,t} \tag{2.24}$$

where $\widetilde{c}_{j,t}$ denotes consumption inclusive of transactions costs,

$$\widetilde{c}_t \equiv (1+\varphi(v_t))\, c_{j,t}$$

and

$$\widetilde{\vartheta}_t \equiv \frac{\vartheta_t (1+\varphi(v_t))}{1+\varphi(v_t)\left(1+\zeta v_t^{-1}\right)}$$

Combining the first order conditions for consumption and hours worked gives:

$$\chi_{j,t} n_{j,t}^{\psi} = \frac{w_t}{c_{j,t}\left[1+\varphi(v_t)\left(1+\zeta v_t^{-1}\right)\right]}$$

Given the specification of $\chi_{j,t}$ we have:

$$\chi n_{j,t}^{\psi} = \frac{w_t}{c_t\left[1+\varphi(v_t)\left(1+\zeta v_t^{-1}\right)\right]}$$

2.B.2 Derivation of the aggregate consumption equation

For the derivation it is useful to define a household's total financial assets:

$$A_{j,t+1}^p \equiv R_{t+1}^M M_{j,t}^p + R_{t+1} B_{j,t}^p \tag{2.25}$$

which represents the total effective monetary and non-monetary obligations of the government, including interest due, in period $t+1$.

Using the definition of assets and $\widetilde{c}$, the household budget constraint can be written as:

$$\begin{aligned}
\frac{A_{j,t+1}^p}{P_t R_{t+1}} + \frac{M_{j,t}^p}{P_t} - \frac{R_{t+1}^M M_{j,t}^p}{R_{t+1} P_t} &= \frac{A_{j,t}^p}{\gamma P_t} + \tilde{w}_{j,t} - \widetilde{c}_{j,t} \\
\frac{a_{j,t+1}^p \pi_{t+1}}{R_{t+1}} + \frac{\left(R_{t+1} - R_{t+1}^M\right) M_{j,t}^p}{R_{t+1} P_t} &= \gamma^{-1} a_{j,t}^p + \tilde{w}_{j,t} - \widetilde{c}_{j,t}
\end{aligned}$$

where the first line uses the fact that (2.25) implies that $B_{j,t}^p = R_{t+1}^{-1} A_{j,t+1}^p - R_{t+1}^{-1} R_{t+1}^M M_{j,t+1}^p$ and the second line uses the definition of real assets $a_{j,t}^p \equiv A_{j,t}^p / P_t$ and the inflation rate $\pi_t \equiv P_t / P_{t-1}$. The final line can be written as:

$$a_{j,t}^p = \frac{\gamma a_{j,t+1}^p \pi_{t+1}}{R_{t+1}} + \gamma\left[\widetilde{c}_{j,t} - \tilde{w}_{j,t} + \frac{R_{t+1} - R_{t+1}^M}{R_{t+1}} m_{j,t}^p\right] \tag{2.26}$$

The real discount factor is defined recursively as:

$$\mathcal{D}_{t+i} = \frac{\gamma\pi_{t+i}}{R_{t+i}}\mathcal{D}_{t+i-1} \tag{2.27}$$

from $\mathcal{D}_t = 1$.

The household's no-Ponzi condition is assumed to be:

$$\lim_{i\to\infty} \mathcal{D}_{t+i}a^p_{j,t+i} \geq 0 \tag{2.28}$$

Iterating the household budget constraint (2.26) gives:

$$a^p_{j,t} = \lim_{i\to\infty} \mathcal{D}_{t+i}\frac{\gamma a^p_{j,t+i}\pi_{t+i+1}}{R_{t+i+1}} + \gamma\sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left(\widetilde{c}_{j,t+i} - \tilde{w}_{j,t+i} + \frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}}m^p_{j,t+i}\right)$$

so that, if the no Ponzi constraint binds with equality (as it will if marginal utility is positive in the limit):

$$a^p_{j,t} = \gamma\sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left(\widetilde{c}_{j,t+i} - \tilde{w}_{j,t+i} + \frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}}m^p_{j,t+i}\right) \tag{2.29}$$

Using the definition of post-tax income in (2.29) gives:

$$a^p_{j,t} = \gamma\sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left(\widetilde{c}_{j,t+i} - w_{t+i}n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i} + \frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}}m^p_{j,t+i}\right) \tag{2.30}$$

The household's money demand equation can be substituted into (2.30) to give

$$a^p_{j,t} = \gamma\sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left(\widetilde{c}_{j,t+i} - w_{t+i}n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i} + \Xi_{t+i}\widetilde{c}_{j,t+i}\right)$$

where

$$\Xi_{t+i} \equiv \frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}}\left(1+\varphi\left(v_{t+i}\right)\right)^{-1}\zeta^{-1}\left[\ln\left(\zeta Z\right) - \ln\frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}}\right]$$

is determined by the relative rates of return on bonds and money.

Rearranging the intertemporal budget constraint gives:

$$\sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left[1 + \Xi_{t+i}\right] \widetilde{c}_{j,t+i} = \gamma^{-1} a^{p}_{j,t} + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{j,t+i} + d_{j,t+i} - \tau_{j,t+i}\right) \quad (2.31)$$

The Euler equation (2.24) implies that

$$\widetilde{c}_{j,t+i} = (\gamma\beta)^{i} \mathcal{D}^{-1}_{t+i} \frac{\widetilde{\vartheta}_{t+i}}{\widetilde{\vartheta}_{t}} \widetilde{c}_{j,t} \quad (2.32)$$

Using (2.62) allows us to write (2.61) in terms of current consumption:

$$\widetilde{c}_{j,t} = \mu_t \left[\gamma^{-1} a^{p}_{j,t} + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{j,t+i} + d_{j,t+i} - \tau_{j,t+i}\right)\right] \quad (2.33)$$

where μ is the marginal propensity to consume from wealth, given by:

$$\mu_t = \left(\sum_{i=0}^{\infty} (\gamma\beta)^{i} \frac{\widetilde{\vartheta}_{t+i}}{\widetilde{\vartheta}_{t}} \left[1 + \Xi_{t+i}\right]\right)^{-1} \quad (2.34)$$

which implies that:

$$\mu_t^{-1} = 1 + \Xi_t + \gamma\beta \frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_{t}} \mu^{-1}_{t+1} \quad (2.35)$$

The consumption function (2.63) now aggregates straightforwardly. This follows from the fact that future income flows are identical for all households. Identical income flows are delivered by the assumption of identical lump sum taxes and dividends for all households together with the specification of the preference shifter $\chi_{j,t}$ to eliminate cohort-specific labor supply effects.

This implies that

$$\widetilde{c}_t = \mu_t \left[a_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{t+i} + d_{t+i} - \tau_{t+i}\right)\right] \quad (2.36)$$

where market clearing for assets is also imposed. The coefficient on assets on the right hand side of (2.36) is unity (rather that γ^{-1}) because it represents a weighted average of the assets held by households that experience an asset reset and those that do not. The former group has a weight of $1-\gamma$ and hold no assets. The latter group has a weight γ and average asset holdings of $\gamma^{-1}a_t$.

Substituting the definition of dividends into the consumption function gives:

$$\begin{aligned}\widetilde{c}_t =&\mu_t \left[a_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i}\left(y_{t+i} - \frac{\Phi}{2}\left(\frac{\pi_{t+i}}{\pi^*} - 1\right)^2 - \tau_{t+i}\right)\right]\\ =&\mu_t \left[a_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i}\left(\widetilde{c}_{t+i} + g_{t+i} - \tau_{t+i}\right)\right]\end{aligned}$$

where the second line uses the market clearing condition to substitute for output.

The final equation implies that aggregate consumption in period $t+1$ is given by:

$$\widetilde{c}_{t+1} = \mu_{t+1}\left[a_{t+1} + \mathcal{D}_{t+1}^{-1}\sum_{i=1}^{\infty}\mathcal{D}_{t+i}\left(\widetilde{c}_{t+i} + g_{t+i} - \tau_{t+i}\right)\right]$$

The consumption functions at dates $t+1$ and t can be combined to eliminate discounted future income flows:

$$\begin{aligned}\mathcal{D}_{t+1}\frac{\mu_t}{\mu_{t+1}}\widetilde{c}_{t+1} - \widetilde{c}_t =&\mathcal{D}_{t+1}\mu_t a_{t+1} - \mu_t a_t - \mu_t\left(\widetilde{c}_t + g_t - \tau_t\right)\\ =&\mathcal{D}_{t+1}\mu_t a_{t+1} - \mu_t\left(\widetilde{c}_t + a_t + g_t - \tau_t\right) \qquad (2.37)\end{aligned}$$

To proceed, the intertemporal budget constraints of the household and the government are combined. In parallel with households, define the total stock of government liabilities as:

$$A_{t+1}^g \equiv R_{t+1}^M M_t^g + R_{t+1}B_t^g$$

Again using lower case notation to denote real-valued asset stocks, these definitions can be used to write the government budget constraint (2.8) as:

$$a_t^g = \frac{a_{t+1}^g \pi_{t+1}}{R_{t+1}} - g_t + \tau_t + \frac{R_{t+1} - R_{t+1}^M}{R_{t+1}} m_t^g \qquad (2.38)$$

This implies that

$$a_t + g_t - \tau_t = \gamma^{-1}\mathcal{D}_{t+1}a_{t+1} + \Xi_t\widetilde{c}_t$$

which can be used in (2.37) to give:

$$\mathcal{D}_{t+1}\frac{\mu_t}{\mu_{t+1}}\widetilde{c}_{t+1} - \widetilde{c}_t = \mathcal{D}_{t+1}\mu_t a_{t+1} - \mu_t\left(\widetilde{c}_t + \gamma^{-1}\mathcal{D}_{t+1}a_{t+1} + \Xi_t\widetilde{c}_t\right)$$

The government budget constraint (2.38), can be written in aggregate terms (imposing asset market equilibrium $a_t^g = a_t^p = a_t, \forall t$) as:

$$\begin{aligned} a_t =& \mathcal{D}_{t+1}^g a_{t+1} - g_t + \tau_t + \frac{R_{t+1} - R_{t+1}^M}{R_{t+1}}m_t \\ =& \gamma^{-1}\mathcal{D}_{t+1}a_{t+1} - g_t + \tau_t + \frac{R_{t+1} - R_{t+1}^M}{R_{t+1}}m_t \\ =& \gamma^{-1}\mathcal{D}_{t+1}a_{t+1} - g_t + \tau_t + \Xi_t\widetilde{c}_t \end{aligned}$$

where the final line substitutes for money demand.

Collecting terms gives:

$$\mathcal{D}_{t+1}\frac{\mu_t}{\mu_{t+1}}\widetilde{c}_{t+1} = \mathcal{D}_{t+1}\mu_t\left(1 - \gamma^{-1}\right)a_{t+1} + \left[1 - \left(1 + \Xi_t\right)\mu_t\right]\widetilde{c}_t$$

which implies that

$$\widetilde{c}_t = \left[1 - \left(1 + \Xi_t\right)\mu_t\right]^{-1}\frac{\gamma\pi_{t+1}}{R_{t+1}}\left[\frac{\mu_t}{\mu_{t+1}}\widetilde{c}_{t+1} + \mu_t\left(1 - \gamma\right)\gamma^{-1}a_{t+1}\right] \tag{2.39}$$

which also uses the fact that $\mathcal{D}_{t+1} = \frac{\gamma\pi_{t+1}}{R_{t+1}}$.

When $\gamma = 1$, equation (2.39) becomes:

$$\widetilde{c}_t = \left[1 - \left(1 + \Xi_t\right)\mu_t\right]^{-1}\frac{\pi_{t+1}}{R_{t+1}}\frac{\mu_t}{\mu_{t+1}}\widetilde{c}_{t+1}$$

and the dependence on assets a_{t+1} (the real balance effect) disappears.

The terms in the marginal propensities to consume can be simplified as follows:

$$
\begin{aligned}
[1-(1+\Xi_t)\,\mu_t]^{-1}\,\frac{\mu_t}{\mu_{t+1}} &= \left[\frac{\mu_{t+1}}{\mu_t}\,(1-(1+\Xi_t)\,\mu_t)\right]^{-1} \\
&= \left[\frac{\mu_t^{-1}}{\mu_{t+1}^{-1}} - \frac{1+\Xi_t}{\mu_{t+1}^{-1}}\right]^{-1} \\
&= \left[\frac{1+\Xi_t+\gamma\beta\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\mu_{t+1}^{-1}}{\mu_{t+1}^{-1}} - \frac{1+\Xi_t}{\mu_{t+1}^{-1}}\right]^{-1} \\
&= \left[\gamma\beta\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\right]^{-1}
\end{aligned}
$$

Similarly,

$$
[1-(1+\Xi_t)\,\mu_t]^{-1}\,\mu_t = \left[\gamma\beta\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\right]^{-1}\mu_{t+1}
$$

Using these results in the aggregate consumption equation gives

$$
\widetilde{c}_t = \frac{\pi_{t+1}}{\beta R_{t+1}}\frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}}\left[\widetilde{c}_{t+1}+(1-\gamma)\,\gamma^{-1}\mu_{t+1}a_{t+1}\right]
$$

or

$$
\widetilde{c}_t = \frac{\pi_{t+1}}{\beta R_{t+1}}\frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}}\left[\widetilde{c}_{t+1}+(1-\gamma)\,\gamma^{-1}\mu_{t+1}\pi_{t+1}^{-1}\left(R_{t+1}^M m_t + R_{t+1}b_t\right)\right] \tag{2.40}
$$

which also uses the definition of total assets:

$$
a_{t+1} = \pi_{t+1}^{-1}\left(R_{t+1}^M m_t + R_{t+1}b_t\right)
$$

When $\gamma = 1$, this can be written as:

$$
\widetilde{c}_{t+1} = \beta\pi_{t+1}^{-1}R_{t+1}\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\widetilde{c}_t
$$

which coincides with the individual household Euler equation (2.24).

2.B.3 Firms and supply side

The objective function for the firm is:

$$\max \sum_{k=t}^{\infty} \bar{\lambda}_k \beta^{k-t} \left[\left(\frac{P_{j,k}}{P_k} - \frac{w_k}{A} \right) \left(\frac{P_{j,k}}{P_k} \right)^{-\eta} y_k - \frac{\Phi}{2} \left(\frac{P_{j,k}}{\pi^* P_{j,k-1}} - 1 \right)^2 \right]$$

where $\bar{\lambda}$ is a stochastic discount factor used to value the flow of profits. In general, there are heterogeneous households (which differ by their date of asset resets) so there is not a unique stochastic discount factor that can be used for valuing the profit flows paid to households. However, the first order condition (2.20) implies that the discount factors of all households satisfy:

$$\frac{\lambda_{j,t}}{\lambda_{j,t+1}} = \beta \frac{R_{t+1}}{\pi_{t+1}}$$

so a discount factor is chosen that satisfies:

$$\bar{\lambda}_{t+1} = \bar{\lambda}_t \frac{\pi_{t+1}}{\beta R_{t+1}}$$

The first order condition for the firm's price is:

$$\begin{aligned} 0 = &- \eta \left(\frac{P_{j,t}}{P_t} - \frac{w_t}{A} \right) \left(\frac{P_{j,t}}{P_t} \right)^{-\eta-1} \frac{\bar{\lambda}_t y_t}{P_t} + \left(\frac{P_{j,t}}{P_t} \right)^{-\eta} \frac{\bar{\lambda}_t y_t}{P_t} \\ &- \frac{\Phi \bar{\lambda}_t}{\pi^* P_{j,t-1}} \left(\frac{P_{j,t}}{\pi^* P_{j,t-1}} - 1 \right) + \beta \frac{\Phi \bar{\lambda}_{t+1} P_{j,t+1}}{\pi^* P_{j,t}^2} \left(\frac{P_{j,t+1}}{\pi^* P_{j,t}} - 1 \right) \end{aligned}$$

which reveals that optimal pricing decisions depend on the stochastic discount factor only through the ratio $\bar{\lambda}_{t+1}/\bar{\lambda}_t$.

In a symmetric equilibrium in which $P_{j,t} = P_t, \forall j, t$, the first order condition simplifies to:

$$\frac{\Phi \pi_t}{\pi^* y_t} \left(\frac{\pi_t}{\pi^*} - 1 \right) = 1 - \eta + \eta \frac{w_t}{A} + \Phi \frac{\pi_{t+1}}{R_{t+1}} \frac{\pi_{t+1}}{\pi^* y_t} \left(\frac{\pi_{t+1}}{\pi^*} - 1 \right) \tag{2.41}$$

The supply side of the model is unchanged from the baseline variant, so the pricing equation is given by:

$$\frac{\Phi \pi_t}{\pi^* y_t} \left(\frac{\pi_t}{\pi^*} - 1 \right) = 1 - \eta + \eta \frac{w_t}{A} + \Phi \frac{\pi_{t+1}}{R_{t+1}} \frac{\pi_{t+1}}{\pi^* y_t} \left(\frac{\pi_{t+1}}{\pi^*} - 1 \right)$$

as before.

Noting that $A = 1$ and that the first order conditions imply that the wage is given by:

$$w_t = \chi n_t^{\psi} c_t \left[1 + \varphi(v_t)\left(1 + \zeta v_t^{-1}\right)\right]$$

allows the pricing equation to be written as:

$$\frac{\Phi\pi_t}{\pi^* y_t}\left(\frac{\pi_t}{\pi^*} - 1\right) = 1 - \eta + \eta\chi y_t^{\psi} c_t \left[1 + \varphi(v_t)\left(1 + \zeta v_t^{-1}\right)\right] + \Phi\frac{\pi_{t+1}}{R_{t+1}}\frac{\pi_{t+1}}{\pi^* y_t}\left(\frac{\pi_{t+1}}{\pi^*} - 1\right)$$

which also uses the fact that the production function implies that $n_t = y_t$.

2.B.4 The model equations

Collecting together the previously derived equations gives:

$$
\begin{aligned}
y_t &= (1+\varphi_t)\, c_t + g^* + \frac{\Phi}{2}\left(\frac{\pi_t}{\pi^*}-1\right)^2 \\
\frac{\Phi\pi_t}{y_t\pi^*}\left(\frac{\pi_t}{\pi^*}-1\right) &= 1-\eta+\eta\chi y_t^{\psi} c_t\left[1+\varphi_t\left(1+\zeta v_t^{-1}\right)\right]+\Phi\frac{\pi_{t+1}}{R_{t+1}}\frac{\pi_{t+1}}{\pi^* y_t}\left(\frac{\pi_{t+1}}{\pi^*}-1\right) \\
\widetilde{c}_t &= \frac{\pi_{t+1}}{\beta R_{t+1}}\frac{\widetilde{\vartheta}_t}{\widetilde{\vartheta}_{t+1}}\left[\widetilde{c}_{t+1}+(1-\gamma)\,\gamma^{-1}\mu_{t+1}\pi_{t+1}^{-1}\left(R_{t+1}^M m_t+R_{t+1}b_t\right)\right] \\
\mu_t^{-1} &= 1+\Xi_t+\gamma\beta\frac{\widetilde{\vartheta}_{t+1}}{\widetilde{\vartheta}_t}\mu_{t+1}^{-1} \\
\Xi_t &= \frac{\varphi_t}{1+\varphi_t}\left[\ln(\zeta Z)-\ln\frac{R_{t+1}-R_{t+1}^M}{R_{t+1}}\right] \\
\widetilde{c}_t &= (1+\varphi_t)\, c_t \\
\widetilde{\vartheta}_t &= \frac{\vartheta_t(1+\varphi_t)}{1+\varphi_t\left(1+\zeta v_t^{-1}\right)} \\
\varphi_t &= \zeta^{-1}\frac{R_{t+1}-R_{t+1}^M}{R_{t+1}} \\
\varphi_t &= Z\exp\left[-\frac{\zeta}{v_t}\right] \\
v_t &= \frac{c_t}{m_t} \\
R_{t+1} &= \max\left\{R\left(\frac{\pi_t}{\pi^*}\right)^{\theta_\pi}\left(\frac{y_t}{y_t^f}\right)^{\theta_y}, \underline{R}\right\} \\
\Delta\ln\vartheta_t &= \rho_\vartheta\Delta\ln\vartheta_{t-1}+\varepsilon_t^\vartheta
\end{aligned}
$$

Conditional on the behavior of flex price output, y_t^f (derived below), the return on money, R_t^M, and government debt, b_t, this system determines the variables $y_t, c_t, \pi_t, R_{t+1}, m_t, v_t, \varphi_t, \vartheta_t, \widetilde{\vartheta}_t, \widetilde{c}_t, \mu_t, \Xi_t$.

The model is closed by assumptions about the return on money and government debt. The gross rate of return on money, R_{t+1}^M, is either held fixed at unity in the conventional approach ($R_{t+1}^M = 1, \forall t$) or adjusted to ensure that households willingly hold a stock of money that is determined by a rule, as in the baseline version of the model. The baseline assumption for government debt is that it is fixed, $b_t = b^* \geq 0$, while the 'debt financing' experiments in Section 2.3.1 incorporate the government budget constraint with temporarily fixed taxes, $\tau_t = \tau$.

2.B.5 Flexible price allocations

Models with transactions frictions present some challenges for defining flexible price allocations. In particular, even if prices and wages are fully flexible, allocations still depend on the level(s) of nominal interest rate(s). Kim and Subramanian (2006) and Ravenna and Walsh (2006) define a 'supply side' flexible price equilibrium, conditional on steady-state nominal returns on money and bonds. An analogous concept is used here.

If nominal returns on bonds and money are at their steady-state levels, of R and 1 respectively, then flexible price velocity and transactions costs are constant, at $\bar{\varphi}$ and $\bar{v}$ respectively, where these values satisfy:

$$\begin{aligned} \zeta\bar{\varphi} &= \frac{R-1}{R} \\ \bar{\varphi} &= Z\exp\left[-\frac{\zeta}{\bar{v}}\right] \end{aligned}$$

Conditional on flexible price velocity and transactions costs, market clearing and the pricing equation can be used to solve for flexible price output and consumption as follows:

$$\begin{aligned} y_t^f &= (1+\bar{\varphi})\, c_t^f + g_t \\ 0 &= 1-\eta+\eta\chi c_t^f \left(y_t^f\right)^{\psi}\left(1+\bar{\varphi}\left(1+\frac{\zeta}{\bar{v}}\right)\right) \end{aligned}$$

where the pricing equation does not feature price adjustment costs under the assumption that inflation is constant at target in the flexible price equilibrium.

2.B.6 Steady state

Steady-state allocations are indicated by the absence of time subscripts.

Long-run government policies are treated as exogenous. This includes the value of the inflation target π^* and the steady-state levels of debt and government spending. Steady-state government spending is set exogenously at $g = g^* y$, with $g^* \in [0, 1)$.

The utility parameter χ and the productivity parameter A are chosen to normalize steady-state hours worked and output at unity: $n = y = 1$. Conditional on $n = 1$, the required value of A is 1. The required value of χ is found by noting that the steady-state Phillips curve implies that:

$$0 = 1 - \eta + \eta\chi \frac{1 - g^*}{1 + \varphi} \left[1 + \varphi \left(1 + \zeta v^{-1}\right)\right]$$

so that delivering $n = 1$ requires:

$$\chi = \frac{\eta - 1}{\eta} \frac{1 + \varphi}{(1 - g^*) \left[1 + \varphi \left(1 + \zeta v^{-1}\right)\right]}$$

and hence the required value of χ depends on steady-state velocity and (hence) transactions costs. These variables will be targeted by an appropriate choice of the transaction cost parameters Z and ζ (detailed below).

In steady state, the consumption equation implies

$$\beta \widetilde{c} = \frac{\pi^*}{R} \left[\widetilde{c} + \mu (1 - \gamma) \frac{1}{\gamma \pi^*} \left(R^M m + Rb\right)\right]$$

or

$$\frac{\beta R}{\pi^*} = 1 + \frac{1 - \gamma\beta}{1 + \Xi} (1 - \gamma) \frac{R^M m + Rb}{\gamma \pi^* \widetilde{c}}$$

(since $\mu = \frac{1 - \gamma\beta}{1 + \Xi}$). This illustrates that the steady state real interest rate is increasing in the steady state ratio of (the value of) government liabilities to consumption.

The steady-state net interest rate on money is assumed to be zero so that $R^M = 1$. As noted above, steady-state velocity, $v \equiv c/m$, is also treated as a calibration target and is therefore 'known'.

This means that the steady-state interest rate satisfies:

$$\frac{\beta R}{\pi^*} = 1 + \frac{(1 - \gamma\beta)(1 - \gamma)}{(1 + \Xi)(1 + \varphi) v \gamma \pi^*} + \frac{1 - \gamma\beta}{1 + \Xi} (1 - \gamma) \frac{Rb}{\gamma \pi^* \widetilde{c}}$$

The steady-state target level of government debt is assumed to satisfy $b = b^* y$, where $b^* \geq 0$ and $y = 1$. This implies that the steady-state interest rate satisfies:

$$\frac{\beta R}{\pi^*} = 1 + \frac{(1-\gamma\beta)(1-\gamma)}{(1+\Xi)(1+\varphi)\, v\gamma\pi^*} + \frac{1-\gamma\beta}{1+\Xi}(1-\gamma)\frac{Rb^*}{\gamma\pi^*(1-g^*)}$$

which also uses the fact that $\widetilde{c} = 1 - g^*$ in steady state.

Collecting terms implies that, conditional on steady-state government spending, government debt, the inflation target and steady-state velocity (which determines φ and Ξ), the value of β consistent with a desired steady state nominal interest rate can be found by setting:

$$\begin{aligned}\beta = &\left[\frac{R}{\pi^*} + \frac{1-\gamma}{(1+\Xi)(1+\varphi)\, v\pi^*} + \frac{(1-\gamma)\, Rb^*}{(1+\Xi)\,\pi^*(1-g^*)}\right]^{-1} \\ &\times \left[1 + \frac{1-\gamma}{(1+\Xi)(1+\varphi)\, v\gamma\pi^*} + \frac{(1-\gamma)\, Rb^*}{(1+\Xi)\,\gamma\pi^*(1-g^*)}\right]\end{aligned}$$

Conditional on ζ, Z is chosen to deliver a target level of steady-state real money balances. This implies that:

$$Z = \exp\left[\frac{\zeta}{v} - \ln\zeta + \ln\left(1 - R^{-1}\right)\right]$$

where steady-state velocity v is chosen based on the target real money balance level.

Finally, to calibrate the price adjustment costs, note that log-linearizing the Phillips curve gives:

$$\hat{\pi}_t = \beta\hat{\pi}_{t+1} + \frac{\eta-1}{\Phi}\hat{w}_t$$

so that the slope of the linearized Phillips curve with respect to marginal cost is $\frac{\eta-1}{\Phi}$. In a model with Calvo (1983) contracts, the slope is:

$$\frac{(1-p)(1-p\beta)}{p}$$

where p is the probability that the price is not adjusted in each quarter (see Galí, 2008). The slope of the linearized Phillips curve can be replicated for a desired value of p by setting:

$$\Phi = \frac{p(\eta-1)}{(1-p)(1-p\beta)}$$

Appendix 2.C Cash in advance variant

This appendix derives a cash in advance variant of the model similar to that used by Ireland (2005). Because in this variant money is held for its transactions services, the net interest rate on money is assumed to be zero in all periods. This permits consideration of cases in which the nominal interest rate on bonds is also equal to zero.

2.C.1 Overview of the differences from the baseline model

In this variant of the model households are subject to a cash in advance constraint so that purchases of consumption goods are constrained by the quantity of cash holdings that the household has access to. The government, monetary policy and firms are modeled in the same was as in the baseline model, so are not discussed in detail here.

The main changes to the model structure affect the timing of events within each period. Such timing assumptions are required to clarify how the cash in advance constraint limits the spending power of the household. These timing assumptions, require a more explicit treatment of the tax and transfer payments made to households. It is also convenient to redefine the price of a one period bond (so that households are assumed to purchase a bond that pays one unit of money in the following period for a price equal to the reciprocal of the gross return on the bond). This renormalization does not have any implications for the equilibrium conditions of the model, but simplifies the exposition and derivation. As noted above, it is also assumed that money pays no interest (that is $R_t^M = 1, \forall t$).

2.C.2 Household budget constraints and timing

As in Ireland (2005), the timing protocol is based on the worker-shopper setup introduced by Lucas (1980). At the start of period t, a household born in period j receives a monetary transfer from the government, $T_{t,s}^m \geq 0$. In addition to this transfer, the household also receives income from maturing one-period bonds purchased in the previous period. Similarly, the household also carries over any money balances that were not used for consumption in the previous period.

Asset markets open at the beginning of the period and the household decides how to allocate its asset income between money and bonds. Only the amount allocated to money can be used to purchase consumption goods, so that the cash in advance constraint is written as:

$$M^p_{j,t-1} + \widetilde{B}^p_{j,t-1} + T^m_{j,t} - \frac{\widetilde{B}^p_{j,t}}{R_{t+1}} \geq P_t c_{j,t} \tag{2.42}$$

The left-hand side of (2.42) is the quantity of money held at the start of period t. As described above, the first two terms represent the stock of previously accumulated money and (matured) one-period bonds, the second term is the monetary transfer from the government and the third term is the households investment in one period bonds that will mature at the start of period $t+1$. As noted above, the notation for bond pricing is such that a bond that pays one unit of money in period $t+1$ is purchased at price R^{-1}_{t+1} in period t. For this reason the notation $\widetilde{B}$ denotes bond holdings, to make it clear that they are distinct from the variable B in the baseline version of the model. As in the baseline variant of the model, the rate of return between periods t and $t+1$ is determined in period t.[42]

The right hand side of (2.42) represents the consumption expenditure of the household. Consumption is carried out by the 'shopper' who splits from the 'worker' at the beginning of the period. At the end of period t, the worker and shopper reunite and pool resources. This pooling of resources gives rise to the end of period budget constraint:

$$M^p_{j,t} \leq M^p_{j,t-1} + \widetilde{B}^p_{j,t-1} + T^m_{j,t} - \frac{\widetilde{B}^p_{j,t}}{R_{t+1}} - P_t c_{j,t} + W_t n_{j,t} + D_{j,t} - T^g_{j,t}$$

which shows that the quantity of money carried forward to period $t+1$ can be no greater than the residual from the cash in advance constraint (that is, the quantity of money remaining after consumption) plus net income. Net income consists of the

[42]The model implicitly allows households to borrow from each other using one period nominal bonds. In equilibrium, the absence of arbitrage opportunities will imply that these bonds trade at the same price as government bonds and the net supply of such bonds across all households will be zero. In fact, given the nature of the equilibrium in the model and the cash in advance constraint, households will typically need to borrow (that is choose $\widetilde{B}^p_{j,t} < 0$) in the periods immediately following an asset reset in order to be able to finance the optimal level of consumption. A similar effect arises in the model studied by Ireland (2005).

wage income and dividends paid to the worker by firms, net of taxes T_t^g levied by the government to finance government spending and its liabilities.

The household budget constraint can be written in real terms as:

$$m_{j,t}^p \leq \pi_t^{-1}\left(m_{j,t-1}^p + \widetilde{b}_{j,t-1}^p\right) + \tau_{j,t}^m - \frac{\widetilde{b}_{j,t}^p}{R_{t+1}} - c_{j,t} + w_t n_{j,t} + d_{j,t} - \tau_{j,t}^g \tag{2.43}$$

where lower case letters denote nominal variables deflated by the price level P_t, inflation is denoted $\pi_t \equiv P_t/P_{t-1}$ as in the main text and $\tau_{j,t}^x \equiv T_{j,t}^x/P_t, x = m, g$.

Defining total assets as $\widetilde{a}_{j,t}^p \equiv \pi_t^{-1}\left(m_{j,t-1}^p + \widetilde{b}_{j,t-1}^p\right)$ (the household's assets at the start of period t) and assuming that the budget constraint binds gives:

$$\pi_{t+1}R_{t+1}^{-1}\widetilde{a}_{j,t+1}^p = \widetilde{a}_{j,t}^p + \tau_{j,t}^m - \frac{R_{t+1}-1}{R_{t+1}}m_{j,t}^p - c_{j,t} + w_t n_{j,t} + d_{j,t} - \tau_{j,t}^g$$

which implies that

$$\begin{aligned}\widetilde{a}_{j,t}^p = &\sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left[\frac{R_{t+i+1}-1}{R_{t+i+1}}m_{j,t+i}^p + c_{j,t+i} - w_{t+i}n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i}^g - \tau_{j,t+i}^m\right]\\ &+ \lim_{i\to\infty}\mathcal{D}_{t+i}\pi_{t+i+1}R_{t+i+1}^{-1}\widetilde{a}_{j,t+i}^p\end{aligned}$$

where the discount factor $\mathcal{D}_{t+i}$ is defined as in the main text. Assuming that the household's no Ponzi condition holds with equality, we have $\lim_{i\to\infty}\mathcal{D}_{t+i}\widetilde{a}_{j,t+i}^p = 0$ so that the intertemporal budget constraint is:

$$\widetilde{a}_{j,t}^p = \sum_{i=0}^{\infty}\mathcal{D}_{t+i}\left[\frac{R_{t+i+1}-1}{R_{t+i+1}}m_{j,t+i}^p + c_{j,t+i} - w_{t+i}n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i}^g - \tau_{j,t+i}^m\right] \tag{2.44}$$

2.C.3 Household optimization

Analogous to the assumptions in the baseline model, the household solves:

$$\max\sum_{t=0}^{\infty}\beta^t\vartheta_t\left[\ln c_{j,t} - \frac{\chi_{j,t}}{1+\psi}n_{j,t}^{1+\psi}\right] \tag{2.45}$$

so that preferences correspond to those in a variant of the baseline model with $\gamma = 1$.

Utility maximization subject to the cash in advance constraint and the budget constraint (2.43) gives first order conditions with respect to bonds, labor supply, consumption and money:

$$\frac{\omega_{j,t}+\xi_{j,t}}{R_{t+1}} = \beta\frac{\omega_{j,t+1}+\xi_{j,t+1}}{\pi_{t+1}} \tag{2.46}$$

$$\vartheta_t\chi_{j,t}n_{j,t}^{\psi} = \omega_{j,t}w_t \tag{2.47}$$

$$\frac{\vartheta_t}{c_{j,t}} = \omega_{j,t}+\xi_{j,t} \tag{2.48}$$

$$\omega_{j,t} = \beta\frac{\omega_{j,t+1}+\xi_{j,t+1}}{\pi_{t+1}} \tag{2.49}$$

where $\omega_{j,t}$ is the multiplier on the budget constraint and $\xi_{j,t}$ is the multiplier on the cash in advance constraint.

Under the assumption that the budget constraint binds (so that $\omega_{j,t}\neq 0, \forall t$) the first order conditions can be rearranged to give:

$$\frac{\vartheta_t}{c_{j,t}} = \beta\frac{R_{t+1}}{\pi_{t+1}}\frac{\vartheta_{t+1}}{c_{j,t+1}} \tag{2.50}$$

$$\chi\bar{c}_t n_{j,t}^{\psi} = \frac{w_t}{R_{t+1}}$$

where the second equation is the labor supply equation. Note that because the marginal value of consumption is affected by the multiplier on the cash in advance constraint, the labor supply relationship depends on the short-term bond rate (the opportunity cost of holding cash) giving rise to a Tobin effect.

As long as the short-term bond rate is strictly positive, the cash in advance constraint binds. This means that (2.43) and (2.42) (written in real terms) can be combined to give:

$$m_{t,j}^{p} = w_t n_{t,j} + d_{j,t} - \tau_{j,t}^{g} \tag{2.51}$$

which can be written as a money demand equation by using the labor supply condition to eliminate hours worked:

$$m_{t,j}^{p} = w_t\left(\frac{w_t}{\chi\bar{c}_t R_{t+1}}\right)^{\frac{1}{\psi}} + d_{j,t} - \tau_{j,t}^{g}$$

Under the assumption that dividends are distributed equally to all households and taxes are levied equally on all households, this implies that all households will hold an identical stock of real money balances, given by:[43]

$$m^p_{j,t} = \bar{m}^p_t = w_t \left(\frac{w_t}{\chi \bar{c}_t R_{t+1}} \right)^{\frac{1}{\psi}} + \bar{d}_t - \bar{\tau}^g_t$$

which is useful for aggregation purposes.

2.C.4 The government budget constraint, monetary and fiscal policies

The period budget constraint of the government in real, aggregate, terms is given by:

$$m^g_t + R^{-1}_{t+1} \widetilde{b}^g_t = \pi^{-1}_t \left(m^g_{t-1} + \widetilde{b}^g_{t-1} \right) + g_t + \tau^m_t - \tau^g_t$$

Monetary and fiscal policies are coordinated. There is a fiscal rule for τ^g_t that ensures that the government's solvency condition is satisfied. The short-term bond rate is set according to a monetary policy rule, subject to a lower bound:

$$R_{t+1} = \max \left\{ R \left(\frac{\pi_t}{\pi^*} \right)^{\theta_\pi} \left(\frac{y_t}{y} \right)^{\theta_y}, 1 \right\} \tag{2.52}$$

which is similar to the monetary policy rule for the baseline version of the model, with $\underline{R} = 1$. The only difference in the specification of the rule is that the output gap is measured relative to the steady state level of output. As described below, the presence of the cash in advance constraint generates a 'cost channel' effect in the Phillips curve which means that the flexible price allocations in this model are affected by the behavior of monetary policy.

When the monetary policy rule (2.52) prescribes a strictly positive interest rate on bonds ($R_{t+1} > 1$), the cash in advance constraint binds and the monetary transfer τ^m_t is chosen to deliver the value of R_{t+1} implied by the monetary policy rule.

[43]Note that this implies that all households hold the same quantity of money at the *end* of the period. Differing consumption levels can be financed while satisfying the cash in advance constraint through the appropriate trading of bonds.

When the policy rule is constrained at $R_{t+1} = 1$, the cash in advance constraint does not bind and the monetary transfer τ_t^m is chosen to ensure that the total money stock satisfies:

$$\frac{M_t}{M_{t-1}} = \left(\frac{M_{t-1}}{M_{t-2}}\right)^{\rho_m} (\pi^*)^{(1-\rho_m)} \exp\left(\varepsilon_t^m\right) \tag{2.53}$$

Fiscal policy is the same as the baseline model. Real government spending and debt are held fixed in current value terms:

$$\begin{aligned} g_t &= g^* \\ \widetilde{b}_t &= R_{t+1} b^* \end{aligned}$$

where the second equation imposes the same fiscal policy as in the baseline model.

2.C.5 Aggregation

Some care is required when aggregating asset stocks because of the alternative timing notation for total assets and the stocks of money and bonds.

To derive the consumption function for the individual household, first note that the Euler equation (2.50) implies that:

$$c_{j,t+1} = \beta \frac{R_{t+1}}{\pi_{t+1}} \frac{\vartheta_{t+1}}{\vartheta_t} c_{j,t}$$

and hence that

$$c_{j,t+i} = (\gamma\beta)^i \mathcal{D}_{t+i}^{-1} \frac{\vartheta_{t+i}}{\vartheta_t} c_{j,t} \tag{2.54}$$

where the discount factor $\mathcal{D}_{t+i}$ is defined in (2.27), identically to the baseline model.

Substituting into the household's intertemporal budget constraint (2.44) gives:

$$\begin{aligned} \widetilde{a}_{j,t}^p &= \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left[\begin{array}{c} \frac{R_{t+i+1}-1}{R_{t+i+1}} m_{j,t+i}^p + (\gamma\beta)^i \mathcal{D}_{t+i}^{-1} \frac{\vartheta_{t+i}}{\vartheta_t} c_{j,t} \\ -w_{t+i} n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i}^g - \tau_{j,t+i}^m \end{array} \right] \\ &= \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left[\frac{R_{t+i+1}-1}{R_{t+i+1}} m_{j,t+i}^p - w_{t+i} n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i}^g - \tau_{j,t+i}^m \right] \\ &\quad + \sum_{i=0}^{\infty} (\gamma\beta)^i \frac{\vartheta_{t+i}}{\vartheta_t} c_{j,t} \end{aligned}$$

which implies that

$$c_{j,t} = \mu_t \left[\widetilde{a}^p_{j,t} + w_{t+i} n_{j,t+i} + d_{j,t+i} + \tau^m_{j,t+i} - \tau^g_{j,t+i} - \frac{R_{t+i+1} - 1}{R_{t+i+1}} m^p_{j,t+i} \right]$$

where the marginal propensity to consume satisfies

$$\mu_t^{-1} = \sum_{i=0}^{\infty} (\gamma\beta)^i \frac{\vartheta_{t+i}}{\vartheta_t} = 1 + \gamma\beta \frac{\vartheta_{t+1}}{\vartheta_t} \mu_{t+1}^{-1}$$

These conventions imply that aggregating the consumption function across households gives:

$$c_t = \mu_t \left[\widetilde{a}^p_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{t+i} + d_{t+i} + \tau^m_{t+i} - \tau^g_{t+i} - \frac{R_{t+i+1} - 1}{R_{t+i+1}} m^p_{t+i} \right) \right]$$

which is valid because all households make identical labor supply decisions, all dividends and taxes are distributed equally across all households and (hence) all households hold the same real money balances.

2.C.6 Equilibrium and parsimonious model representation

The goods market clearing condition and production function equations are identical to the baseline model:

$$y_t = c_t + g_t + \frac{\Phi}{2} \left(\frac{\pi_t}{\pi^*} - 1 \right)^2 = c_t + g^* + \frac{\Phi}{2} \left(\frac{\pi_t}{\pi^*} - 1 \right)^2$$
$$y_t = n_t$$

The labor supply condition and production function can be combined to show that the real wage satisfies:

$$w_t = \chi y_t^{\psi} c_t R_{t+1}$$

Using this expression in the pricing equation (which is identical to the baseline model), gives:

$$\frac{\Phi \pi_t}{y_t \pi^*} \left(\frac{\pi_t}{\pi^*} - 1 \right) = 1 - \eta + \eta \chi y_t^{\psi} c_t R_{t+1} + \frac{\Phi}{y_t} \frac{\pi_{t+1}}{R_{t+1}} \frac{\pi_{t+1}}{\pi^*} \left(\frac{\pi_{t+1}}{\pi^*} - 1 \right)$$

which implies that the cash in advance constraint gives rise to a 'cost channel' in the Phillips curve.

Using the definition of dividends and imposing goods and asset market clearing conditions to the consumption function gives:

$$\begin{aligned}
c_t &= \mu_t \left[\widetilde{a}_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(y_{t+i} - \frac{\Phi}{2} \left(\frac{\pi_{t+i}}{\pi^*} - 1 \right)^2 + \tau^m_{t+i} - \tau^g_{t+i} - \frac{R_{t+i+1} - 1}{R_{t+i+1}} m_{t+i} \right) \right] \\
&= \mu_t \left[\widetilde{a}_t + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(c_{t+i} + g_{t+i} + \tau^m_{t+i} - \tau^g_{t+i} - \frac{R_{t+i+1} - 1}{R_{t+i+1}} m_{t+i} \right) \right]
\end{aligned}$$

so that

$$c_{t+1} = \mu_{t+1} \left[\widetilde{a}_{t+1} + \mathcal{D}^{-1}_{t+1} \sum_{i=1}^{\infty} \mathcal{D}_{t+i} \left(c_{t+i} + g_{t+i} + \tau^m_{t+i} - \tau^g_{t+i} - \frac{R_{t+i+1} - 1}{R_{t+i+1}} m_{t+i} \right) \right]$$

and hence

$$\begin{aligned}
\mathcal{D}_{t+1} \frac{\mu_t}{\mu_{t+1}} c_{t+1} - c_t =& \mathcal{D}_{t+1} \mu_t \widetilde{a}_{t+1} - \mu_t \widetilde{a}_t - \mu_t \left(c_t + g_t + \tau^m_t - \tau^g_t - \frac{R_{t+1} - 1}{R_{t+1}} m_t \right) \\
=& \mathcal{D}_{t+1} \mu_t \widetilde{a}_{t+1} - \mu_t \left(c_t + \widetilde{a}_t + g_t + \tau^m_t - \tau^g_t - \frac{R_{t+1} - 1}{R_{t+1}} m_t \right)
\end{aligned}$$

Evaluating the government budget constraint at asset market equilibrium gives:

$$m_t + R^{-1}_{t+1} \widetilde{b}_t = \pi^{-1}_t \left(m_{t-1} + \widetilde{b}_{t-1} \right) + g_t + \tau^m_t - \tau^g_t = \widetilde{a}_t + g_t + \tau^m_t - \tau^g_t$$

which implies that

$$\begin{aligned}
\mathcal{D}_{t+1} \frac{\mu_t}{\mu_{t+1}} c_{t+1} - c_t =& \mathcal{D}_{t+1} \mu_t \widetilde{a}_{t+1} - \mu_t \left(c_t + m_t + R^{-1}_{t+1} \widetilde{b}_t - \frac{R_{t+1} - 1}{R_{t+1}} m_t \right) \\
=& \mathcal{D}_{t+1} \mu_t \pi^{-1}_{t+1} \left(\widetilde{b}_t + m_t \right) - \mu_t \left[c_t + R^{-1}_{t+1} \left(\widetilde{b}_t + m_t \right) \right] \\
=& \gamma R^{-1}_{t+1} \mu_t \left(\widetilde{b}_t + m_t \right) - \mu_t \left[c_t + R^{-1}_{t+1} \left(\widetilde{b}_t + m_t \right) \right] \\
=& \left(\gamma - 1 \right) R^{-1}_{t+1} \mu_t \left(\widetilde{b}_t + m_t \right) - \mu_t c_t
\end{aligned}$$

Re-arranging for c and using the definition of $\mathcal{D}_{t+1}$ gives:

$$c_t = \left(1 - \mu_t \right)^{-1} \left[\frac{\gamma \pi_{t+1}}{R_{t+1}} \frac{\mu_t}{\mu_{t+1}} c_{t+1} + \left(1 - \gamma \right) R^{-1}_{t+1} \mu_t \left(\widetilde{b}_t + m_t \right) \right]$$

Pinning down equilibrium in the money market, requires explicit assumptions about how the fiscal policy instruments τ^g and τ^m are determined. Specifically, τ_t^g is adjusted to finance government spending and the interest payments on debt required to hold the debt stock constant. That is:

$$\tau_t^g = g^* + \left(\pi_t^{-1} R_t - 1\right) b^*$$

which implies that τ^m is used to finance money creation:

$$\tau_t^m = m_t - \pi_t^{-1} m_{t-1}$$

When the cash in advance constraint binds, aggregating equation (2.51) and imposing market clearing gives:

$$m_t = w_t n_t + d_t - \tau_t^g = y_t - \frac{\Phi}{2}\left(\frac{\pi_t}{\pi^*} - 1\right)^2 - \tau_t^g = c_t + g_t - \tau_t^g$$

and imposing the rule for τ_t^g gives:

$$m_t = c_t - \left(\pi_t^{-1} R_t - 1\right) b^*$$

The model equations can be collected into two blocks. The first block of equations hold in all periods (assuming that the household budget constraint binds):

$$\begin{aligned}
y_t &= c_t + g^* + \frac{\Phi}{2}\left(\frac{\pi_t}{\pi^*} - 1\right)^2 \\
\frac{\Phi \pi_t}{y_t \pi^*}\left(\frac{\pi_t}{\pi^*} - 1\right) &= 1 - \eta + \eta \chi y_t^{\psi} c_t R_{t+1} + \frac{\Phi}{y_t}\frac{\pi_{t+1}}{R_{t+1}}\frac{\pi_{t+1}}{\pi^*}\left(\frac{\pi_{t+1}}{\pi^*} - 1\right) \\
c_t &= (1 - \mu_t)^{-1}\left[\frac{\gamma \pi_{t+1}}{R_{t+1}}\frac{\mu_t}{\mu_{t+1}} c_{t+1} + (1 - \gamma) R_{t+1}^{-1} \mu_t \left(R_{t+1} b^* + m_t\right)\right] \\
\mu_t^{-1} &= 1 + \gamma \beta \frac{\vartheta_{t+1}}{\vartheta_t} \mu_{t+1}^{-1} \\
\Delta \ln \vartheta_t &= \rho_\vartheta \Delta \ln \vartheta_{t-1} + \varepsilon_t^\vartheta
\end{aligned}$$

which provide solutions for the sequences $\{y_t, c_t, \pi_t, \mu_t, \vartheta_t\}_{t=0}^\infty$, conditional on $\left\{\varepsilon_t^\vartheta\right\}_{t=0}^\infty$ and solutions for the sequences $\{m_t, R_{t+1}\}_{t=0}^\infty$.

The second block of equations can be solved for the sequence $\{m_t, R_{t+1}\}_{t=0}^\infty$. There are two variants, depending on whether the monetary policy rule is constrained by the zero bound. If the monetary policy rule is unconstrained by the zero bound, then the interest rate is determined by the Taylor rule. In this case, the cash in advance constraint binds and this determines real money balances:

$$\begin{aligned} R_{t+1} &= R\left(\frac{\pi_t}{\pi^*}\right)^{\theta_\pi}\left(\frac{y_t}{y}\right)^{\theta_y} \\ m_t &= c_t - \left(\pi_t^{-1}R_t - 1\right)b^* \end{aligned}$$

Alternatively, the short-term bond rate may be constrained by the zero lower bound, in which case the cash in advance constraint does not bind and the quantity of real money balances is determined by the rule (2.53), which can be written as:

$$\begin{aligned} R_{t+1} &= 1 \\ \frac{m_t\pi_t}{m_{t-1}} &= \left(\frac{m_{t-1}\pi_{t-1}}{m_{t-2}}\right)^{\rho_m}(\pi^*)^{(1-\rho_m)}\exp\left(\varepsilon_t^m\right) \end{aligned}$$

2.C.7 Steady state

As for the baseline model, government policy is considered as exogenous, so the steady state is conditional on $g^* \in [0,1)$, $b^* > 0$ and π^*. The steady state is one in which the zero bound is not binding (by virtue of the inflation target being sufficiently high). In steady state, $\vartheta = 1$ and $\mu = 1 - \beta$. The monetary policy rule ensures that $\pi = \pi^*$.

The parameter χ is chosen to normalize steady-state output to unity ($y = 1$, so that $c = 1 - g^*$), which requires:

$$0 = 1 - \eta + \eta\chi\left(1 - g^*\right)R$$

or

$$\chi = \frac{\eta - 1}{\eta}\frac{1}{R\left(1 - g^*\right)}$$

In a steady state with a positive interest rate, the cash in advance constraint will bind, implying that

$$\begin{aligned} m &= c - \left(\left(\pi^*\right)^{-1}R - 1\right)b^* \\ &= 1 - g^* - \left(\left(\pi^*\right)^{-1}R - 1\right)b^* \end{aligned}$$

As for the baseline model, the calibration approach is to calibrate β with reference to a target rate of return on short-term bonds. Treating R as a 'parameter' therefore implies that steady-state real money balances are fully determined.

In steady state, the consumption equation implies:

$$c = (\gamma\beta)^{-1}\left[\frac{\gamma\pi^*}{R}c + (1-\gamma)\,R^{-1}\,(1-\gamma\beta)\,(Rb^* + m)\right]$$

so that

$$\gamma\beta = \frac{\gamma\pi^*}{R} + (1-\gamma)\,(1-\gamma\beta)\left(\frac{b^*}{1-g^*} + \frac{m}{R\,(1-g^*)}\right)$$

which implies that:

$$\beta = \frac{\frac{\gamma\pi^*}{R} + (1-\gamma)\left(\frac{b^*}{1-g^*} + \frac{m}{R(1-g^*)}\right)}{\gamma\left(1 + (1-\gamma)\left(\frac{b^*}{1-g^*} + \frac{m}{R(1-g^*)}\right)\right)}$$

Appendix 2.D Additively separable money demand

This variant of the model abstracts from transactions frictions and assumes that household derive utility from holding real money balances. The maximization problem is:

$$\max\sum_{t=0}^{\infty}(\gamma\beta)^t\,\vartheta_t\left[(1-\alpha)\ln c_{j,t} + \alpha\ln\left(\frac{M^p_{j,t}}{P_t}\right) - \frac{\chi_{j,t}}{1+\psi}n^{1+\psi}_{j,t}\right] \tag{2.55}$$

subject to:

$$\frac{M^p_{j,t}}{P_t} + \frac{B^p_{j,t}}{P_t} = \gamma^{-1}\left[\frac{R^M_t M^p_{j,t-1}}{P_t} + \frac{R_t B^p_{j,t-1}}{P_t}\right] + \tilde{w}_{j,t} - c_{j,t} \tag{2.56}$$

The first order conditions deliver an Euler equation for consumption:

$$c_{j,t+1} = \beta\frac{\vartheta_{t+1}}{\vartheta_t}\frac{R_{t+1}}{\pi_{t+1}}c_{j,t} \tag{2.57}$$

and a money demand function:

$$m^p_{j,t} = \frac{\alpha}{1-\alpha} \frac{R_{t+1}}{R_{t+1} - R^M_{t+1}} c_{j,t} \tag{2.58}$$

Making the same assumptions about the properties of $\chi_{j,t}$ as the baseline model gives rise to the following labor supply relationship:

$$\chi n^{\xi}_{j,t} = (1-\alpha) \frac{w_t}{c_t} \tag{2.59}$$

As in the baseline model, the intertemporal household budget constraint is:

$$a^p_{j,t} = \gamma \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(c_{j,t+i} - w_{t+i} n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i} + \frac{R_{t+i+1} - R^M_{t+i+1}}{R_{t+i+1}} m^p_{j,t+i} \right) \tag{2.60}$$

Substituting for the household's money demand equation gives:

$$a^p_{j,t} = \gamma \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(c_{j,t+i} - w_{t+i} n_{j,t+i} - d_{j,t+i} + \tau_{j,t+i} + \frac{\alpha}{1-\alpha} c_{j,t+i} \right)$$

which can be rearranged to:

$$(1-\alpha)^{-1} \sum_{i=0}^{\infty} \mathcal{D}_{t+i} c_{j,t+i} = \gamma^{-1} a^p_{j,t} + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{j,t+i} + d_{j,t+i} - \tau_{j,t+i} \right) \tag{2.61}$$

The Euler equation (2.57) implies that

$$c_{j,t+i} = (\gamma\beta)^i \, \mathcal{D}^{-1}_{t+i} \frac{\vartheta_{t+i}}{\vartheta_t} c_{j,t} \tag{2.62}$$

Using (2.62) allows (2.61) to be written in terms of current consumption:

$$c_{j,t} = (1-\alpha) \, \mu_t \left[\gamma^{-1} a^p_{j,t} + \sum_{i=0}^{\infty} \mathcal{D}_{t+i} \left(w_{t+i} n_{j,t+i} + d_{j,t+i} - \tau_{j,t+i} \right) \right] \tag{2.63}$$

where μ is the marginal propensity to consume from wealth, given by:

$$\mu_t = \left(\sum_{i=0}^{\infty} (\gamma\beta)^i \, \frac{\vartheta_{t+i}}{\vartheta_t} \right)^{-1} \tag{2.64}$$

which implies that:

$$\mu_t^{-1} = 1 + \gamma\beta \frac{\vartheta_{t+1}}{\vartheta_t} \mu_{t+1}^{-1} \tag{2.65}$$

The same aggregation and manipulation of the consumption function detailed in Appendix 2.B.2 for the baseline model deliver an aggregate consumption equation:

$$c_t = (1-\mu_t)^{-1} \frac{\gamma \pi_{t+1}}{R_{t+1}} \left[\frac{\mu_t}{\mu_{t+1}} c_{t+1} + (1-\alpha)\,\mu_t\,(1-\gamma)\,\gamma^{-1} a_{t+1} \right] \tag{2.66}$$

The aggregate pricing equation in this variant is given by:

$$\frac{\Phi \pi_t}{\pi^* y_t} \left(\frac{\pi_t}{\pi^*} - 1 \right) = 1 - \eta + \eta \frac{\chi c_t y_t^{\psi}}{1-\alpha} + \Phi \frac{\pi_{t+1}}{R_{t+1}} \frac{\pi_{t+1}}{\pi^* y_t} \left(\frac{\pi_{t+1}}{\pi^*} - 1 \right)$$

where the difference from the baseline model reflects the change in the labor supply relationship.

The flexible price economy is one in which there are no price adjustment costs $\Phi = 0$ and monetary policy is set to ensure that inflation is at target: $\pi_t = \pi^*, \forall t$.

Under these assumptions, the resource constraint and Phillips curve become:

$$y_t^f = c_t^f + g_t \tag{2.67}$$

$$0 = 1 - \eta + \eta \frac{\chi c_t^f \left(y_t^f \right)^{\psi}}{1-\alpha} \tag{2.68}$$

where the f superscript denotes a quantity from the flexible price economy. These two equations can be solved jointly for the levels of consumption and output that would prevail under flexible prices. It is immediate that preference shocks ϑ have no effect on flexible price output and consumption.

To ensure that consumption and output do not respond, the short-term nominal interest rate and (flexible price) money demand must adjust accordingly. The required movements can be found by (jointly) solving flexible price analogues of the consumption and money demand equations. In particular:

$$(1-\mu_t)\, c_t^f = \frac{\gamma \pi^*}{R_{t+1}^f} \frac{\mu_t}{\mu_{t+1}} c_{t+1}^f + (1-\alpha)\,\mu_t\,(1-\gamma) \left(R_{t+1}^f \right)^{-1} \left(m_t^f + R_{t+1}^f b^* \right)$$

$$m_t^f = \frac{\alpha}{1-\alpha} \frac{R_{t+1}^f}{R_{t+1}^f - R_{t+1}^M} c_t^f$$

where flexible price allocations are computed under the assumption that the flexible price return on money is the same as the actual return on money (R^M_{t+1}) and the bond stock is held fixed.[44]

[44]These assumptions can be relaxed at the expense of increasing the size of the flexible price block of the model.

Chapter 3

Flexible inflation targeting under fiscal dominance

Abstract

I study optimal time consistent monetary policy in a simple New Keynesian model with long-term government debt. Fiscal policy is 'active' so that stabilization of the government debt stock is a binding constraint on monetary policy. Away from the zero bound, optimal monetary policy does not fully offset the effects of shocks to the natural rate of interest, reducing welfare. At the lower bound, recessionary shocks increase debt and generate the anticipation of higher future inflation, to reduce real debt. These higher inflation expectations mitigate the effects of recessionary shocks. For sufficiently long debt duration, improved performance at the lower bound may outweigh the welfare losses in normal times.

'Central banks are often accused of being obsessed with inflation. This is untrue. If they are obsessed with anything, it is with fiscal policy.' Mervyn King, 1995.

3.1 Introduction

The interaction between monetary and fiscal policies has become an area of renewed focus following the financial crisis. In particular, there have been intense debates over the extent to which fiscal policy can or should support monetary policy when the latter is constrained by the zero lower bound. While a number of countries undertook discretionary fiscal stimulus in response to financial crisis, many of those policies were subsequently reversed. Concerns over rising government debt levels were often cited as a key motivation for these 'austerity' policies (Blyth, 2013).

My focus is on how optimal monetary policy is affected by the nature of fiscal policy behavior. Much of the literature on optimal monetary policy has assumed that the monetary policymaker can safely ignore fiscal policy developments. I explore the implications of relaxing this assumption. In doing so, this chapter contributes to a long and rich literature studying the interactions between monetary and fiscal policy, reviewed in Section 3.2.

My approach aims to explore the implications of two very stark assumptions about fiscal policy. In the textbook New Keynesian model, fiscal policy is 'passive' (Leeper, 1991). It is assumed that the government will always take appropriate action to satisfy its intertemporal budget constraint in response to any changes in financing costs or real debt generated by monetary policy decisions. So taxes and/or spending are adjusted to ensure that the real government debt stock is stabilized for any path of prices. This behavior means that the government's intertemporal budget constraint is irrelevant for the monetary policymaker.

In contrast, I study the polar opposite assumption: real tax revenues are held fixed. This means that fiscal policy is 'active' and monetary policy decisions must ensure that the real value of (nominal) government debt is stabilized. This policy configuration is sometimes called 'fiscal dominance'.

While stark, this assumption is a natural starting point for analyzing the implications of the so-called 'fiscal theory of monetary policy' (see Section 3.2.4) for *optimal* monetary policy behavior. The simplicity of the assumed fiscal policy behavior also enables me to derive some key results analytically. Moreover, this assumption has some real world relevance as it implies all government spending changes are 'unfunded', which has some parallels with recent fiscal policy behavior in the United States.

The analytical framework for my analysis is a textbook New Keynesian model (for example, Galí, 2008; Woodford, 2003) extended to include long-term nominal government debt. The presence of this debt has no implications for optimal monetary policy under the textbook assumption of passive fiscal policy.

However, as highlighted by Sims (2011) and Cochrane (2018b), under 'active' fiscal policy, the presence of long-term government debt allows the price level adjustments required to stabilize government debt to be spread out over time. Material movements in long-term bond prices can be achieved by relatively small, but persistent changes in the expected paths of inflation and short-term interest rates.

A loss function based on a quadratic approximation to household utility is used to assess the welfare implications of alternative variants of the model. Under the baseline assumption of active fiscal policy, monetary policy must be set in a way that minimizes welfare losses (expressed in terms of squared inflation and output gap deviations) subject to the additional constraint that the the real value of government debt is stabilized. Two parameterizations of the model are studied. The baseline parameterization sets the duration of government debt equal to an average across OECD countries. An alternative 'long duration' parameterization is based on data from the United Kingdom.

I study optimal time consistent policy in a log-linearized version of the model. I first ignore the lower bound on the short-term interest rate. This gives rise to a standard linear-quadratic optimal policy problem that can be studied analytically. I demonstrate that there is a unique stable Markov perfect equilibrium.

I show that the equilibrium behavior of the output gap and inflation is determined by a key coefficient: the equilibrium elasticity of the debt stock with respect to debt in the previous period. This elasticity increases with the duration of the government debt stock. As a result, the variant of the model with long-duration government debt exhibits smaller initial responses in the output gap and inflation to shocks, but the responses are more persistent. Achieving these outcomes typically requires larger movements in the short-term nominal interest rate.

The duration of government debt in the model therefore underpins the extent of the "debt stabilization bias" (Leith and Wren-Lewis, 2013; Leeper and Leith, 2016). Longer duration debt allows for a slower adjustment of debt to steady state following a shock and reduces the extent of fluctuations in the output gap and inflation required to stabilize the debt stock.

Two further results are evident by comparing the model to the textbook New

Keynesian model with passive fiscal policy. First, the so-called 'divine coincidence' no longer holds. In the textbook model, shocks that affect the economy through their effect on the natural rate of interest are perfectly stabilized under optimal time-consistent policy. In my model, this result no longer holds because the monetary policymaker must ensure that the government debt stock is stabilized. This additional constraint requires the policymaker to allow deviations of output from potential and inflation from target. These fluctuations generate welfare losses, so that active fiscal policy incurs higher welfare costs in response to these shocks.

The second result is that welfare losses generated by cost-push shocks – that generate a trade-off between stabilizing the output gap and inflation – may be *smaller* under active fiscal policy than under the textbook assumption of passive fiscal policy. This is because a positive cost-push shock that raises inflation in the near term reduces the real value of existing nominal government debt. Stabilizing the debt stock requires future policymakers to deliver lower inflation in the future. This reduces expected inflation in the near term and makes the trade-off between stabilizing the output gap and inflation more favorable. In equilibrium, the policymaker delivers smaller welfare losses in response to cost push shocks for both parameterizations of the model with active fiscal policy, relative to the textbook New Keynesian model.

The results from the linear quadratic analysis also provide intuition for the behavior of the model under optimal time-consistent policy in the presence of the zero lower bound on the short-term interest rate. The non-linearity induced by the zero bound requires the model to be solved by projection methods.

Welfare losses are *smaller* in the variant of the model with long duration government debt than the textbook New Keynesian model with passive fiscal policy. This result is driven by the balance between two effects. Away from the zero bound, welfare losses are larger under active fiscal policy, since the 'divine coincidence' result of the textbook model described above does not hold. However, when the economy is constrained by the zero bound, the combination of active fiscal policy and long-duration debt reduces welfare losses. This is because deflationary shocks that drive the policy rate to the zero bound raise the real value of government debt. This requires future policymakers to generate higher inflation to stabilize the debt stock, which increases inflation expectations. Higher inflation expectations at the zero bound reduces the real interest rate, stimulating spending and offsetting the recessionary effects of the deflationary shock.

For the long duration parameterization, the reduction in welfare losses at the zero bound is substantial enough to offset the larger losses away from the zero bound (relative to the textbook model). However, for the baseline parameterization, the improvement in outcomes at the zero bound is not sufficient to offset poorer performance in normal times.

The equilibrium distribution of government debt plays an important role in delivering these results. For both the baseline model and long duration variant, the average debt stock is above the deterministic steady state. This is because the presence of the zero bound limits the degree to which monetary policy is able to offset the effects of deflationary shocks that increase the real value of debt. However, the mean of the distribution of government debt is much higher for the long duration parameterization than for the baseline model.[1] The higher average debt stock means that, on average, agents expect higher future inflation to return debt to the steady state when the zero bound is encountered. The increase in average inflation expectations helps to cushion the effect of the zero bound on expected real interest rates, limiting the scale of recessions and generating a substantial reduction in welfare losses.

Finally, I consider the effects of a *risk* that fiscal policy becomes active during a debt reduction scenario. To do so, I use a variant of the model in which fiscal policy behavior is characterized by empirically-motivated fiscal rules. I first demonstrate that, under optimal monetary policy, a debt reduction program amounting to 10pp of annual GDP can generate large inflationary effects when the fiscal policy rule is active. In the textbook New Keynesian model with passive fiscal policy, the same debt reduction program has no effects on the output gap or inflation.

This scenario is extended to consider the case in which fiscal policy is initially

[1]The debt stock is more persistent under the long duration parameterization, so that rises in the real value of debt caused by low inflation outcomes at the zero bound are more persistent. In contrast, falls in the real value of debt generated by high inflation outcomes are moderated by a more aggressive (i.e., unrestricted) policy tightening.

passive, but there is a risk that it will become active in future periods. I develop a solution algorithm to solve for optimal policy in the presence of a time-varying probability of switching permanently to an active fiscal policy rule in each period.

I first consider the case in which the probability of a switch to active fiscal policy is exogenous and constant over time. Even when the probability is small enough that it is most likely that fiscal policy will remain passive, a debt reduction program in the presence of such fiscal risk generates an expectation that inflation will overshoot the target. This is because a switch to active fiscal policy will generate a large increase in inflation. Even though the probability of a switch to active fiscal policy is small, the rise in inflation in those states of the world is large enough to increase expected inflation by an economically meaningful amount.

For the baseline model, even a small probability of a switch to active fiscal policy induces the monetary policymaker to accommodate some of the rise in expected inflation and to allow actual inflation to increase. It is optimal to do so because this inflation erodes the real value of debt and this reduces the losses incurred in states of the world in which the fiscal policy rule becomes active.

However, for the long duration debt variant, the upward pressure on inflation expectations acts like a cost push shock, increasing the rate of inflation consistent with a closed output gap. The optimal response is to tighten monetary policy and trade off a smaller rise in inflation with a slightly negative output gap. In this case, the optimal monetary policy response to the rise in inflation expectations involves tighter monetary policy than would be warranted in the absence of the risk. This result is driven by the fact that output gap movements are much more persistent in the long-duration debt variant. Thus expected positive output gaps in the distant future are sufficient to increase inflation today, even if the output gap is negative today. As such, the rise in inflation expectations more closely resembles a cost-push disturbance.

I next consider the case in which the probability of a switch to active fiscal policy is endogenous. Inspired by the literature on fiscal limits (for example Davig, Leeper, and Walker, 2011; Bi, 2012; Leeper, 2013), the probability of a switch to active fiscal policy is an increasing function of the primary surplus. One interpretation of this assumption is that higher primary surpluses entail a greater financial burden on households, making it more likely that the fiscal authority will abandon its use of taxation as the primary mechanism through which the level of debt is reduced. The solution algorithm is extended to allow the monetary policymaker to account for the effects of their actions on future probabilities of a switch to active fiscal policy.

I first study an example constructed so that the *ex ante* probability of a switch to active fiscal policy in some future period is the same as the exogenous risk scenario. Relative to the case of exogenous risk, the responses of output and inflation are more 'front loaded', since a switch to active fiscal policy is (relatively) more likely when the primary surplus is higher, which occurs in the earlier part of the simulation. This endogenous profile of fiscal risk implies that the switch to active fiscal policy is relatively high in the first half of the debt reduction program, when the debt stock is still relatively high. Since the inflationary impact of a switch to the active fiscal rule is larger when the debt stock is high, inflation expectations are higher than the exogenous uncertainty example. This makes the stabilization problem faced by the monetary policymaker more difficult and a larger deviation of the output gap and inflation are (optimally) accommodated.

I examine the role of time consistency in generating the observed behavior under optimal monetary policy. Welfare losses are larger under time consistent policy relative to a counterfactual case in which the monetary policymaker and private sector make the decisions that would be optimal if there was no risk. This is the case even though ignoring the risk generates a higher path for the primary surplus (and hence a greater probability of a switch to active fiscal policy) and also increases the welfare losses incurred when a switch does occur.

This result shows that the welfare cost of allowing some inflation along the most likely path of the economy is not outweighed by the benefits of risk reduction because monetary policy is constrained to be time consistent. It is not possible to coordinate on a set of beliefs that would deliver the outcomes if policymakers and the private sector ignored the risk. Unilaterally deviating to behave in this way generates higher welfare losses than the time consistent policy.

The rest of this chapter is structured as follows. Section 3.2 provides a brief literature review, discussing the fiscal theory of the price level, the 'fiscal theory of monetary policy' and jointly optimal monetary and fiscal policies. Section 3.3 sets out the model, welfare-based loss function and the baseline and 'long duration' parameterizations. Section 3.4 presents the analysis of optimal time-consistent policy, assuming that there is no lower bound on the short-term interest rate. Section 3.5 examines optimal time-consistent policy when the zero lower bound on the short-term interest rate is accounted for. Section 3.6 extends the model to incorporate empirically-motivated fiscal policy rules and studies the optimal monetary policy responses to a debt reduction scenario. That model is used to consider the effects of fiscal risk: the effects of the possibility that the fiscal policy rule may switch to become 'active'. Section 3.7 concludes.

3.2 Literature review

The literature on the interactions between monetary and fiscal policies is vast. This section aims to highlight some of the key contributions to that literature of particular relevance for the focus of this chapter.

3.2.1 Government debt accumulation, sustainability and stabilization

Government debt is central to the questions studied in this chapter. To fix ideas and concepts, consider a simple model of government finance. Suppose the government finances real government spending, g, using a combination of real (lump sum) taxes τ and one period nominal bonds, B. The government's debt accumulation in period t is given by:

$$B_t = R_{t-1}B_{t-1} + P_t\left(g_t - \tau_t\right) \tag{3.1}$$

where R_t is the nominal return on one-period bonds (payable in period $t+1$) and P_t is the price level.

Equation (3.1) is an extremely stylized representation of government debt accumulation. In particular, it assumes that all government liabilities are in the form of one period bonds, thus abstracting from base money, longer-term bonds and index-linked debt. It also abstracts from economic growth, which has been a key element of the fiscal policy debate in recent years, given the slow recoveries from the global financial crisis in many economies.[2] Nevertheless, this representation contains the key elements of government financing required for the following discussion. In particular, the evolution of government bonds will depend on the behavior of taxes, government spending, the short-term nominal interest rate and the price level.

An important question in international public policy is whether government debt stocks are sustainable. That is, conditional on the likely paths of taxes, spending, interest rates and inflation, will government debt (relative to GDP) stabilize, shrink or grow without bound? Indeed, a number of international institutions (for example, the International Monetary Fund and European Commission) conduct regular assessments of debt sustainability.

Importantly, the concept of debt sustainability encompasses a much broader range of behavior of debt (and its determinants) than the concept of *debt stabilization* used in much of the literature on monetary and fiscal policy (and in this chapter). To see this, iterate (3.1) forward to give:

$$\frac{R_{t-1}B_{t-1}}{P_t} = \mathbb{E}_t \sum_{s=0}^{\infty} \widetilde{R}_{t,t+s}^{-1} \left(\tau_{t+s} - G_{t+s}\right) + \mathbb{E}_t \lim_{s\to\infty} \widetilde{R}_{t,t+s}^{-1} \frac{B_{t+s+1}}{P_t}$$

where $\widetilde{R}_{t,t+s}$ represents the compounded real return between periods t and $t+s$:

$$\widetilde{R}_{t,t+s} \equiv \prod_{j=1}^{s} \frac{R_{t+j}P_{t+j-1}}{P_{t+j}} = \frac{R_{t+s}P_{t+s-1}}{P_{t+s}} \widetilde{R}_{t,t+s-1}$$

with $\widetilde{R}_{t,t} = 1$ and $\mathbb{E}_t$ represents the expectations operator, conditional on information available in period t.

[2] A key question for debt sustainability is whether the average real interest rate is higher or lower than the trend real growth rate of the economy. Since the financial crisis, both real interest rates and real growth rates have been low by historical standards. If the average real interest rate is lower than the trend real growth rate, then it may be possible for government debt to have positive value even if the government runs primary deficits forever. Indeed Blanchard (2019) argues that this constellation of interest rates and growth rates may be the norm. I abstract from this possibility in this chapter, adopting the conventional assumption that the average real interest rate is positive (and determined by the inverse of the household discount factor) and hence strictly greater than the trend growth rate of zero.

If the real debt stock grows more slowly than the real interest rate, the expected discounted value of the terminal debt stock is zero:

$$\mathbb{E}_t \lim_{s \to \infty} \widetilde{R}_{t,t+s}^{-1} \frac{B_{t+s+1}}{P_t} = 0 \tag{3.2}$$

and the present-valued government budget constraint becomes:

$$\frac{R_{t-1}B_{t-1}}{P_t} = \mathbb{E}_t \sum_{s=0}^{\infty} \widetilde{R}_{t,t+s}^{-1} \left(\tau_{t+s} - G_{t+s}\right) \tag{3.3}$$

Bohn (2007) studies the restrictions on spending, taxes and government debt that are consistent with (3.2) and (hence) the intertemporal budget constraint (3.3). Perhaps surprisingly, he demonstrates that the transversality condition is satisfied for a wide variety of non-stationary debt processes. Specifically, an integrated process for B_t of any (finite) order will satisfy (3.2).[3] Bohn (2007) uses this result to argue that it is thus impossible to infer whether government debt is sustainable by using unit root or cointegration tests.

Following the existing literature (reviewed below), I use a more restricted concept that fiscal and/or monetary policy must ensure *debt stabilization*. This requires the real value of the debt stock to be stationary and specifically to be returned to a fixed steady-state target level of debt following a disturbance. Indeed, given the difficulty of identifying sustainability from the stochastic properties of government debt, Bohn (2007) argues for a similar criterion.[4]

A key focus of the debt sustainability analyses mentioned above is the likelihood that a government may choose to default on its debt. Allowing for default gives rise to a richer set of interactions between monetary and fiscal policy (where the latter includes a decision on whether or not to default). For example, Uribe (2006)

[3] The result is driven by the fact that the discounting of future debt stocks in (3.2) is exponential. In contrast, the expectation of an m-th order integrated stochastic processes at date t can be written as an m-th order polynomial of t. The exponential rate at which the discount factor shrinks dominates the polynomial growth of expected debt.

[4] "A second strategy [to assess debt sustainability] is to consider stronger conditions on policy, e.g., upper bounds on debt motivated by a limited capacity to service debt. Then *stationarity in levels is the most relevant econometric condition*, and additional restrictions may apply." Bohn (2007, p1846, emphasis added).

presents a model in which strict adherence to an inflation target will ultimately drive the government to default if government finances are unsustainable. Bi, Leeper, and Leith (2018) examine the interplay of debt sustainability and default in a richer setting.

In this chapter, I rule out default. It is assumed that the government will always repay its debt. This focuses attention on the additional constraints that fiscal sustainability may place on the optimal conduct of monetary policy.

3.2.2 Monetary and fiscal policy configurations

It has been long understood that monetary and fiscal policies are intertwined. This fact is evident from consideration of the government debt accumulation equation (3.1). Tax and spending decisions affect the primary surplus. Nominal interest rate decisions affect the cost of issuing nominal liabilities. The rate of inflation affects the real value of nominal government liabilities. This suggests that not all types of monetary and fiscal policy behavior will be compatible with stabilization of real government debt and/or inflation.

Leeper (1991) confirms this logic and develops a taxonomy of policy configurations that are consistent with stable real government debt and determinate inflation. Leeper labels monetary and fiscal policies as 'passive' or 'active' depending on whether or not they are constrained to respond to the level of (or disturbances to) real government debt.

Two of the four possible policy configurations are compatible with real government stabilization and determinate inflation. In an 'active monetary, passive fiscal' (AMPF) configuration, monetary policy stabilizes inflation without regard to government debt. In this configuration, fiscal policy passively adjusts taxes and/or spending to ensure that the real government debt stock is stabilized. In contrast, a 'passive monetary, active fiscal' (PMAF) configuration is one in which the response of taxes and spending to changes in the real value of government debt is insufficiently strong to ensure that the real debt stock is stabilized. In this case, monetary policy must passively adjust the short-term interest rate to ensure that the real government debt stock is stabilized.[5]

Leeper's taxonomy of policy configurations has shaped much of the subsequent research on monetary and fiscal policy interactions.[6] Woodford (2001, 2003) distinguishes between 'Ricardian' and 'non-Ricardian' fiscal policies, which correspond to Leeper's passive and active specifications respectively. Although Woodford's terminology is well known, I use Leeper's active/passive distinction in the remainder of this chapter.

A passive monetary, active fiscal (PMAF) regime is sometimes referred to as 'fiscal dominance', reflecting the notion that the monetary policymaker must ensure that their policy actions are consistent with stabilization of the real government debt stock. This terminology has a long history: for example, (Sargent and Wallace, 1981, p2) and (King and Plosser, 1985, p172) discuss "[cases in which] fiscal policy dominates monetary policy" and "fiscal dominance" respectively.

3.2.3 Monetary and fiscal policies and the 'consensus assignment'

Much of the 'New Keynesian' literature on optimal monetary policy assumes that monetary policy is active and fiscal policy is passive. This is, for example, the baseline assumption for the textbook treatments of Woodford (2003) and Galí (2008).[7]

[5]If both policies are passive (PMPF), then both fiscal and monetary policymakers attempt to adjust their instrument to ensure that real government debt is stabilized. There are many combinations of policy settings that are consistent with real debt stabilization. However, these correspond to different paths of the price level, rendering the inflation rate indeterminate. If both policies are active (AMAF), then real government debt is not stabilized, since neither policymaker is constrained to adjust their policy instrument to ensure that it is. See Leeper (1991, p. 138–139).

[6]While this general taxonomy applies to a wide range of macroeconomic models, the details may be model specific. For many models (as in Leeper (1991)) a simple monetary policy will be active if the coefficient on inflation in a Taylor (1993) rule is greater than unity and fiscal policy will be passive if the elasticity of the primary surplus with respect to the outstanding debt stock is greater than the steady-state real interest rate. However, different restrictions may determine regions of active and passive policy in other models.

[7]Woodford (2003, Chapter 4.4) does analyze the implications of Ricardian and non-Ricardian (passive and active) fiscal regimes.

This focus can be justified by appealing to the institutional arrangements in most economies in the years preceding the financial crisis. Central banks were typically assigned responsibility for price stability (often in the form of an inflation target). Fiscal authorities were responsible for controlling public debt, often with explicit targets for the government debt to GDP ratio. Kirsanova, Leith, and Wren-Lewis (2009) refer to this policy configuration as the "consensus assignment".

The financial crisis called the consensus assignment into question. With short-term policy rates forced to their effective lower bounds, monetary policy was unable to respond to weak activity and inflation in the conventional manner. This prompted consideration of whether an alternative policy configuration may be appropriate. Much of this debate hinged on the extent to which the effects of government spending on activity are larger than normal when economic activity is weak and/or monetary policy is constrained by the zero bound (Christiano, Eichenbaum, and Rebelo, 2011; Auerbach and Gorodnichenko, 2012; DeLong and Summers, 2012).

3.2.4 Fiscal theories

Leeper's taxonomy of policy configurations implies that a combination of passive monetary policy and active fiscal policy can deliver determinate inflation and stabilize the real value of government debt. This policy configuration has been explored in a strand of literature sometimes labeled as the 'fiscal theory of the price level' (FTPL).

Early expositions of the FTPL were often intended to to explain the idea clearly (see, for example, Sims, 1994). These contributions often used a variant of equation (3.3), which says that the real value of outstanding one period bonds must be equal to the expected discounted value of real primary surpluses (tax revenue less government spending). Under the simplifying assumption that prices are fully flexible, the real discount factor $\widetilde{R}^{-1}_{t,t+s}$ is independent of the price level.[8] If spending and taxes are exogenous, the only way that (3.3) can be satisfied in response to disturbances to current and future taxes and spending is for the price level P_t to adjust.

[8]In a model with an optimizing representative household, the real discount factor will be pinned down by disturbances that determine the household's intertemporal marginal rate of substitution.

The result that the price level must adjust to implement the government's transversality condition (3.2) is both striking and difficult to reconcile with the real-world institutional arrangements consistent with the consensus assignment.

However, some of the results from early FTPL models are implications of the specific assumptions employed. Sims (2011) demonstrates that the stark behavior of the price level is driven by the (unrealistic) assumption that all government debt is in the form of one period nominal bonds. Extending the model to include long-term government debt means that the government's intertemporal budget constraint is satisfied by adjustments in the entire sequence of price levels (which are important determinants of the price of long-dated debt), rather than just the current price level.

Cochrane (2018b) extends Sims's model further and demonstrates that striking neo-Fisherian results – that inflation rises in response to exogenous increases in the short-term interest rate – stem from the assumption of flexible prices. Incorporating standard sticky price assumptions (for example, Calvo (1983) or Rotemberg (1982) pricing) delivers more conventional results in response to temporary changes in the short-term nominal interest rate. Cochrane (2018b) argues that incorporating these features delivers a fiscal theory of *monetary policy* (FTMP). Cochrane (2018a) argues that such a model provides a more convincing explanation of macroeconomic dynamics in the recent (post-crisis) period. In particular, zero lower bound episodes are not as costly as 'New Keynesian' (active monetary, passive fiscal) models would suggest and prolonged interest rate 'pegs' have not generated indeterminacy in inflation rates.

3.2.5 A new consensus?

The development of the FTMP may could be interpreted as a convergence of approaches that had previously been regarded as mutually exclusive. The inclusion of New Keynesian modeling features into models with a FTPL focus (in particular the assumption of a PMAF policy configuration) allows for a closer comparison of alternative models. However, a key implication of this line of work is that the important implications for macroeconomic variables stem primarily from the assumption of the policy configuration (i.e., AMPF versus PMAF). The remaining details of the model (for example, the way that aggregate demand and inflation are determined) are not critical features of the debate between 'New Keynesians' and 'fiscal theorists'.[9] So 'fiscal theorists' do not necessarily assume that prices are flexible or that demand is determined by optimizing agents with rational expectations (see, for example, Sims, 2016).

This narrative of convergence between fiscal theorists and New Keynesians may appear strange to those studying jointly optimal monetary and fiscal policies. In that case, monetary and fiscal policies are chosen jointly to maximize welfare, subject to the constraint imposed by the structure of the economy. The optimal policy configuration emerges endogenously and may depend on the nature of the instruments available to the policymaker. For example, Kirsanova et al. (2009) demonstrate that jointly optimal monetary and fiscal policy in their New Keynesian model typically leads to a consensus assignment (i.e., an AMPF configuration). More generally, surveys of the state of the art in monetary and fiscal policy coordination discuss at length the importance of the monetary/fiscal policy configuration (see, for example, Canzoneri, Cumby, and Diba, 2010; Leith and Wren-Lewis, 2013).

3.3 Model

As in Chapter 1, the government issues both long-term and short-term bonds. However, in this chapter I abstract from portfolio frictions in government bond markets. This section describes the key elements of the model structure. A detailed derivation is provided in Appendix 3.A.

3.3.1 Households

The representative household maximizes a utility function defined over consumption, c, and hours worked, n, subject to a budget constraint that defines how proceeds

[9] One exception is the emphasis on long-term government debt by advocates of the fiscal theory (evidenced in the work of John Cochrane and Chris Sims discussed in Section 3.2.4). In a textbook New Keynesian model with an AMPF policy configuration and lump sum taxation, the maturity structure of government debt has no implications for macroeconomic outcomes.

from wage and profit income, net of taxes are allocated to short-term and long-term government bonds.

The optimization problem is

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left\{ \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right\}$$

subject to

$$V_t D_t + B_t = (\varrho + \chi V_t) D_{t-1} + R_{t-1} B_{t-1} + W_t n_t - T_t + F_t - P_t c_t \tag{3.4}$$

where P is the price of consumption, W is the nominal wage, T is a lump sum tax and F represents dividend payments from firms. The household may invest in one period government bonds B or long-term government debt, D. To keep the model as close to the textbook New Keynesian benchmark, I abstract from money and consider a 'cashless limit' economy, following Woodford (2003).[10]

As in Chapter 1, long-term debt is a security that pays a sequence of nominal coupons that decay geometrically at rate $\chi < 1$.[11] The nominal value of a newly issued bond at date t is V_t and such a bond pays a coupon stream of $\varrho, \varrho\chi, \varrho\chi^2, \ldots$ in periods $t+1, t+2, t+3, \ldots$. The importance of the initial coupon $\varrho > 0$ is discussed in Section 3.3.3.

Preferences are subject to an exogenous shock, ϕ_t which follows the process

$$\ln \phi_t = \rho_\phi \ln \phi_{t-1} + \sigma_\phi \varepsilon_t^\phi \tag{3.5}$$

where ε_t^ϕ is an iid normally-distributed shock with unit variance.

[10] Early explorations of fiscal dominance studied the implications for money growth and seigniorage (see, for example, Sargent and Wallace, 1981). More recent treatments have abstracted from money as government debt represents the vast majority of outstanding government liabilities in most countries (see, for example, Cochrane, 2018b).

[11] A key benefit of this setup is that the value of a bond issued at date $t-j$ is equal to $\chi^j V_t$ so that holdings of all previously issued bonds can be summarized in terms of an equivalent quantity of newly issued bonds, simplifying aggregation.

3.3.2 Firms

There is a set of monopolistically competitive producers indexed by $j \in (0,1)$ that produce differentiated products that form a Dixit-Stiglitz bundle that is purchased by households. Preferences over differentiated products are given by

$$y_t = \left[\int_0^1 y_{j,t}^{1-\eta_t^{-1}} dj \right]^{\frac{1}{1-\eta_t^{-1}}}$$

where y_j is firm j's output and the elasticity of demand η_t varies over time according to

$$\ln \eta_t - \ln \eta = \rho_\eta \left(\ln \eta_{t-1} - \ln \eta \right) + \sigma_\eta \varepsilon_t^\eta \tag{3.6}$$

Firms produce using a constant returns production function in the single input (labor):

$$y_{j,t} = A_t n_{j,t}$$

where A_t is an exogenous productivity process that follows:

$$\ln A_t - \ln A = \rho_A \left(\ln A_{t-1} - \ln A \right) + \sigma_A \varepsilon_t^A \tag{3.7}$$

Firms set prices according to the a Calvo (1983) staggered pricing scheme, with a probability $1-\alpha$ of changing price each period. A fixed production subsidy ensures that the steady state is efficient.

3.3.3 Government

The nominal government flow budget constraint is:

$$B_t + V_t D_t = R_{t-1} B_{t-1} + \left(\varrho + \chi V_t \right) D_{t-1} + G_t - T_t$$

To focus on the case in which the government issues only long-term debt, I first assume that short-term and long-term debt move in fixed proportions:

$$B_t = \delta^{-1} D_t$$

for $\delta > 1$ and then take the limit as $\delta \to \infty$. This gives:

$$V_t D_t = (\varrho + \chi V_t)\, D_{t-1} + G_t - T_t$$

which can also be written in terms of the primary surplus, $S \equiv T - G$:

$$V_t D_t = (\varrho + \chi V_t)\, D_{t-1} - S_t$$

Denoting real quantities with lower case letters means that the real-valued government budget constraint is :

$$V_t d_t = (\varrho + \chi V_t)\, \pi_t^{-1} d_{t-1} - s_t \tag{3.8}$$

Appendix 3.A.1 demonstrates that the steady-state value of price of debt, V, is equal to unity if the initial coupon satisfies $\varrho = \beta^{-1} - \chi$. Invoking this assumption allows me to interpret d as the (real) par value of long-term debt. This is useful for calibration purposes, since most empirical data on government debt stocks are measured at par, rather than market value.

The baseline assumption for fiscal policy is that lump sum taxes are held fixed at $\tau_t = \tau$ in real terms (with $\tau_t \equiv T_t/P_t$). This means that the real primary surplus, s is determined entirely by movements in real government spending, $g\,(\equiv G/P)$.[12] Real government spending is assumed to evolve according to a simple exogenous process around its long-run steady state level, $g\,(< \tau)$:

$$\bar{g}_t = \rho_g \bar{g}_{t-1} + \varepsilon_t^g \tag{3.9}$$

where $\bar{g}_t \equiv g_t - g$ denotes the linear deviation of spending from steady state (rather than the log-deviation).

[12] The experiments in Section 3.6 use empirically motivated rules for the primary surplus.

3.3.4 Log-linearized model

Appendix 3.A contains the derivation of the log-linearized approximation of the model around the efficient steady state. The log-linear model equations are:

$$\hat{x}_t = \mathbb{E}_t\hat{x}_{t+1} - \widetilde{\sigma}\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} - r_t^*\right] \tag{3.10}$$

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t \tag{3.11}$$

$$\hat{d}_t = \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) - (1-\chi)\hat{V}_t + \zeta^{-1}\bar{g}_t \tag{3.12}$$

$$\hat{V}_t = -\hat{R}_t + \chi\beta\mathbb{E}_t\hat{V}_{t+1} \tag{3.13}$$

where ζ is the steady-state ratio of government debt to output $(\frac{d}{y})$ and the parameters $\widetilde{\sigma}$ and κ satisfy:

$$\widetilde{\sigma} \equiv \sigma(1-g)$$
$$\kappa \equiv \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi + \widetilde{\sigma}^{-1}\right)$$

The natural real interest rate, r^*, and cost-push shock, u_t, are given by:

$$r_t^* = \mathbb{E}_t\left[-\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right) + \frac{1+\psi}{1+\psi\widetilde{\sigma}}\left(\hat{A}_{t+1} - \hat{A}_t\right) - \frac{\psi}{1+\psi\widetilde{\sigma}}\left(\bar{g}_{t+1} - \bar{g}_t\right)\right]$$
$$u_t = -\frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\frac{\eta}{\eta-1}\hat{\eta}_t$$

Equations (3.10) and (3.11) are the familiar New Keynesian IS and Phillips curves (Galí, 2008; Woodford, 2003). Equation (3.12) is the government debt accumulation equation. Equation (3.13) is a log-linearized version of the no-arbitrage condition between long-term and short-term bonds.

When fiscal policy is active, the government budget constraint is a constraint on monetary policy. Variations in the monetary policy instrument (the short-term bond rate $\hat{R}$) influence the evolution of long-term debt via their effects on the price of long-term debt ($\hat{V}$) and inflation. Monetary policy must be set so that the government debt stock is stabilized.

3.3.5 Welfare-based loss function

Appendix 3.B derives a loss function based on a second-order approximation to household utility. The model structure is the very similar to that considered in Chapter 1, but without portfolio adjustment frictions. The resulting welfare-based loss function is therefore identical to that considered in Chapter 1, but without portfolio adjustment costs (that is, $\tilde{\nu} = \tilde{\xi} = 0$) and accounting for the presence of government spending (by replacing σ with $\widetilde{\sigma}$):

$$\mathcal{L}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2\right] \tag{3.14}$$

where

$$\omega = \kappa \eta^{-1}$$

This result means that, relative to the standard New Keynesian assumption of passive fiscal policy, active fiscal policy affects the constraints upon the monetary policymaker, but not their objectives. Following Vestin (2006), I interpret the time-consistent pursuit of (3.14) as representing 'flexible inflation targeting'.

3.3.6 A 'textbook New Keynesian' benchmark

A textbook New Keynesian model is a natural benchmark against which to assess the implications of active fiscal policy. In the textbook model, a fiscal policy reaction function ensures that primary surpluses are adjusted to ensure that the trajectory for government debt consistent with (3.12) does not explode. The fiscal solvency condition is satisfied for any path of the price level, regardless of the actions of the monetary policymaker. Since government spending is exogenous, passive fiscal policy requires that lump sum taxes adjust to ensure that the intertemporal government budget constraint is satisfied.

In the New Keynesian benchmark model, then, only the IS curve (3.10) and Phillips curve (3.11) are constraints on the monetary policymaker. Indeed, in the absence of the zero lower bound, the IS curve is not a binding constraint and, in this case, the optimal time-consistent monetary policy delivers the following targeting rule

$$\omega \hat{x}_t + \kappa \hat{\pi}_t = 0 \tag{3.15}$$

as will be demonstrated below (see also, Galí, 2008; Woodford, 2003).

Under passive fiscal policy, the precise formulation of the fiscal reaction function for the lump sum tax rate does not affect equilibrium outcomes for the output gap and inflation. However, for the purposes of comparison, I assume that the lump sum tax rate is adjusted to hold the stock of government debt constant at its steady-state level at all times: $\hat{d}_t = 0, \forall t$.

3.3.7 Parameter values

Table 3.1 shows the baseline values used in the experiments in the remainder of the chapter. Given the similarity of the model to the one studied in Chapters 1 and 2, the values for parameters that appear in those model are set with reference to the same sources and motivations. The parameters governing the persistence of the exogenous shocks (ρ_a, ρ_g, ρ_ϕ) are set equal to the posterior mean estimates of the analogous shocks in Burgess, Fernandez-Corugedo, Groth, Harrison, Monti, Theodoridis, and Waldron (2013) and Del Negro, Giannoni, and Schorfheide (2015b) as appropriate. The persistence of the markup shock ρ_η is set to 0, given that Burgess et al. (2013) assume that markup shocks are white noise.

The parameter χ is important for the present study as it determines the maturity of government debt in the model. I consider two values for this parameter using the OECD data on the Macaulay duration of domestic government debt shown in Table 3.2. The table shows that the United Kingdom is a clear outlier, issuing longer-term debt relative to the rest of the sample (Ellison and Scott (2017) show that this is a long-standing feature of UK debt issuance). Indeed, data from the UK debt management office (DMO) suggests that the duration of UK government debt may be even longer than implied by Table 3.2.[13]

[13]The DMO report estimates of modified duration. While Macaulay duration and modified duration are different concepts – measuring the weighted average *time in years* until repayment

Table 3.1: Baseline parameter values

	Value	Source/motivation
σ	1	Log utility (as in Chapter 1)
β	0.9926	Steady-state annual real interest rate $\approx 3\%$
g	0.2	Sims and Wolff (2013)
ζ	2	Reinhart et al. (2012) (advanced economies, pre-crisis)
η	7.88	Rotemberg and Woodford (1997)
α	0.855	Implies $\kappa \approx 0.05$ (as in Chapter 1)
ψ	0.55	Smets and Wouters (2007)
ρ_η	0	Burgess et al. (2013)
ρ_g	0.91	Burgess et al. (2013) (ρ_G)
ρ_A	0.96	Del Negro et al. (2015b) (ρ_z)
ρ_ϕ	0.71	Burgess et al. (2013) (ρ_B)
χ	0.945 ($\equiv \bar{\chi}$)	'Average duration' variant (see text)
	0.976 ($\equiv \chi_L$)	'Long duration' variant (see text)

Table 3.2: Macaulay duration of domestic government debt, selected countries

Country	Sample	Average	Minimum	Maximum
Austria	2000–2010	5.5	4.1	7
Denmark	2000–2010	4.7	3.7	7.3
Finland	2000–2010	2.8	2.4	3.5
France	2001–2004	4.5	4.3	4.8
Hungary	2000–2010	2.3	1.4	2.8
Italy	2000–2010	4.2	3.4	4.9
Norway	2000–2010	3.0	1.9	3.5
Spain	2000–2010	4.7	3.9	5.2
Sweden	2000–2005	2.8	2.7	3.1
United Kingdom	2000–2010	8.0	6.9	9.0
United States	2000–2010	3.5	3.4	4.0

Macaulay duration is measured in years. *Source:* OECD statistics library (https://stats.oecd.org/). Downloaded 24 November 2018.

and the *percentage* change in price for a unit change in yield respectively – their numerical values are usually similar. The DMO report average modified duration of conventional debt equal to around 11%, suggesting a similar Macaulay duration in years. Data for 2018Q4 were downloaded from https://www.dmo.gov.uk/data/.

The ‘long duration’ calibration for χ is based on the UK case and is selected to deliver a Macaulay duration of eight years. The ‘average duration’ calibration is selected to deliver a Macaulay duration of four years, slightly above the cross-country average (excluding the United Kingdom) of 3.8 years from Table 3.2. Appendix 1.D (equation (1.44)) shows that the steady-state Macaulay duration is given by $(1-\beta\chi)^{-1}$. So if the desired Macaulay duration is M years, the required value of χ is given by $\chi = \beta^{-1}\left(1-(4M)^{-1}\right)$, which incorporates the fact that each time period in the model is one quarter.

3.4 Time-consistent policy without a zero bound

In this section, I examine the behavior of the model under time-consistent optimal monetary policy. I focus on time-consistent optimal policy for two reasons. First, optimal commitment policy in this class of models tends to generate an extreme form of time inconsistency. Appendix 3.E demonstrates that the optimal commitment policy implies that government debt follows a random walk allowing the policymaker to increase welfare in the near term by inducing permanent movements in debt, output and inflation (Leith and Wren-Lewis, 2013, demonstrate similar results in a similar model). Second, there is evidence that monetary policymakers have doubts over their ability to credibly pre-commit to future policy actions (see, for example, Nakata, 2015).

As in Chapter 1, the policymaker at date t is treated as a Stackelberg leader with respect to both private agents at date t and policymakers (and private agents) in dates $t+i, i \geq 1$. The equilibrium Markov perfect policy is one in which optimal decisions are a function only of the payoff relevant state variables in the model $\left\{\eta_t, \bar{g}_t, \hat{A}_t, \hat{\phi}_t, \hat{d}_{t-1}\right\}$. The policymaker recognizes that future allocations will satisfy time-invariant policy functions with this property. Current policy decisions affect future outcomes through their impact on the endogenous state variable, which in the context of the present model is the stock of government debt.

To derive insights that can be studied analytically, I ignore the lower bound on the short-term bond rate. Given the quadratic objective function and fully linear constraints, the Markov perfect policy functions are linear functions of the state variables. Section 3.5 examines the behavior of the model when the presence of the lower bound on the short-term bond rate is accounted for.

3.4.1 The optimal policy problem

The policymaker's optimization problem is characterized by the following Lagrangean:

$$\begin{aligned}\widetilde{\mathcal{L}}_t =& \frac{1}{2}\left[\hat{\pi}_t^2 + \omega\hat{x}_t^2\right]\\ &- \mu_t^x\left[\hat{x}_t - \mathbb{E}_t\hat{x}_{t+1} + \sigma\left(1-g\right)\left(\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} - r_t^*\right)\right]\\ &- \mu_t^\pi\left[\hat{\pi}_t - \kappa\hat{x}_t - \beta\mathbb{E}_t\hat{\pi}_{t+1} - u_t\right]\\ &- \mu_t^d\left[\hat{d}_t - \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) + \left(1-\chi\right)\hat{V}_t - \zeta^{-1}\bar{g}_t\right]\\ &- \mu_t^V\left[\hat{V}_t + \hat{R}_t - \chi\beta\hat{V}_{t+1}\right]\\ &+ \beta\mathbb{E}_t\widetilde{\mathcal{L}}_{t+1}\end{aligned}$$

The first order conditions for minimization are:

$$0 = \hat{\pi}_t - \mu_t^\pi - \beta^{-1}\mu_t^d \tag{3.16}$$

$$0 = \omega\hat{x}_t - \mu_t^x + \kappa\mu_t^\pi \tag{3.17}$$

$$\begin{aligned}0 =& \mu_t^x\left[\frac{\partial\mathbb{E}_t\hat{x}_{t+1}}{\partial\hat{d}_t} + \sigma\left(1-g\right)\frac{\partial\mathbb{E}_t\hat{\pi}_{t+1}}{\partial\hat{d}_t}\right] + \beta\mu_t^\pi\frac{\partial\mathbb{E}_t\hat{\pi}_{t+1}}{\partial\hat{d}_t}\\ &- \mu_t^d + \chi\beta\mu_t^V\frac{\partial\mathbb{E}_t\hat{V}_{t+1}}{\partial\hat{d}_t} + \beta\frac{\partial\mathbb{E}_t\widetilde{\mathcal{L}}_{t+1}}{\partial\hat{d}_t}\end{aligned} \tag{3.18}$$

$$0 = -\left(1-\chi\right)\mu_t^d - \mu_t^V \tag{3.19}$$

$$0 = -\sigma\left(1-g\right)\mu_t^x - \mu_t^V \tag{3.20}$$

Derivatives of $\mathbb{E}_t\widetilde{\mathcal{L}}_{t+1}$ can be eliminated by noting that:

$$\frac{\partial\widetilde{\mathcal{L}}_t}{\partial\hat{d}_{t-1}} = \beta^{-1}\mu_t^d \Rightarrow \frac{\partial\mathbb{E}_t\widetilde{\mathcal{L}}_{t+1}}{\partial\hat{d}_t} = \beta^{-1}\mathbb{E}_t\mu_{t+1}^d$$

The linear-quadratic nature of the problem and the focus on Markov-perfect equilibria implies that equilibrium allocations are linear functions of the state variables.

This means that:

$$\frac{\partial \mathbb{E}_t Z_{t+1}}{\partial \hat{d}_t} \equiv F_Z$$

for some (fixed) coefficient F_Z for any variable Z.

These observations can be used to write (3.18) as:

$$\mu_t^d = \left[F_{\hat{x}} + \sigma\left(1-g\right)F_{\hat{\pi}}\right]\mu_t^x + \beta F_{\hat{\pi}}\mu_t^\pi + \chi\beta F_{\hat{V}}\mu_t^V + \mathbb{E}_t\mu_{t+1}^d \tag{3.21}$$

A straightforward, but tedious, application of the method of undetermined coefficients can be used to characterize the solutions of the coefficients $\{F_{\hat{\pi}}, F_{\hat{x}}, F_{\hat{V}}, F_{\hat{d}}\}$. Appendix 3.C contains the details and demonstrates that (conditional on solutions for $F_{\hat{\pi}}$ and $F_{\hat{d}}$):

$$\begin{aligned} F_{\hat{x}} &= \kappa^{-1}F_{\hat{\pi}} - \kappa^{-1}\beta F_{\hat{\pi}}F_{\hat{d}} \\ F_{\hat{V}} &= \left(1-\chi\right)^{-1}\beta^{-1} - \left(1-\chi\right)^{-1}\beta^{-1}F_{\hat{\pi}} - \left(1-\chi\right)^{-1}F_{\hat{d}} \end{aligned}$$

Solving for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ involves solving a coupled system of quadratic equations. The quadratic equation for $F_{\hat{\pi}}$ has a solution (conditional on $F_{\hat{d}}$) given by the following function:

$$F_{\hat{\pi}} = m\left(F_{\hat{d}}\right) \equiv \frac{1+\chi-\left(1+\beta\chi\right)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1} + \left(1-\chi\right)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)} \tag{3.22}$$

where $\Xi \equiv \left(1-\chi\right)\widetilde{\sigma}^{-1} + \kappa\beta^{-1}$.

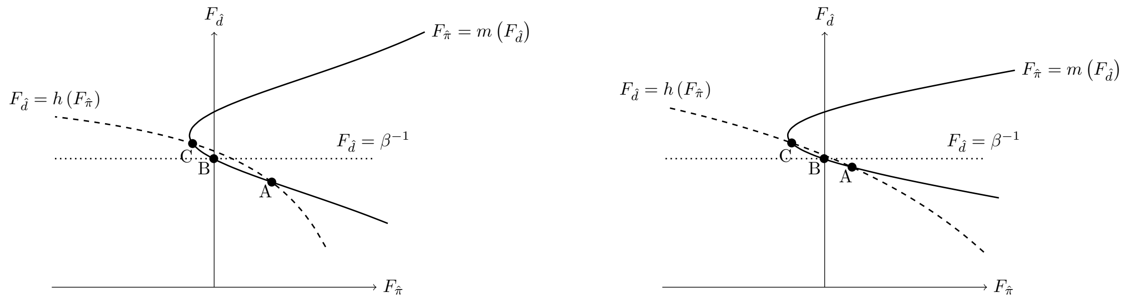
The quadratic equation for $F_{\hat{d}}$ can be factorized. One solution is shown to be $F_{\hat{d}} = \beta^{-1}$. Equation (3.22) then implies that $F_{\hat{\pi}} = \frac{\kappa}{\Xi\beta}\left(1-\beta^{-1}\right) < 0$. The other solution, conditional on $F_{\hat{\pi}}$, satisfies:

$$F_{\hat{d}} \equiv h\left(F_{\hat{\pi}}\right) = \frac{\left(1+\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\right)F_{\hat{\pi}} - \kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}}{F_{\hat{\pi}} - \chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}} \tag{3.23}$$

The model has a unique, stable ('determinate') solution if the debt stock returns to steady state from any initial condition, which requires that $F_{\hat{d}} < 1$. Blake and Kirsanova (2012) demonstrate the the presence of endogenous state variables

Figure 3.1: Solutions for $F_{\hat{\pi}}$ and $F_{\hat{d}}$

(a) Baseline model ($\chi = 0.945$) (b) 'Long duration' variant ($\chi = 0.976$)



Notes: Each panel plots the functions m and h defined by equations (3.22) and (3.23) respectively. Panel (a) shows the baseline model and panel (b) shows the variant with long-duration government debt. In each case, point A denotes the solution for the coefficients $F_{\hat{\pi}}$ and $F_{\hat{d}}$ consistent with the unique Markov perfect equilibrium.

can generate multiple stable Markov perfect equilibria for time-consistent linear-quadratic optimal policy problems.[14]

Figure 3.1 provides a graphical analysis of the candidate equilibria for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ for the baseline model (panel (a)) and 'long duration' parameterization (panel (b)). In both cases there are three candidate equilibria, labeled A, B and C. Of these, B and C generate an explosive trajectory for debt (since $F_{\hat{d}} \geq \beta^{-1} > 1$). Point A is the unique stable solution and is the equilibrium used for the experiments in the next subsections.[15]

Comparing panels (a) and (b) reveals that intersection point A in panel (b) lies to the North-West of the corresponding intersection point in panel (a). So the equilibrium trajectory for government debt is more persistent in the long duration variant. The longer-term debt variant also has the property that inflation depends less strongly on previously accumulated debt.

Appendix 3.C demonstrates that the first order conditions can be combined into

[14]Indeed, one of their motivating examples adds (one period) government debt to a textbook New Keynesian model.

[15]Figure 3.1 provides a 'local' analysis, plotting the m and h functions in the vicinity of the intersection points. Figure 3.21 in Appendix 3.D expands the range over which the functions are plotted to present a global picture, demonstrating that no other candidates exist.

a targeting rule:

$$\omega\hat{x}_t + \kappa\hat{\pi}_t = \Xi\mu_t^d \tag{3.24}$$

Equation (3.24) reveals that the value of the multiplier on the government budget constraint, μ_t^d, affects the optimal achievable combination of the output gap and inflation. If the government budget is not a constraint on monetary policy (as is the case under passive fiscal policy) then $\mu_t^d = 0, \forall t$ and (3.24) collapses to the targeting rule in the New Keynesian benchmark model, (3.15), as previously claimed.

3.4.2 Impulse responses

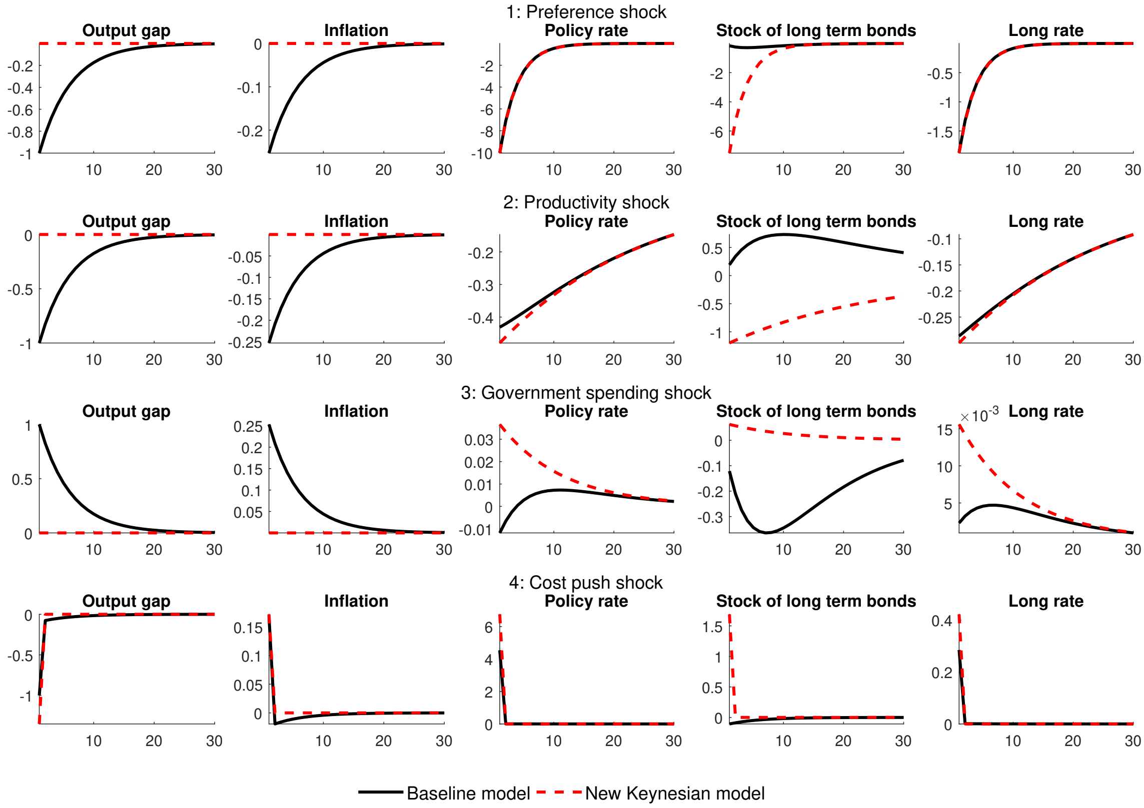
This section examines the impulse responses of the model to shocks, comparing the baseline ('average duration') model to the textbook New Keynesian benchmark and to the 'long duration' variant.

To examine the implications for debt, I plot the responses of the par value of debt, $\hat{d}$ and the long-term bond rate, denoted $\hat{\mathcal{R}}$. As in Chapter 1, the long-term bond rate is computed as the the yield to maturity:

$$\hat{\mathcal{R}}_t = \chi\beta\mathbb{E}_t\hat{\mathcal{R}}_{t+1} + (1-\chi\beta)\,\hat{R}_t$$

Figure 3.2 plots responses of the model to a positive innovation in each of the shocks (solid black lines) alongside the responses of the benchmark New Keynesian model (red dashed lines). To aid the comparison, the scale of each shock is chosen so that it has an identical impact effect on the output gap in the baseline model (1 percentage point in absolute terms).

Figure 3.2: Responses to shocks: baseline model and New Keynesian benchmark



Notes: Impulse responses to shocks to the baseline model (solid black lines) and the 'New Keynesian' variant described in Section 3.3.6 (dashed red lines). The scale of all shocks is normalized to deliver a 1% response of the output gap in the baseline variant. Policy rate and long rate plotted in annualized units. All variables are shown in percentage point deviations from steady state.

Two key results emerge from Figure 3.2.

First, the so-called 'divine coincidence' (Blanchard and Galí, 2007) disappears when fiscal policy is active. The divine coincidence result refers to the fact that optimal time-consistent policy in the *textbook New Keynesian model* (dashed red lines) achieves complete stability of the output gap and inflation in response to government spending, preference and productivity shocks (rows 1–3). For that variant, the relevant targeting criterion (3.15) can be achieved with $x_t = \pi_t = 0, \forall t$. This is achievable because, in that variant, these shocks affect the economy solely through their effects on the natural real interest rate, r^*. The time-consistent policy is to track movements in r^*_t with the short-term nominal interest rate $\hat{R}_t$, delivering a zero output gap and (hence) zero inflation.[16]

However, this policy response is not feasible in the baseline model (with active fiscal policy). Tracking exogenous movements in r^* with the nominal interest rate would not stabilize the government debt stock.[17] A necessary condition for doing so is that the interest rate responds to the debt stock. Because the nominal interest rate must respond to the debt stock, it cannot fully insulate the economy from the effects of exogenous changes in r^* and costly fluctuations in the output gap and inflation cannot be avoided.

The second key result is that the (absolute) inflation and output gap responses to government spending, preference and productivity shocks are identical (rows 1–3). In all cases, the responses of inflation and the output gap satisfy the targeting criterion (3.24). Appendix 3.C.1 shows that, *in the absence of cost push shocks*, this targeting criterion can be combined with the first order condition for government debt (3.21) and the IS curve (3.10) to deliver a second order difference equation for inflation. Appendix 3.C.1 further demonstrates that the solution to that difference equation implies that inflation and the output gap follow AR(1) processes given by:

$$\hat{\pi}_{t+1} = F_{\hat{d}}\hat{\pi}_t$$
$$\hat{x}_{t+1} = F_{\hat{d}}\hat{x}_t$$

for $t \geq 1$.

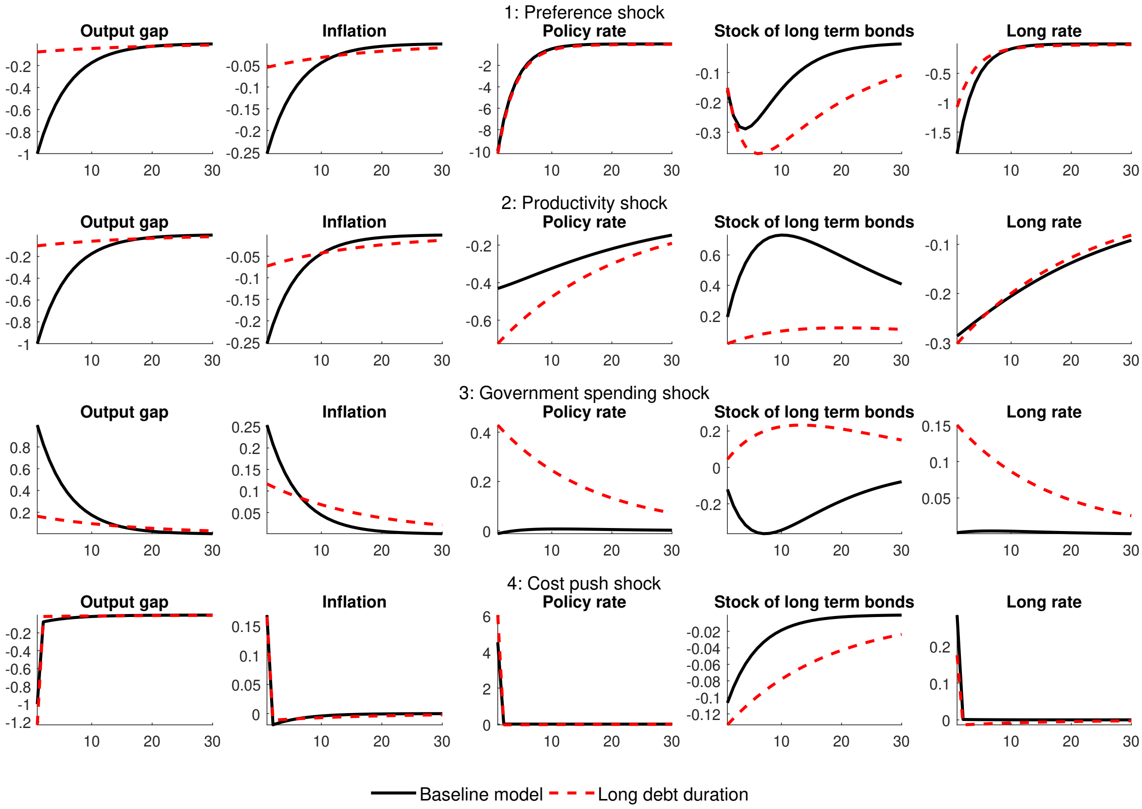
So both inflation and the output gap follow identical (to scale) AR(1) processes in response to preference, productivity and government spending shocks. The initial response of the output gap in period 1 is equal to unity in all cases, by virtue of the normalization assumption, so the (absolute) responses of both the output gap and inflation are identical for these shocks. The fact that the AR(1) parameter is equal to $F_{\hat{d}}$ is an important result, discussed below.

[16]See Galí (2008) for a complete analysis of optimal monetary policy in the textbook New Keynesian model.

[17]An informal proof by contradiction is as follows. Suppose that tracking exogenous movements in r^* with $\hat{R}$ does stabilize inflation $\hat{\pi}_t = 0, \forall t$. Note now that equation (3.13) implies that the value of long-term debt will be a function of the exogenous fluctuations in r^* (since $\hat{R}_t = r_t^*$). Inspecting (3.12) reveals than an exogenous impulse to $\hat{V}_t$ with $\hat{\pi}_t = 0, \forall t$ generates an explosive trajectory for $\hat{d}_t$ given that $\beta^{-1} > 1$. So full stabilization of inflation by tracking r^* with the policy rate does not also ensure that the debt stock returns to steady state.

Figure 3.3 compares responses from the baseline model (solid black lines) with the 'long duration' variant (red black lines).

Figure 3.3: Responses to shocks under 'average' and 'long' debt duration



Notes: Impulse responses to shocks to the baseline model (solid black lines) and variant with long debt duration (dashed red lines). The scale of all shocks is normalized to deliver a 1% response of the output gap in the baseline variant. Policy rate and long rate plotted in annualized units. All variables are shown in percentage point deviations from steady state.

A key result is that the output gap and inflation are better stabilized in response to preference, productivity and government spending shocks for the long duration variant (rows 1–3). The responses of the output gap and inflation are also more persistent in this variant, compared with the baseline calibration. A corollary of the smaller responses of inflation and the output gap is that the policy rate must adjust by more in response to each of these shocks.

The greater persistence of the output gap and inflation responses in the long-duration variant follows from the result that these variables both follow AR(1) processes with parameter $F_{\hat{d}}$. As shown in Figure 3.1, $F_{\hat{d}}$ is larger for the long-duration variant.

These results relate to previous findings that optimal time-consistent policy may exhibit a "debt stabilization bias". Leith and Wren-Lewis (2013) study jointly optimal monetary and fiscal policy in a similar model.[18] In their model, optimal time consistent policy rapidly returns the government debt stock back to steady state following a shock. That in turn requires large movements in output and inflation to achieve the required change in real debt values. Leeper, Leith, and Liu (2019) study the responses of a non-linear model with optimal monetary and fiscal policy for alternative assumptions about government debt duration. They also find that longer duration debt dampens the responses of macroeconomic variables to shocks under optimal policy.

The presence of long-term debt reduces the degree to which large and immediate movements in inflation will reduce the real value of government debt. Instead, it is possible to stabilize the debt stock through smaller but more persistent movements in inflation (which may nonetheless have a sizable effect on the market value of debt). The presence of government debt with a longer duration allows even smaller, and even more persistent, changes in inflation to be used to bring the debt stock back to steady state following a shock.

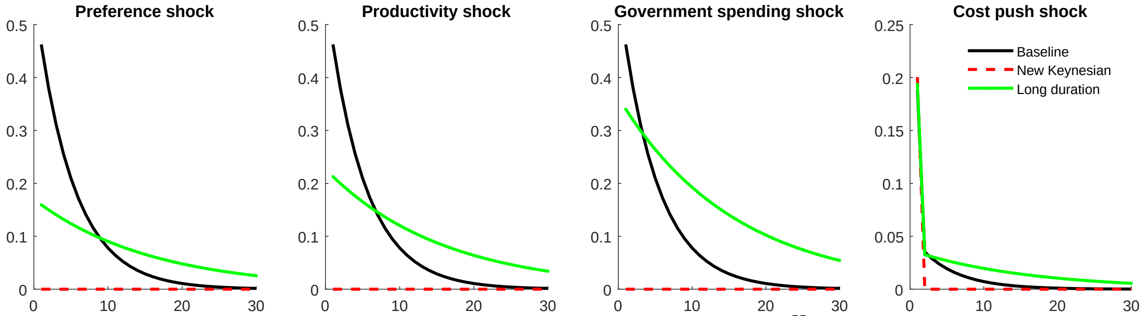
Figure 3.4 examines the welfare implications of the shocks across model variants. In each panel an approximation to the square root of the loss $\mathcal{L}_t$ is plotted for each model variant.[19] The square root transformation facilitates comparison of the model variants, but obviously understates the true welfare differences between them.

As expected, the divine coincidence result for the New Keynesian variant implies that there are no welfare losses from preference, productivity or government spending shocks in that variant (red dashed lines). For these shocks, losses are, initially, much smaller for the long duration variant (green lines) compared with the baseline model (solid black lines). As the shocks dissipate, however, losses are larger for the long duration variant.

[18]The main differences are the inclusion of distortionary taxation and the assumption that the government finances its activities using one-period debt.

[19]The approximation is to compute losses over a finite horizon, H: $\hat{\mathcal{L}}_t \equiv \mathbb{E}_t \sum_{i=0}^{H} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2\right]$. Results are shown for $H = 200$, but are not sensitive to this assumption.

Figure 3.4: Normalized losses for each model variant



Notes: Normalized losses are defined as $\sqrt{\hat{\mathcal{L}}_t}$, where $\hat{\mathcal{L}}_t \equiv \mathbb{E}_t \sum_{i=0}^{H} \beta^i \left[\hat{\pi}_{t+i}^2 + \omega \hat{x}_{t+i}^2\right]$, a finite-horizon approximation to the loss defined in (3.14). The horizon H is set to 200.

These results are consistent with the observation from Figure 3.3 that inflation and the output gap exhibit muted and persistent responses to preference, productivity and government spending shocks in the long duration variant. The presence of longer duration debt increases the extent to which the monetary policymaker is able to smooth welfare losses across time. The existence of longer duration debt means that bond prices ($\hat{V}$) can be materially affected by relatively small, but very persistent movements in inflation. The optimal monetary policy exploits this mechanism to mitigate welfare losses in the near term, at the expense of larger losses in the longer term.

For cost-push shocks, the welfare ranking across model variants is, initially, reversed. On impact, losses are greatest for the New Keynesian variant and, initially, the baseline model also generates smaller losses than the long-duration variant. Because the cost-push shock has no persistence, losses in the New Keynesian variant are zero from period 2 onwards.[20]

The smaller initial losses in the variants with active fiscal policy can be understood by observing that inflation is below target from period 2 onwards in these variants (Figures 3.2 and 3.3). Relative to the New Keynesian variant, therefore, the Phillips curve trade-off in the *first* period is improved, because inflation expectations are lower. This allows the policymaker to achieve a less costly mix of inflation and the output gap in period 1. From period 2 onwards, losses are higher than

[20]The textbook New Keynesian model has no endogenous state variables, so the absence of any disturbance from period 2 onward allows complete stabilization of the output gap and inflation (Galí, 2008, Figure 5.1).

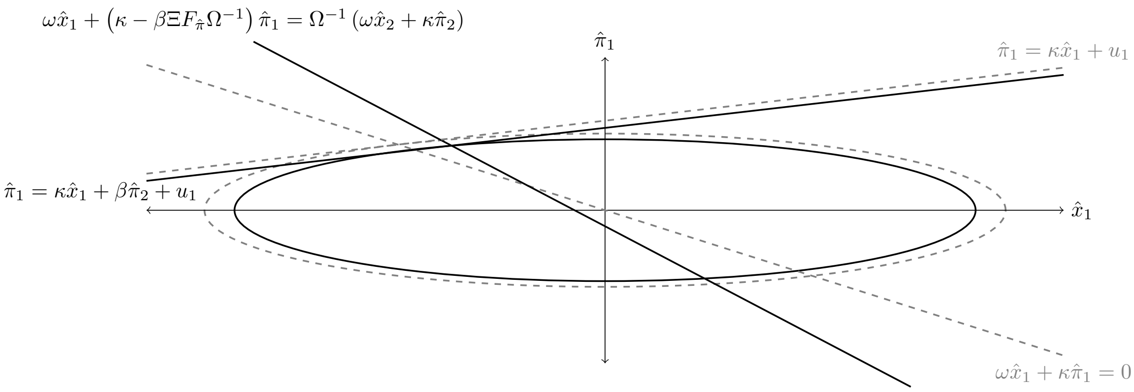
the textbook New Keynesian model because the requirement that monetary policy stabilizes the government debt stock requires a persistent deviation of inflation and the output gap. The net effect on the present value loss, $\mathcal{L}_1$ depends on whether the gains from the improved trade-off in period 1 outweigh the future losses.

Figure 3.5 shows outcomes in period 1 for the baseline model and the textbook New Keynesian model. The solid black lines show the baseline model and grey dashed lines show the New Keynesian variant. In both cases, the inflation-output gap trade-off is determined by the intersection of an upward- sloping Phillips curve (3.11) and a downward-sloping optimal policy criterion. The optimal policy criterion for the New Keynesian variant is given by (3.15). Appendix 3.C.1 demonstrates that the targeting criterion in the baseline model can be written as:

$$\omega\hat{x}_t + \left(\kappa - \beta\Xi F_{\hat{\pi}}\Omega^{-1}\right)\hat{\pi}_t = \Omega^{-1}\mathbb{E}_t\left(\omega\hat{x}_{t+1} + \kappa\hat{\pi}_{t+1}\right) \tag{3.25}$$

where $\Omega = \chi + F_{\hat{\pi}}\left(1 - (1-\beta)\frac{1-\chi}{\kappa\tilde{\sigma}}\right) > 0$.

Figure 3.5: Optimal responses to cost-push shock in period 1



Notes: The diagram shows the optimal policy decisions when a cost-push shock arrives in period 1, for the baseline model (solid black lines) and the New Keynesian variant described in Section 3.3.6 (dashed grey lines). The upward-sloping lines are the Phillips curve, (3.11), conditional on expected outcomes in period 2. The downward-sloping lines are the optimal trade-off criteria: equation (3.25) for the baseline model and equation (3.15) for the New Keynesian variant. The ellipses are iso-loss lines, tracing out combinations of the output gap ($\hat{x}$) and inflation ($\hat{\pi}$) that deliver the same value of the period 1 loss, $L_1 \equiv \hat{\pi}_1^2 + \omega\hat{x}_1^2$.

As noted above, the optimal response in the baseline model implies that inflation is negative from period 2 onward. The spike in inflation in period 1 reduces the real value of debt and negative inflation thereafter helps to stabilize real debt. The negative inflation in period 2 implies that the Phillips curve in the baseline model lies below the Phillips curve in the New Keynesian variant.[21]

Other things equal, a downward shift in the Phillips curve allows the monetary policy maker to achieve a better outcome (a smaller increase in inflation and a smaller reduction in the output gap). The optimal combination of the output gap and inflation that is chosen, however, depends on the trade-off criterion (3.25). Relative to the New Keynesian model, the trade-off criterion in the baseline model features a downward shift and an clockwise tilt. The downward shift reflects the fact that the right hand side of (3.25) is negative.[22] The tilt occurs because $(\kappa - \beta \Xi F_{\hat{\pi}} \Omega^{-1}) < \kappa$. The resulting optimal combination of the output gap and inflation features a similar inflation rate to the New Keynesian case, but a noticeably smaller negative output gap.

The ellipses in Figure 3.5 are iso-loss curves, showing combinations of the output gap and inflation in period 1 that satisfy $\hat{\pi}_1^2 + \omega \hat{x}_1^2 = L$. The ellipse for the baseline model (solid black line) lies within the ellipse for the New Keynesian variant (dashed grey), so that the losses incurred *in period 1* are lower in the baseline model. While per-period losses are larger from period 2 onwards, the gain in period 1 is sufficient for the discounted loss $\mathcal{L}_1$ to be lower in the baseline model than the New Keynesian variant.

3.5 Time-consistent monetary policy at the lower bound

The analysis in this section accounts for the existence of a lower bound on the short-term bond rate. To do so, the model is solved using projection methods. To reduce the number of state variables (and hence the dimensionality of the problem), I abstract from government spending shocks and productivity shocks.

The motivation for ignoring government spending shocks is that the precise

[21]The upward-sloping black line in Figure 3.5 lies below the upward-sloping grey dashed line. Inflation expectations in the textbook New Keynesian model are zero ($\hat{\pi}_2 = 0$).

[22]The output gap and inflation in period 2 are both negative.

nature of the effect of government spending on debt is determined by the particular (extreme) assumption that real lump sum taxes are held fixed. Productivity and preference shocks both influence the model only through their effects on the natural rate of interest, r^*. If these two shocks had identical persistence ($\rho_A = \rho_\phi$), then their effects are identical, up to scale.

These simplifications imply that the natural real interest rate, r^* can be treated as a 'primitive' disturbance and I assume that it follows a simple autoregressive process:

$$r_t^* = \rho_r r_{t-1}^* + \sigma_r \varepsilon_t^r \tag{3.26}$$

I assume that the cost push shock process follows:

$$u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_t^u \tag{3.27}$$

Both ε^r and ε^u are i.i.d., normally distributed and have unit variance.

The persistence and standard deviations of the shocks are important parameters for the model solution. I use the parameterization in Table 3.3.

Table 3.3: Shock process parameters

	Description	Value
ρ_r	Natural rate persistence	0.85
$100 \times \sigma_r$	Natural rate shock standard deviation	0.225
ρ_u	Cost-push shock persistence	0
$100 \times \sigma_u$	Cost-push shock standard deviation	0.135

The shock processes are calibrated with reference to the assumptions in Chapter 1. Relative to that calibration, the variance of the disturbance to r^* is reduced slightly. This is because the calibration in Chapter 1 was designed to present a substantial stabilization problem to the policymaker in the presence of the zero bound. The addition of the requirement that monetary policy also stabilizes the government debt stock makes the policy problem more difficult, other things equal.[23]

[23]Another difference between the model specification used here and that in Chapter 1 is that the effective slope of the IS curve is equal to $\widetilde{\sigma} = \sigma(1-g) < \sigma$. This means that the effect of policy rate changes on aggregate demand is smaller, again making the stabilization problem more challenging.

The assumed persistence of the r^* process is close to the average persistence of the shock processes (productivity, preference and government spending) used in Section 3.4.

3.5.1 Optimal policy problem and solution

The policymaker's optimization problem is the same as considered in Section 3.4.1, with the addition of a constraint on the short-term bond rate:

$$\hat{R}_t \geq 1 - \beta^{-1} \tag{3.28}$$

where the lower bound on the nominal interest rate is assumed to be zero.[24]

The first order conditions (3.16)–(3.19) are unchanged. The first order condition for the short-term nominal rate becomes

$$0 = -\sigma (1 - g) \mu_t^x - \mu_t^V - \mu_t^Z$$

where μ_t^Z is the Lagrange multiplier on the constraint (3.28). The fact that the bound on the policy rate binds occasionally gives rise to a contemporary slackness condition reported in Appendix 3.F.1.

I solve the model using projection methods. To reduce the dimensionality of the state space, I approximate the stochastic processes (3.26) and (3.27) using finite state Markov processes with transition matrices derived using the Rouwenhorst (1995) method.[25] The approach is described in detail in Appendix 3.F.

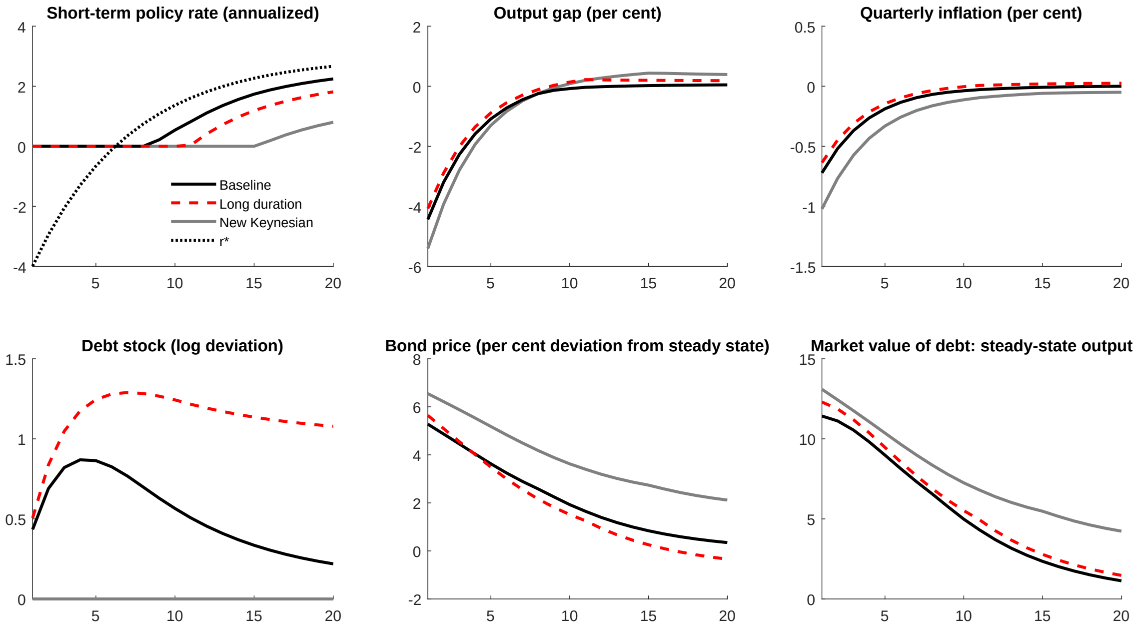
[24]Variables in the model are measured relative to steady state, so $1 - \beta^{-1}$ measures the difference between the steady state gross nominal interest rate β^{-1} and a gross nominal rate of 1 (corresponding to a net nominal interest rate of zero).

[25]Kopecky and Suen (2010) demonstrate that this approach generates accurate approximations to autoregressive process with high persistence. In the context of the present model, this reduces the computational burden of analyzing the case in which shocks to the natural rate of interest r^* are very persistent.

3.5.2 Outcomes at the zero bound

To examine the behavior of the model at the zero bound, I construct a recessionary scenario. In period 0, the model is assumed to be at its deterministic steady state. In period 1, the natural real interest rate is initialized at a negative value (-4% on an annualized basis) and is assumed to follow the process (3.26) (with $\varepsilon_t^r = 0, t = 1, \dots$). The values of the cost-push state are set to zero throughout the simulation ($u_t = 0, t = 1, \dots$). Conditional on the initial value of the natural rate, r_1^*, the exogenous states $\{u_t, r_t^*\}$ follow their most likely paths. However, in each periods outcomes for the endogenous variables account for the *risk* that future shocks arrive, including those that would prolong the time spent at the zero bound.

Figure 3.6: A recessionary scenario that causes the ZLB to bind



Notes: The panels show outcomes from simulations of the baseline model (solid black lines), the variant with long-duration debt (dashed red lines) and the textbook New Keynesian model (grey lines). In each case, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* (plus the deterministic steady-state interest rate) is set to -4% on an annualized basis. Thereafter r^* follows the process (3.26), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \dots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \dots$. The initial debt stock is at the deterministic steady state: $\hat{d}_0 = 0$.

Figure 3.6 shows the effects of the recessionary scenario. The solid black lines show the baseline model, the red dashed lines show the variant with long duration debt and the grey lines show the textbook New Keynesian model.[26] The dotted line in the top left panel shows the trajectory of the natural real interest rate, r^*.

Relative to the textbook New Keynesian model (grey lines), active fiscal policy reduces the scale of the recession and allows the short-term policy rate to lift off from the zero bound earlier. The deflationary effect of the recession increases the real value of government debt. Other things equal, this increases inflation expectations, as higher future inflation will be required to stabilize the real debt stock. This mechanism reduces expected real interest rates, stimulating spending and supporting inflation. In turn, that mitigates the recessionary effects of the fall in the natural real interest rate.

Comparing the results from the baseline and 'long duration' variants reveals that longer duration debt is associated with a (slightly) smaller recession and a later liftoff from the zero bound. The results in Section 3.4.2 reveal that, away from the zero bound, optimal policy is able to better stabilize the output gap and inflation with long duration debt. However, achieving this improved stabilization performance requires larger movements in the policy rate.

This implies that, with long duration debt, the zero bound is a more binding constraint on the setting of the policy rate required to deliver smaller output and inflation responses to a recessionary shock. For the particular shock examined in Figure 3.6, the net effect is a slight improvement in output and inflation stabilization. However, this improvement requires the policy rate to remain at the zero bound for an additional two quarters compared with the baseline model.

These effects are also apparent in the simulated distributions of key variables, for which Table 3.4 provides a summary.[27]

The results for the textbook New Keynesian model demonstrate the familiar result that the zero lower bound induces a downward skew in the distributions for inflation and the output gap, both of which have a negative mean.[28] The mean of the policy rate is below the deterministic steady-state value of 3%, as the downward

[26] That is, the variant with passive fiscal policy: see Section 3.3.6.

[27] Each model variant was simulated for $260,000$ periods, with the first $10,000$ periods discarded.

[28] See also Chapter 1, Section 1.6.

Table 3.4: Summary statistics from alternative model variants

	Baseline		Long duration		New Keynesian	
	Mean	Std dev	Mean	Std dev	Mean	Std dev
Quarterly inflation, %	0.01	0.17	0.04	0.14	-0.08	0.15
Output gap, %	0.00	0.80	0.01	0.66	-0.01	0.68
Annualized policy rate, %	3.0	2.9	3.2	3.4	2.7	3.5
Debt stock, % deviation from SS	0.18	0.22	1.3	0.31	0	0
Loss per period ($\hat{\pi}_t^2 + \omega \hat{x}_t^2$)	0.034	–	0.023	–	0.031	–
When at ZLB:	0.057	–	0.023	–	0.058	–
When not at ZLB:	0.026	–	0.023	–	0.010	–
ZLB incidence, %	24		32		45	

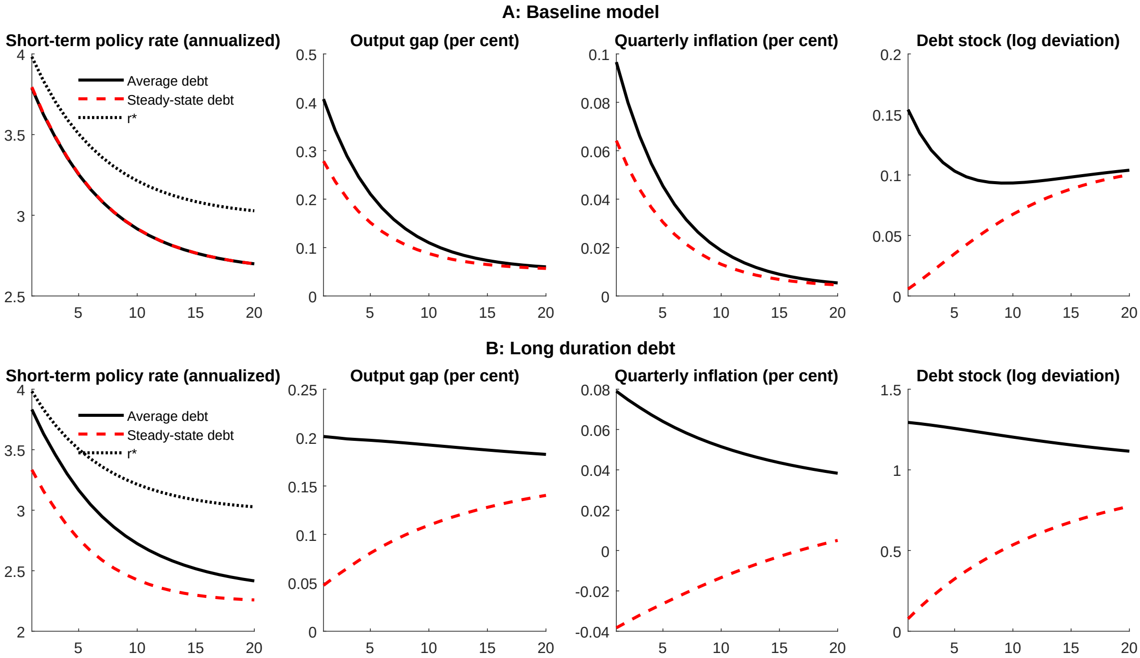
skew in the distribution of expected inflation dominates the positive effect on the mean from truncation of the distribution at zero.

In contrast, the average policy rate is at or above the deterministic steady-state value of 3% for the variants of the model with active fiscal policy. In these cases, the truncation effect of the zero bound is dominant, pushing up on the mean policy rate. One driver of this result is that the policy rate is more variable when fiscal policy is active, particularly for the long duration debt variant (see Section 3.4.2 and Table 3.4). Higher variability of the policy rate increases the truncation effect, other things equal.

Another driver of this result is the distribution of debt under active fiscal policy. Table 3.4 shows that average debt is above the deterministic steady-state for both the baseline and long duration model variants. This is because recessionary shocks that cause the zero bound to bind generate increases in the debt stock via the debt deflation mechanism described above (Figure 3.6). As there is no upper bound on the policy rate, the debt deflation mechanism does not operate in reverse for large expansionary shocks and the debt distribution shifts to the right (relative to a symmetric distribution around the deterministic steady state).

The implications of initial debt levels for the responses to an expansionary shock are explored in Figure 3.7. The top panel shows the responses of the baseline model to a scenario in which the initial level of (annualized) r^* is 1pp above steady state under two assumptions for the initial stock of debt (d). The solid black lines show a case in which the initial stock of debt is equal to the mean of the stochastic

Figure 3.7: An expansionary shock with for different assumptions about initial debt



Notes: The top panels (A) show outcomes from simulations of the baseline model. The lower panels (B) show outcomes from simulations of the variant with long-duration debt. In all cases, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* is set to 1% above the deterministic steady-state (on an annualized basis). Thereafter r^* follows the process (3.26), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \ldots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \ldots$. The dashed red lines show the case in which the initial value of the debt stock is set to the deterministic steady state, $\hat{d}_0 = 0$. The solid black lines show the case in which the initial value of the debt stock is equal to the mean of the stochastic distribution reported in Table 3.4.

distribution (from table 3.4). The dashed red lines show the case in which the initial debt level is equal to the deterministic steady state. The bottom panel repeats this experiment for the variant of the model with long duration debt.

Figure 3.7 shows that the expansionary scenario generates a larger positive output gap and more inflation when the initial level of debt is at its average level, compared to the case in which the initial debt level is at the deterministic steady state. When debt is relatively high, additional inflation is required to stabilize the debt stock. As a result, the rightward shift in the distribution of debt generates a rightward shift in the distributions of the output gap and inflation. Indeed, average inflation and output gaps are *positive* under active fiscal policy.[29] Unsurprisingly,

[29]The mean output gap for the baseline model is slightly positive but rounds to zero to two

the rightward shift in these distributions is particularly evident for the long duration variant, for which debt is higher on average.

The results so far indicate that, relative to the New Keynesian variant (with passive fiscal policy), active fiscal policy generates a rightward shift in the distributions of the output gap and inflation. As noted in the discussion of Figure 3.6, this increases inflation expectations and hence mitigates the effects of recessionary shocks when the policy rate is constrained at the zero bound. On the other hand, the analysis in Section 3.4.2 revealed that, absent the zero bound, welfare would be higher in the textbook New Keynesian model (with passive fiscal policy).

Which of these effects dominates?

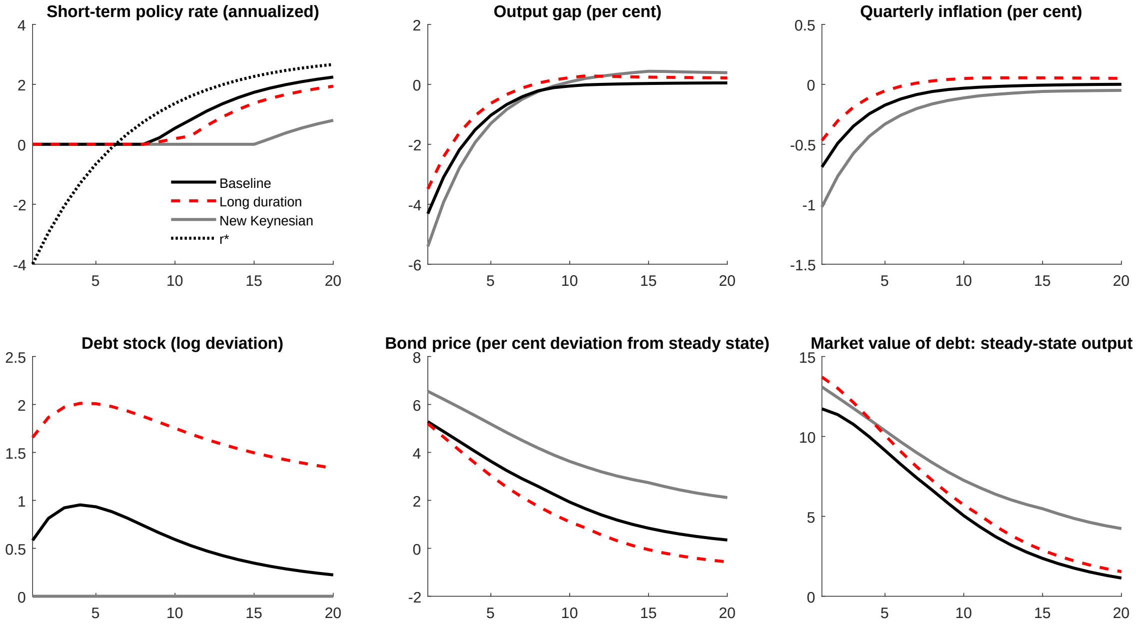
The policy rate is at the zero bound 24% of the time in the baseline model, compared with 45% for the textbook New Keynesian model (Table 3.4). Conditional on being at the zero lower bound, losses are slightly lower on average. However, these performance improvements are outweighed by higher losses when policy is not constrained by the zero bound. Relative to the textbook New Keynesian model, average losses are therefore slightly higher in the baseline model.

For the variant of the model with long duration debt, the policy rate is at the zero bound around a third of the time, midway between the baseline and textbook New Keynesian models. Conditional on being at the zero bound, losses are considerably lower than the other variants. This performance improvement is sufficient to compensate for higher losses (compared with the textbook New Keynesian model) away from the zero bound.

The material welfare improvements at the zero bound for the long duration debt variant are not evident from Figure 3.6. Once again, the distribution of government debt matters. Figure 3.8 repeats the experiment from Figure 3.6, but under the assumption that initial debt stocks are equal to the mean of the stochastic distribution. This reveals greater performance improvements in the model with long-duration debt. Higher average debt generates higher inflation expectations, reducing real interest rates and stimulating aggregate demand. In this case, liftoff

decimal places, as reported in Table 3.4.

Figure 3.8: Recessionary scenario under average debt levels



Notes: The panels show outcomes from simulations of the baseline model (solid black lines), the variant with long-duration debt (dashed red lines) and the textbook New Keynesian model (grey lines). In each case, the simulation is constructed from the policy functions solved by projection methods. The initial value of the natural rate state r_1^* (plus the deterministic steady-state interest rate) is set to -4% on an annualized basis. Thereafter r^* follows the process (3.26), with shocks set to their most likely value of zero, $\varepsilon_t^r = 0, t = 2, \ldots$. The cost-push state is set equal to its most likely value, $u_t = 0, t = 1, \ldots$. The initial value of the debt stock is set equal to the mean of the stochastic distribution reported in Table 3.4.

from the zero bound occurs at the same time as the baseline model, though the policy path remains slightly lower after liftoff.

3.6 Time-consistent monetary policy with fiscal uncertainty

In this section, I study scenarios in which the initial debt stock is considered 'too high' and must therefore be reduced. This is motivated by recent debates over the extent to which current high government debt levels are sustainable. A full examination of those issues would require an analysis of the effects of fiscal tightening at (or close to) the zero bound, in a model with distortionary taxation and a more meaningful role for government. My objective is less ambitious and focuses on the effects of expectations of future fiscal actions on time-consistent monetary policy. A stark comparison case is the textbook New Keynesian model with passive fiscal policy, in which substantial debt reductions using lump sum taxation have no implications for output or inflation.

I first modify the model to include fiscal reaction functions (for the primary surplus) based on empirical estimates. This allows for a richer analysis of the likely fiscal behavior in the scenarios considered later. I then consider a simple debt reduction scenario for two alternative empirical estimates of the fiscal reaction function: one 'active' and the other 'passive'. Finally, I consider the implications for time-consistent monetary policy of the *risk* that fiscal policy switches from passive to active during a debt reduction scenario.

3.6.1 Fiscal reaction functions

Consistent with many empirical studies, I assume that the fiscal reaction function is defined in terms of the primary surplus (rather than the tax rate). Specifically, the reaction function takes the form:

$$\bar{s}_t - \bar{s}_{t-1} = \phi_s \left(\bar{s}_{t-1} - \rho_s \bar{d}_t \right) \tag{3.29}$$

where $\bar{s}$ denotes the absolute deviation of the primary surplus from steady state and $\bar{d}_t \equiv \zeta \hat{d}_t$ is the absolute deviation of the par value of debt from steady state. The parameter values satisfy $\phi_s \leq 0$ and $\rho_s \geq 0$.

Schoder (2014) estimates a similar reaction function, though in his specification the primary surplus and debt stock are measured as ratios to GDP. The estimates can be mapped to the model, by noting that $\bar{s}$ and $\bar{d}$ can be interpreted as ratios to steady-state output (which is normalized to unity).[30] I use Schoder's estimates for a pooled group of non-EMU OECD countries to parameterize two fiscal reaction functions, shown in Table 3.5.

[30]Since the reaction function (3.29) is expressed in terms of deviations from steady state, estimated constants are ignored. Schoder (2014) also includes additional dynamic terms (though coefficient estimates are not reported). I abstract from such dynamics for simplicity.

Table 3.5: Fiscal reaction function parameter values

	'Passive' 1980Q1–1996Q4	'Active' 1997Q1–2010Q4
ρ_s	0.032	0.005
ϕ_s	-0.223	-0.084

Notes: Estimates are from Schoder (2014, Table 2).

Schoder's estimates for the early sample (1980–1996) feature a strong feedback coefficient on the debt stock (ρ_s) and hence delivers a 'passive' fiscal policy in the model. So, under this parameterization, the debt stock is stabilized for any path of the price level and the government budget constraint is not a binding constraint on the monetary policymaker. In contrast, the estimate for the later sample (1997–2010) implies that the response of the surplus to the debt stock is much weaker, giving rise to an 'active' fiscal policy configuration.[31]

To incorporate these reaction functions into the model, I replace equation (3.12) with

$$\hat{d}_t = \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) - (1-\chi)\,\hat{V}_t - \zeta^{-1}\bar{s}_t \tag{3.30}$$

which (as shown in Appendix 3.A) is the general formulation of the government budget constraint written in terms of the primary surplus. Equation (3.12) incorporates the specific assumption that lump sum taxes are held fixed so that all variations in the primary surplus are generated by changes in government spending. In contrast, in this section I assume that variations in $\bar{s}_t$ are implemented via changes in lump sum taxation, so the path of government spending and hence the natural real interest rate, r_t^* does not change.

These modifications imply that the primary surplus becomes an endogenous state variable in the model, so that the characterization of time-consistent policy in Section 3.4.1 is no longer valid. For the analysis in this section, the model under

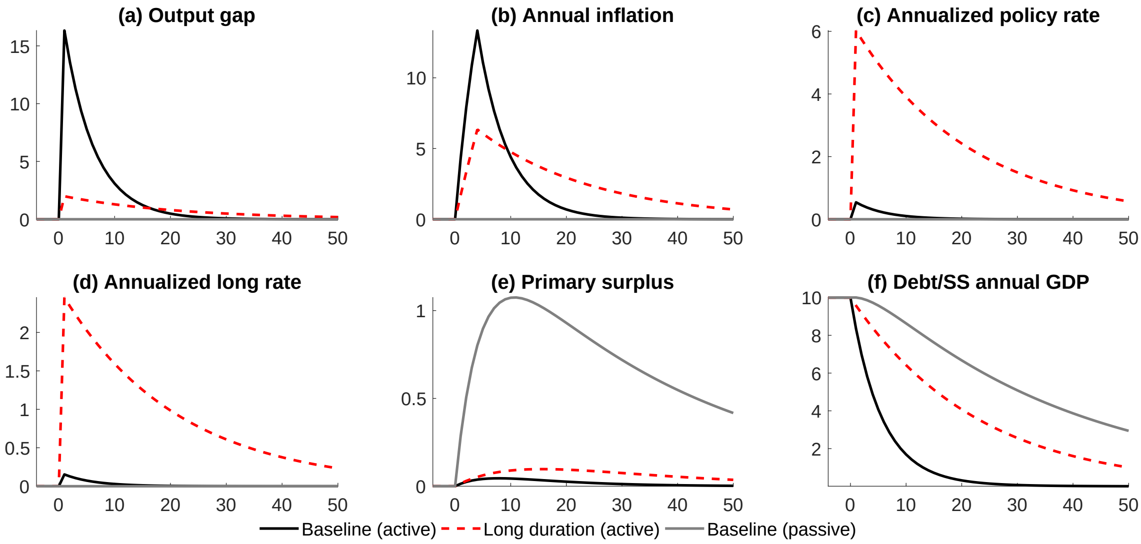
[31] The active fiscal policy specification may be most relevant to the United Kingdom at the present time. Schoder (2014) estimates a full sample value of ρ_s for the United Kingdom that is *negative*. Afonso and Toffano (2013) estimate a regime switching model of fiscal behavior that suggests that the United Kingdom entered an active fiscal regime at around the time of the financial crisis.

time-consistent optimal policy is solved numerically, using the algorithm of Dennis (2007).

3.6.2 A simple debt reduction scenario

In period 0, the economy is at steady state. At the start of period 1, a credible announcement is made that government debt will be reduced by ten percentage points of *annual* GDP. Note that a permanent reduction in the debt to GDP ratio also necessitates an appropriate reduction in ζ.[32]

Figure 3.9: A debt reduction scenario



Notes: The panels show the result of a credible announcement at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at an initial steady state. Results are shown for the baseline model (solid black lines), the long-duration variant (dashed red lines) and the variant with passive fiscal policy (solid grey lines). All variables are plotted as percentage point deviations from the final steady state.

The results of the scenario are shown in Figure 3.9, with responses shown relative to the new steady state.

The grey lies show the responses of the baseline model under the passive fiscal rule from Table 3.5. For this variant, the reduction in the debt stock is implemented

[32]The baseline value for ζ is 2, representing a 50% annual debt to (annual) GDP ratio. In the debt reduction scenario, ζ is set equal to 1.6, commensurate with a 10pp reduction in the annual debt to GDP ratio.

by a strong increase in primary surpluses (implemented via higher lump sum taxes). The reduction in debt (panel (f)) occurs through a steady retirement of the number of outstanding bonds (financed by higher taxes) and requires no change in the policy rate (and hence no change in the bond price). As a result, the purely passive fiscal adjustment has no implications for activity or inflation. So, as expected, under a passive specification of the fiscal reaction function, the model behaves exactly like a textbook New Keynesian model: movements in the primary surplus and government debt stock have no implications for monetary policy.

In contrast, when fiscal policy is active the debt reduction scenario has substantial implications for activity and inflation. Focusing first on the baseline model (solid black lines), the primary surplus (panel (e)) barely responds to the debt reduction scenario. This places a greater burden on bond prices and inflation to ensure that the debt stock is reduced as announced. The optimal response is to generate a burst of inflation (panel (b)), which creates fiscal space to finance the retirement of bonds (panel (f)). Since the policymaker's influence on inflation is via the Phillips curve (3.11), the inflation is created by generating a strongly positive output gap (panel (a)). A small increase in the policy rate (panel (c)), so that expected *real* interest rates fall, supports the boom.[33]

For the variant of the model with long-duration debt (red dashed lines), the broad patterns of the responses in Figure 3.9 are similar. The primary surplus does not increase materially in response to the planned reduction of government debt and a prolonged period of high inflation helps to create fiscal space to reduce the debt stock.

However, two important differences emerge for the variant with longer duration debt, relative to the baseline model. First, the rise in the output gap and inflation are more moderate, but also much more persistent. Second, the fact that the optimal response is to generate a prolonged boom influences the *relative* magnitudes of the output gap and inflation responses.[34]

[33]Although the *real* interest rate declines (supporting a positive output gap), the persistent inflation response implies that the policy rate must rise, in line with the Fisher identity. The responses therefore exhibit some 'neo-Fisherian' qualities, often associated with equilibrium outcomes under the fiscal theory of the price level.

[34]The Phillips curve (3.11) can be iterated forwards to express current inflation as the expected

3.6.3 The importance of expectations

The results from the previous subsection show that the macroeconomic effects of permanent changes in government debt stocks depend critically on whether the government pursues an active or passive fiscal policy. As noted by Cochrane (2018b, p356), the extent to which primary surpluses will be adjusted sufficiently to ensure that the government debt stock is stabilized depends in large part on expectations about *future* fiscal policy behavior.[35]

It is not unreasonable to assume that, in some situations, there may be uncertainty over the government's willingness and/or ability to stabilize government debt for any path of the price level. For example, at the time of writing, there is an active debate about the extent to which the Italian government's fiscal plans are sustainable.[36]

The idea that policy behavior may change over time has been studied extensively. In particular, the sharp distinctions between the implications of 'passive' and 'active' policy regimes studied by Leeper (1991) become more nuanced if those regimes change over time. For example, in many monetary models the coefficient on inflation in a simple Taylor (1993) rule must be greater than unity for monetary policy to be 'active'.[37] This condition is sometimes called the 'Taylor principle'.[38] However, Davig and Leeper (2007) demonstrate that a unique stable rational expectations equilibrium may exist even if the coefficient on inflation violates the Taylor principle in some periods. A unique rational expectations equilibrium can be delivered if there is a large enough probability of the coefficient satisfying the Taylor principle for a sufficiently long duration.

discounted value of current and future output gaps. The fact that the output gap response is more persistent in the long-duration parameterization increases the *relative* response of inflation.

[35]This point has been long understood (see, for example, Barro, 1979).

[36]See, for example, Financial Times (2018).

[37]The Taylor (1993) rule is an equation for the policy rate as a linear function of a constant, the inflation rate and the output gap.

[38]The general requirement, for a very wide range of models, is that the responsiveness of the policy rate to inflation is 'strong enough' to deliver a unique rational expectations equilibrium. So there is a typically a threshold value for the coefficient on inflation in a simple rule, above which the monetary policy rule is 'active'. Whether this threshold value is equal to unity depends on the details of the model.

The Davig and Leeper (2007) result is obtained in a model in which the values of the coefficients in the monetary policy rule depend evolve according to a Markov process. This means that the probability of transitioning from one set of coefficient values to another is determined only by the current policy 'regime'. The Markov property facilitates the development of solution algorithms that can be applied to a wide range of linear rational expectations models (see, for example, Svensson and Williams, 2008; Farmer, Waggoner, and Zha, 2009, 2011; Blake and Zampolli, 2011).

These Markov-switching rational expectations methods have been applied to a wide range of applications studying uncertainty about fiscal policy behavior. For example, Francesco Bianchi, Leonardo Melosi and coauthors have used this type of framework to study the past and possible future behavior of inflation in the United States (see, for example, Bianchi, 2012; Bianchi and Ilut, 2017; Bianchi and Melosi, 2014, 2017, 2018a,b).[39] Many of these studies explore the possibility that US inflation dynamics may have been driven by changing expectations of the fiscal policy regime, in particular whether future policy would be active or passive.

The Markov switching approach has also been used to explore the prospects for inflation in the aftermath of the Global Financial Crisis, which led to a large increase in government debt in many countries and subsequent fiscal tightening in many cases. For example, Bi, Leeper, and Leith (2013) study the effects of uncertainty about the timing, pace and composition of a planned fiscal consolidation.

The majority of these studies assume that policy behavior is described using simple policy rules. Notable exceptions are Svensson and Williams (2008) and Blake and Zampolli (2011), who provide algorithms to solve for optimal policy behavior when structural parameters of the model economy evolve according to a Markov process. However, even in these cases, the probability of a structural change is exogenous with respect to macroeconomic variables.

The following subsection outlines an approach to solve for time-consistent optimal monetary policy in the presence of a *time-varying* probability of a change in fiscal policy behavior. The solution approach is extended to allow for the probability

[39]Some of these investigations use expanded versions of the Markov switching framework in which the policy regime may be imperfectly observed by private agents, requiring specialized solution approaches such as that developed by Bianchi and Melosi (2016).

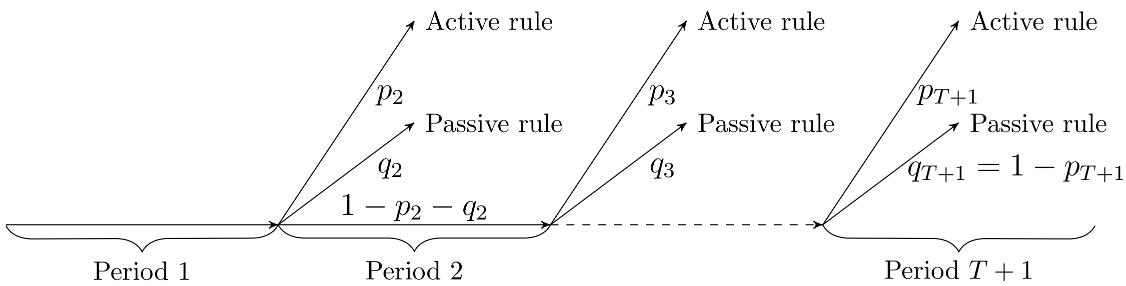
to depend on the values of macroeconomic variables. This allows the analysis of the (plausible) situation in which the probability of a change in fiscal regime depends on the state of the public finances.

3.6.4 Fiscal uncertainty: a simple framework

This subsection describes a solution algorithm to explore the effects of uncertainty about future fiscal policy behavior. An important innovation relative to studies using the Markov-switching approach is that the probabilities of a switch in fiscal policy may change over time and may also respond to the state of the economy.

The specific environment is illustrated in Figure 3.10.

Figure 3.10: The fiscal risk environment



The scenario starts in period 1, with fiscal policy assumed to be conducted using the passive rule from Table 3.5. In each period $2, \ldots, T$, the fiscal rule may *permanently* switch to one of two alternatives. If transition does not occur in a particular period, then the fiscal policy rule remains passive until the next period, when there is once more a set of exit probabilities associated with switching to a new fiscal rule. Note that the probabilities of switching to each of the fiscal policy rules are allowed to change over time.

For example, at the start of period 2, the economy will shift to an active fiscal rule with probability p_2 and to the passive rule with probability q_2. In both of these cases, uncertainty is fully resolved as the switches are permanent. Alternatively, fiscal policy remains passive with probability $1 - p_2 - q_2$ and the risk of a switch in fiscal policy behavior remains in period 3. All uncertainty is resolved in period $T + 1$ so that $q_{T+1} = 1 - p_{T+1}$.

This structure is intended to capture the effects of a persistent risk that fiscal policy may switch to an active rule.[40] However, the environment is also restrictive in some respects. For example, all uncertainty is resolved beyond a finite horizon $(T + 1)$, though this could be calibrated to be arbitrarily large.

The fact that a switch in fiscal rule is assumed to be permanent, once it has occurred, is also somewhat restrictive. The benefit is that the states to which the model switches are absorbing states. As shown in Appendix 3.G this permits solution of the model under optimal time-consistent policy using a backward-induction method. It also allows the probabilities of a switch to an active policy rule to vary over time. Both of these innovations create a richer environment than some that have been studied with Markov-switching models using simple monetary policy rules, such as those discussed in Section 3.6.3. Moreover, the solution method could be extended quite easily to incorporate more persistent uncertainty by assuming that fiscal rule switches involve transitions to variants with Markov switching fiscal policy.[41]

3.6.5 Exogenous fiscal uncertainty

The first experiment reconsiders the debt reduction experiment of Section 3.6.2 when there is a risk that the fiscal rule switches to become active. As before, in period 1 it is announced that the debt stock will be reduced by 10pp of annual (steady-state) GDP. The debt reduction is perfectly credible, but there is uncertainty over the fiscal rule that will be place to deliver that reduction.

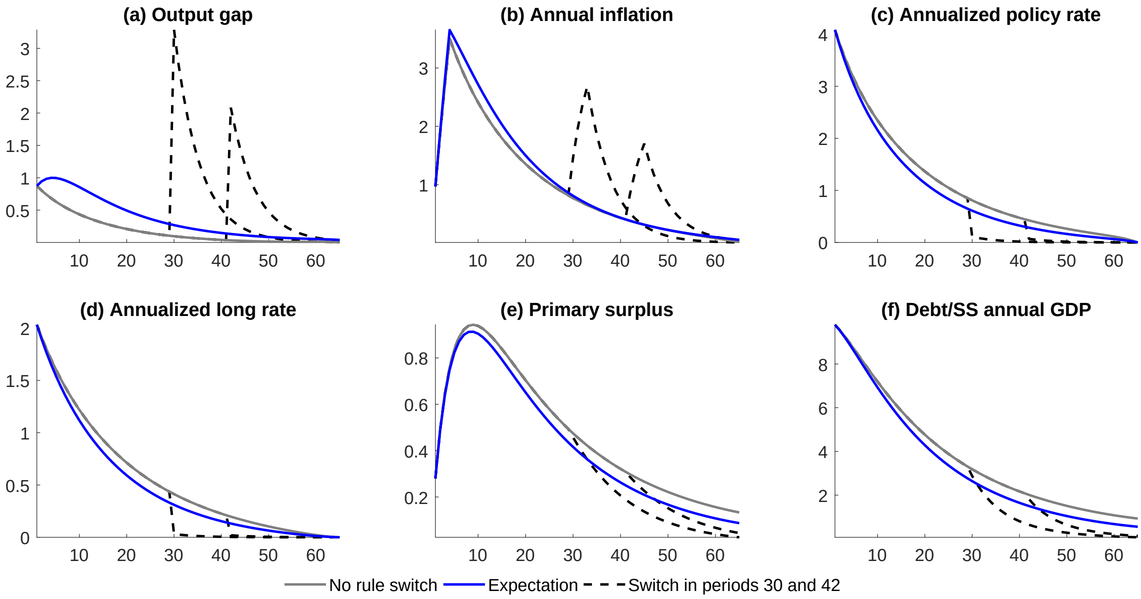
I consider the case in which the probability of switching to the active fiscal policy rule is 1% per quarter for $T = 65$ quarters. Specifically $p_t = 0.01, t = 2, \ldots, T$ and $q_t = 0, t = 2, \ldots, T$ with $p_{T+1} = 0$ and $q_{T+1} = 1$. This means that, until period T, there is a 1% chance each quarter that fiscal policy permanently switches to the

[40] The fact that one of the possible switches involves a shift to the passive fiscal rule may seem puzzling, given that the baseline, 'no switch', assumption (moving along the horizontal arrowed lines) is also that the fiscal policy rule is passive. The rationale this setup is simply to allow for the case in which the risk is temporary. Doing so requires the passive fiscal rule to be an 'exit state', given the assumption that all uncertainty is resolved in period $T + 1$.

[41] The solution of such a model under time-consistent policy has a representation that allows the key objects in the solution algorithm described in Appendix 3.G to be easily computed (Blake and Zampolli, 2011).

'active' rule. If period T is reached without a switch to the active policy rule, then the fiscal policy rule remains passive with certainty thereafter (since $p_{T+1} = 0$ and $q_{T+1} = 1$). This parameterization implies that the probability that the fiscal rule does *not* switch to the active rule is $(1 - 0.01)^{64} \approx 0.53$. So it is more likely than not that the fiscal rule will remain passive throughout the debt reduction process.

Figure 3.11: Debt reduction with fiscal risk: baseline model



Notes: The panels show the result of a debt reduction scenario with exogenous fiscal risk in the baseline model. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. For $t = 2, \ldots, T$, there is a 1% probability that the fiscal policy rule permanently switches to active, conditional on a switch not having already occurred. If a switch has not occurred by period T, then fiscal policy remains passive from period $T+1$ onwards with certainty. The solid grey line shows the outcomes conditional on a switch never occurring. The solid blue line shows the expected path. The dashed black lines show outcomes in which a switch in the fiscal rule occurs in periods 30 and 42. All variables are plotted as percentage point deviations from the final steady state.

Figure 3.11 shows the results. The grey lines depict the outcomes when no fiscal rule switch occurs. Since the *ex ante* probability of observing these trajectories is around 0.53, this is (by far) the most likely outcome. The blue lines depict the expected outcomes, which are the probability weighted averages of the trajectories corresponding to a switch to an active fiscal rule in periods $2, \ldots, T$. The dashed black lines show two illustrative trajectories corresponding to the case in which the fiscal rule becomes active in periods 30 and 42.

Recall that, if fiscal policy is known to be passive with certainty, the debt reduction has no effect on the output gap or inflation (see the grey lines in Figure 3.9). Figure 3.11 shows that the presence of fiscal risk overturns this result. The grey lines show the most likely trajectories for the key variables, the 'modal path'. Along the modal path, the output gap and inflation are both positive. The modal path for the policy rate also lies above the steady state.

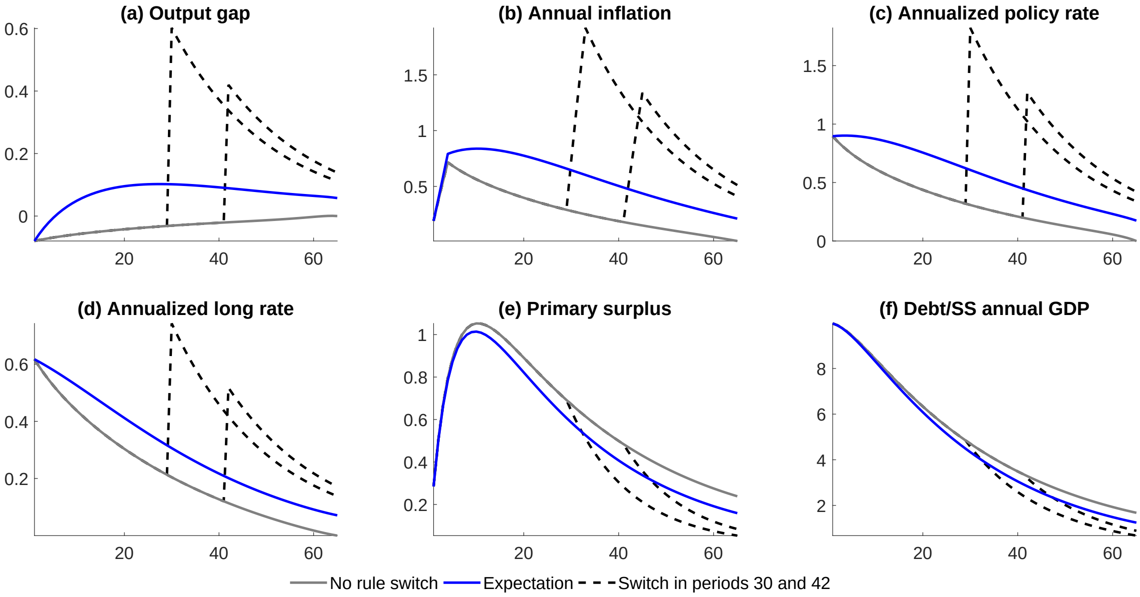
To understand the modal path outcomes, first note that the *expected* path (blue line) for inflation lies above the modal path (grey line). This reflects the fact that, each period, there is a risk that the fiscal rule becomes active, in which case inflation rises markedly: the dashed lines show the cases in which such a switch occurs in periods 30 and 42. Higher inflation expectations are consistent with a positive expected path for the output gap. Although the policy rate rises above steady state, the expected path of the *real* interest rate is below steady state, providing the stimulus required for the output gap to rise above zero.

The results indicate that the policymaker is willing to accommodate some of the effects of higher inflation expectations in the form of higher actual inflation in the near term. It is optimal to do so because this inflation erodes the real value of debt and this reduces the losses incurred in states of the world in which the fiscal policy rule becomes active.

In the event that a switch to active fiscal policy occurs (dashed black lines) the pattern of the responses are qualitatively similar to those studied under certainty in Figure 3.9 (solid black lines). Unsurprisingly, the surge in inflation and the output gap is smaller if the switch occurs later, since by this time the debt stock has already fallen somewhat towards the new steady state.

Figure 3.12 repeats the experiment for long duration parameterization. The broad contours for many variables are similar to the baseline model. The fiscal risk increases inflation expectations and the policymaker accommodates this by allowing positive inflation along the modal path. Unsurprisingly, given the results of Figure 3.9, the increases in inflation associated with a fiscal rule switch (dashed black lines) are somewhat smaller than the baseline model. This has a smaller effect on inflation expectations and so the increases in inflation and the policy rate along the modal

Figure 3.12: Debt reduction with fiscal risk: long duration debt variant



Notes: The panels show the result of a debt reduction scenario with exogenous fiscal risk in the variant of the model with long-duration government debt. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. For $t = 2, \ldots, T$, there is a 1% probability that the fiscal policy rule permanently switches to active, conditional on a switch not having already occurred. If a switch has not occurred by period T, then fiscal policy remains passive from period $T+1$ onwards with certainty. The solid grey line shows the outcomes conditional on a switch never occurring. The solid blue line shows the expected path. The dashed black lines show outcomes in which a switch in the fiscal rule occurs in periods 30 and 42. All variables are plotted as percentage point deviations from the final steady state.

path are also smaller than in the baseline model.

One important difference in the results for the long duration variant is that the modal path for the output gap is *negative*. The expected path of the output gap is slightly positive beyond the first few periods, consistent with positive inflation and expected inflation. In this case, the upward pressure on inflation expectations acts like a cost push shock, increasing the rate of inflation consistent with a closed output gap. The optimal response is to tighten monetary policy and trade off a smaller rise in inflation against a slightly negative output gap.

The relative magnitudes of the output gap and inflation responses in the event that fiscal policy switches to become active are important determinants of this result. In the long duration variant, the rise in annual inflation in the event of a switch is relatively large, compared with the rise in the output gap. As discussed previously, the long-duration variant generates more persistent movements in the output gap and inflation. A small but very persistent positive expected output gap is sufficient to raise inflation by a (relatively) large amount. As a result, the rise in inflation expectations is driven by expectations of a higher output gap in the more distant future, rather than the current output gap. As described above, the rise in inflation expectations therefore resembles a cost-push disturbance which induces the policymaker to generate a negative output gap along the modal path.

3.6.6 Endogenous fiscal uncertainty

The fiscal risk environment allows the probability of switching to an active fiscal policy rule to change over time. A particular case of interest is one in which the probability is related to macroeconomic outcomes. For example, the probability of a switch to active policy may be related to the size of the debt stock or primary surplus.

Appendix 3.G.9 demonstrates how the solution algorithm can be extended so that the monetary policymaker optimally accounts for the effects of their policy actions on the probability of a switch in the fiscal rule. To provide some intuition for that extension, consider a simplified two-period example. Suppose the policymaker's objective is to choose the instrument x to minimize:

$$\mathcal{L} \equiv L^P(x_1, x_0) + \left[p_2 L^A(x_2, x_1) + (1 - p_2) L^P(x_2, x_1)\right]$$

where L^P and L^A denote losses under 'passive' and 'active' fiscal policy rules. These losses are different because the structure of the economy, and hence the achievable macroeconomic outcomes, depend on the fiscal policy regime.

Losses in period 2 depend on the current (x_2) and lagged (x_1) instrument in the presence of endogenous state variables. The term in square brackets represents the expected loss in period 2 when the probability of switching to the active fiscal policy in period 2 is p_2. For now, it is assumed that the probability is treated as exogenous by the policymaker.

As in the general algorithm presented in Appendix 3.G, this problem can be solved by backward induction. In period 2, the fiscal policy rule will either be active or passive. In each case, there will be an optimal policy response function for the instrument setting at date 2.[42] Specifically, let $x_2 = f_A(x_1)$ and $x_2 = f_P(x_1)$ be the best response functions under active and passive fiscal policies respectively.

Then the policy problem is to minimize:

$$\mathcal{L} \equiv L^P(x_1, x_0) + \left[p_2 L^A(f_A(x_1), x_1) + (1 - p_2) L^P(f_P(x_1), x_1)\right]$$

with respect to x_1.

The first order condition is:

$$\begin{aligned} 0 = {} & L_1^P(x_1, x_0) + p_2\left[L_1^A(f_A(x_1), x_1) f'_A(x_1) + L_2^A(f_A(x_1), x_1)\right] \\ & + (1 - p_2)\left[L_1^P(f_P(x_1), x_1) f'_P(x_1) + L_2^P(f_P(x_1), x_1)\right] \end{aligned} \tag{3.31}$$

where L_i^A and L_i^P represent the partial derivatives of L^A and L^P with respect to arguments $i = 1, 2$ and f'_A and f'_P represent the first derivatives of f_A and f_P respectively. The algorithm presented in Appendix 3.G demonstrates how to find the best response functions $x_2 = f_A(x_1)$ and $x_2 = f_P(x_1)$ and the optimal decision x_1 by backward induction (for $T > 2$ this involves solving a sequence of first order conditions).

Now suppose that the probability of switching to active fiscal policy in period 2 is endogenous and specifically that it depends on x_1, so that $p_2 = p_2(x_1)$.

In this case, the first order condition is:

$$\begin{aligned} 0 = {} & L_1^P(x_1, x_0) + p_2(x_1)\left[L_1^A(f_A(x_1), x_1) f'_A(x_1) + L_2^A(f_A(x_1), x_1)\right] \\ & + p'_2(x_1) L^A\left(f^A(x_1), x_1\right) \\ & + (1 - p_2(x_1))\left[L_1^P(f_P(x_1), x_1) f'_P(x_1) + L_2^P(f_P(x_1), x_1)\right] \\ & - p'_2(x_1) L^P\left(f^P(x_1), x_1\right) \end{aligned} \tag{3.32}$$

[42] As discussed in Appendix 3.G.2, it is assumed that it is revealed at the start of each period whether or not a switch in fiscal policy rule has taken place. This information is revealed before the policymaker and private agents make their decisions for that period.

where the policymaker now accounts for the effects of their policy decision in period 1 on the probability of a fiscal rule switch in period 2. This effect is captured by the terms $p_2'(x_1) \equiv \frac{\partial p_2(x_1)}{\partial x_1}$.

Comparing equations (3.31) and (3.32) reveals that treating the probabilities as if they are exogenous (that is, using (3.31)) when they are in fact endogenous omits the following term from the first order condition:

$$\mathcal{T} = p_2'(x_1)\left[L^A\left(f^A(x_1), x_1\right) - L^P\left(f^P(x_1), x_1\right)\right]$$

Appendix 3.G.9 demonstrates how to compute the equivalent of $\mathcal{T}$ for the general problem and specifies an iterative process for incorporating it into the first order conditions for optimal policy.

To incorporate endogenous uncertainty into the debt reduction scenario, I suppose that the probability of a switch is determined by an exponential distribution function:

$$p_t = f(\bar{s}_{t-1}) = \begin{cases} 0 & \text{if } \bar{s}_{t-1} \leq 0 \\ 1 - \exp\left(-\xi^{-1}\bar{s}_{t-1}\right) & \text{if } \bar{s}_{t-1} > 0 \end{cases} \tag{3.33}$$

with $\xi > 0$. Since the timing protocol is such that agents learn whether or not a switch in period t occurs before they take their period t decisions, p_t is a function of the primary surplus in period $t-1$.

This specification means that the likelihood of a switch to the active fiscal rule is increasing in the primary surplus. The motivation for this assumption is that a higher primary surplus increases the fiscal burden on households, since taxes are high and/or government spending is low. A higher fiscal burden on households may increase the likelihood that the government adopts an active fiscal policy.[43] Under active fiscal policy, primary surpluses can be reduced, since they are not required to adjust to stabilize the debt stock, easing the tax burden on households.

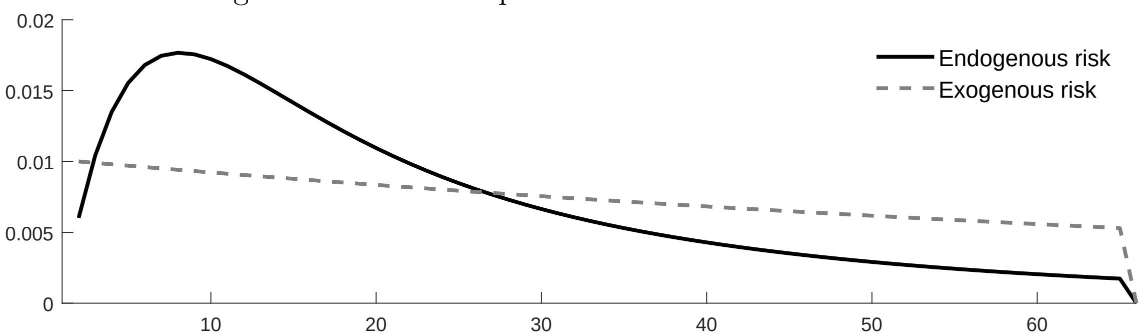
While analysis of sovereign default often considers default probabilities that depend on the level of debt (for example Corsetti, Kuester, Meier, and Müller, 2013),

[43] That could come about through political pressure, a change in administration or the government acting preemptively to avoid such an outcome.

I assume that the probability of a fiscal rule switch depends on the primary surplus. One motivation for this assumption is the literature on 'fiscal limits', which posit that there is a maximal level of the primary surplus that can be achieved by the government (Davig et al., 2011; Bi, 2012). If the value of outstanding debt is larger than the present discounted value of the maximal surplus, then the debt stock cannot be financed by future surpluses. Allowing the probability of a fiscal rule switch to depend on the level of the primary surplus could therefore be interpreted in terms of uncertainty over the maximal level of the primary surplus and hence the fiscal limit.[44]

The parameter ξ controls the strength of the mapping from the primary surplus to the probability of switching to the active fiscal rule. For example, a small value of ξ implies that small primary surpluses are sufficient to imply a high probability of switching to the active fiscal policy rule. The simulation below uses $\xi = 46$, which is chosen so that the *ex ante* probability of *not* switching to the active fiscal rule by period T is approximately 0.53. That is the same probability as the exogenous fiscal uncertainty simulation in Section 3.6.5.

Figure 3.13: *Ex ante* probabilities of fiscal rule switch



Notes: The figure plots *ex ante* probabilities of a switch to the active fiscal rule. These probabilities are computed as $p_t \prod_{s=1}^{t-1} (1 - p_s)$, where p_t is the probability of a switch to active policy in period t conditional on a switch not having occurred previously. The solid black line shows the probabilities under endogenous fiscal risk and the dashed grey line shows the probabilities under exogenous risk (with a fixed conditional probability $p_t = 0.01, t = 2, \ldots, T$).

Figure 3.13 shows the rule switching probabilities associated with this simulation, compared to the exogenous risk simulation from Section 3.6.5. The figure shows the

[44] The fiscal limit would also depend on the level of outstanding debt and economic fundamentals such as productivity (see Bi, 2012).

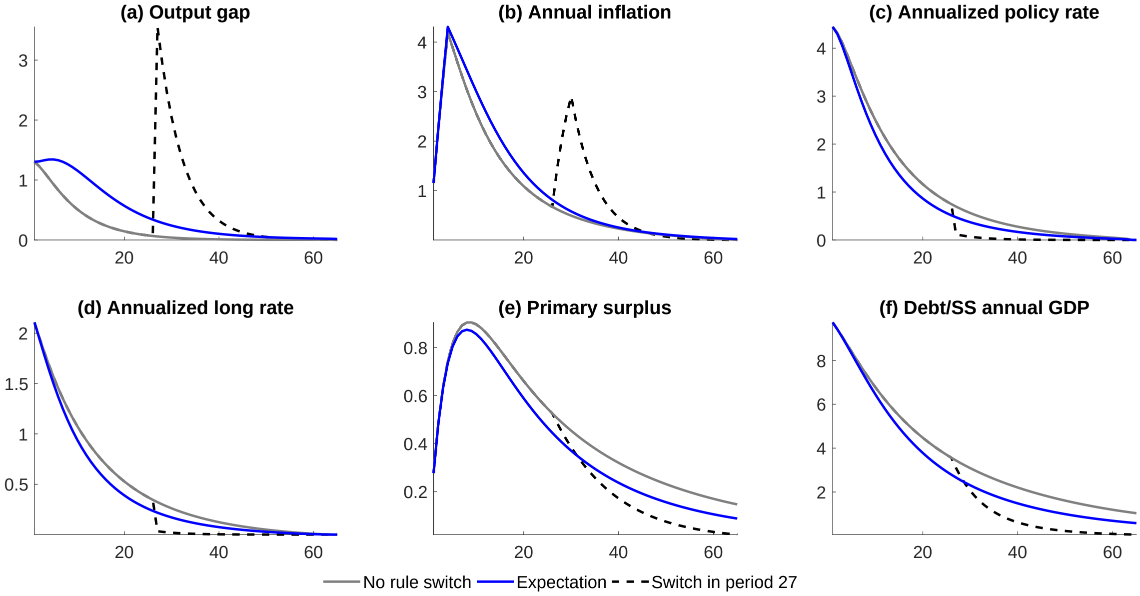
ex ante probability (i.e., from the perspective of period 1 of the simulation) of a switch in each period. Given that $q_t = 0, t = 2, \ldots, T$, the *ex ante* probability of a switch to the active fiscal rule in period $t \geq 2$ is given by $p_t \prod_{s=1}^{t-1} (1 - p_s)$.

The exogenous probability case (grey line) from Section 3.6.5 assumed a 1% probability of a switch occurring, conditional on a switch not having already occurred (i.e., $p_s = 0.01$). The *ex ante* probability therefore declines from a peak value of 0.01 in period 2. In contrast, the *ex ante* probabilities for the endogenous risk profile have a hump-shaped profile, reflecting the endogenous dependence on the primary surplus, $\bar{s}_{t-1}$. The choice of ξ ensures that the integrals beneath the black and grey lines are equal, so that the *ex ante* probability of a fiscal rule switch *not* occurring is the same. The difference, therefore, is the time profile of that risk.

Figure 3.14 shows the results of the simulation under endogenous fiscal uncertainty. As previously, the grey lines show the most likely outcome (with *ex ante* probability 0.53) that a switch to active fiscal policy does *not* occur. The blue line shows the expected trajectory (based on information in period 1). The dashed line shows the case in which a switch to active fiscal policy occurs in period 27. This period is chosen because the unconditional probability of a fiscal switch in the endogenous and exogenous experiments are roughly equal at about 0.8% (Figure 3.13).

Comparing the results with those under exogenous fiscal uncertainty (Figure 3.11) reveals that the responses of output and inflation along the modal and expected paths are more 'front loaded'. This reflects the fact that a switch to active fiscal policy is more likely when the primary surplus is higher, which occurs in the earlier part of the simulation. However, relative to the exogenous probability experiment, the probability of a switch between periods 3 and 25 is relatively high (Figure 3.13). The inflationary impact of a switch to the active fiscal rule is larger when the debt stock is high. So higher probabilities of a switch when the debt stock is high increases inflation expectations, relative to the exogenous probability example. This makes the stabilization problem faced by the monetary policymaker more difficult and a larger deviation of the output gap and inflation are optimally accommodated.

Figure 3.14: Debt reduction with endogenous fiscal risk: baseline model, $\xi = 46$



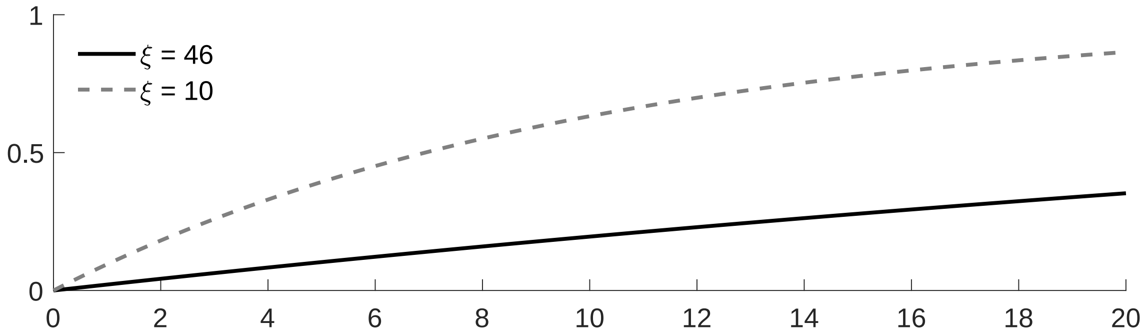
Notes: The panels show the result of a debt reduction scenario with *endogenous* fiscal risk in the baseline model. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. For $t = 2, \ldots, T$, the probability of a switch to active fiscal policy in period t, conditional on a switch not having already occurred, is determined by (3.33), with $\xi = 46$. If a switch has not occurred by period T, then fiscal policy remains passive from period $T + 1$ onwards with certainty. The solid grey line shows the outcomes conditional on a switch never occurring. The solid blue line shows the expected path. The dashed black lines show outcomes in which a switch in the fiscal rule occurs in period 27.

3.6.7 Risk reduction and time-consistent policy

As noted previously, the endogenous probabilities of a switch to active fiscal policy are in part determined by the choices of the monetary policymaker. The policymaker sets policy to minimize expected losses, taking into account the effects of their decisions on future outcomes. As discussed above, the effects of policy choices on the primary surplus have two effects. First, the modal path of the primary surplus affects the probability of a switch to active fiscal policy. Second, the modal path of the primary surplus affects the modal path of the debt stock and hence the impact of a switch to active fiscal policy.

To examine these effects further, I consider an alternative calibration for (3.33): setting $\xi = 10$. This implies a higher probability of a switch to active fiscal policy for a given primary surplus ($\bar{s}$). Figure 3.15 plots (3.33) for $\xi = 10$ against the $\xi = 46$ value used previously.

Figure 3.15: Endogenous probability function, (3.33), for $\xi = 10$ and $\xi = 46$

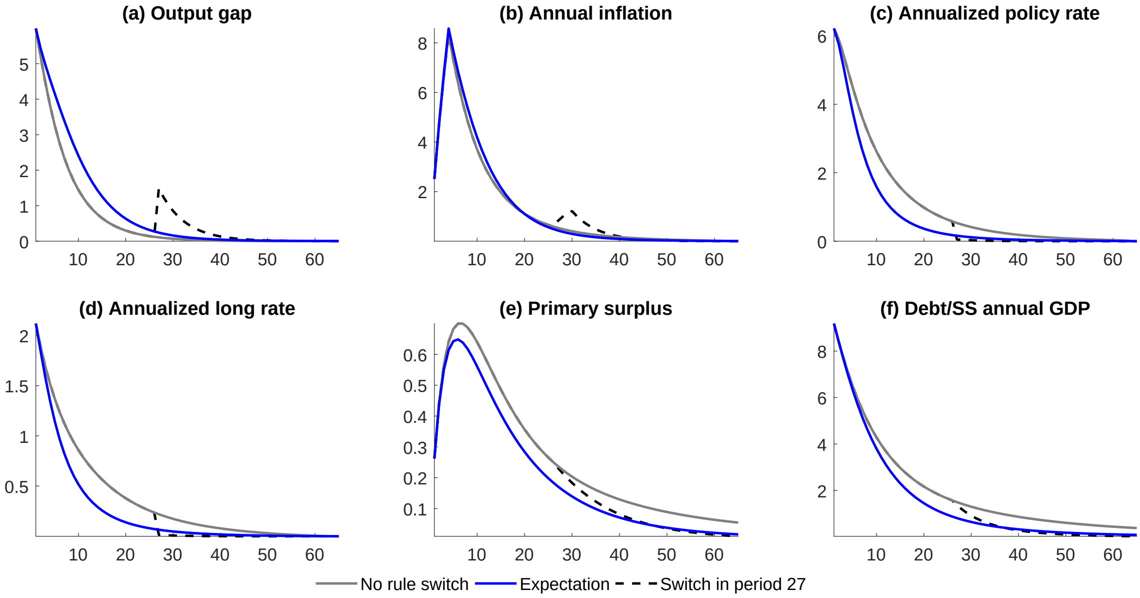


Notes: The solid black line plots (3.33) with $\xi = 46$ and the dashed grey dashed line for $\xi = 10$. The x-axis measures $\bar{s}$.

Figure 3.16 shows the outcome of the endogenous risk simulation using $\xi = 10$. Compared with the case with $\xi = 46$ (Figure 3.14), the probability of a switch to active fiscal policy is markedly higher in the first few periods of the simulation. The probability that a switch to active fiscal policy takes place at some point during the simulation is 81%, substantially higher than the 53% in Figure 3.14. However, it is still the case that the modal path is the one in which a switch to active fiscal policy never occurs.

The results of Figure 3.16 demonstrate that a higher risk of a switch to active fiscal policy increases the amount of inflation that the policymaker accommodates in the near term. The modal paths for the output gap and inflation in Figure 3.16 are materially higher than those in Figure 3.14. However, the effects of a switch to active fiscal policy (dashed black lines) are smaller, should they occur. As noted above, this result is delivered by the fact that higher inflation along the modal path reduces the debt stock more rapidly so that a switch to active fiscal policy is less costly. A higher risk of switching to active fiscal policy therefore leads to higher welfare losses along the modal path and lower losses in the event that a switch actually occurs.

Figure 3.16: Debt reduction with endogenous fiscal risk: baseline model, $\xi = 10$



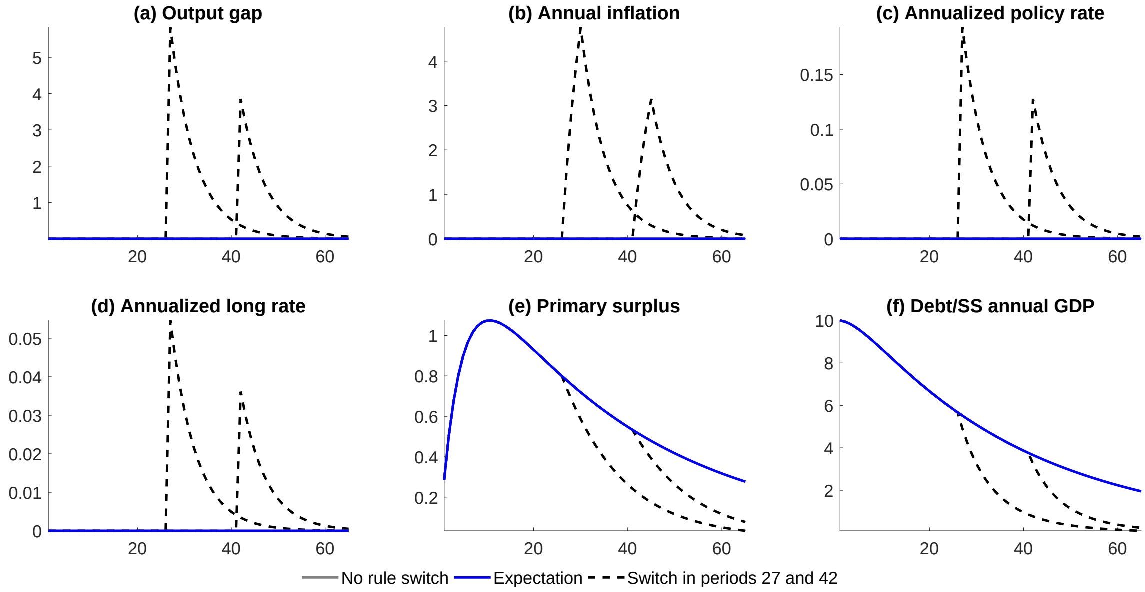
Notes: The panels show the result of a debt reduction scenario with *endogenous* fiscal risk in the baseline model. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. For $t = 2, \ldots, T$, the probability of a switch to active fiscal policy in period t, conditional on a switch not having already occurred, is determined by (3.33), with $\xi = 10$. If a switch has not occurred by period T, then fiscal policy remains passive from period $T + 1$ onwards with certainty. The solid grey line shows the outcomes conditional on a switch never occurring. The solid blue line shows the expected path. The dashed black lines show outcomes in which a switch in the fiscal rule occurs in period 27.

To examine the contributions of losses in the central case versus losses when a switch to active fiscal policy occurs, I consider a hypothetical experiment. In this experiment, all agents (including the policymaker) believe that the risk of a switch to active fiscal policy is always zero (the case considered in Section 3.6.2). But, in fact, the probability of a switch to active fiscal policy is given by equation (3.33). If it occurs, a switch to active fiscal policy is a complete surprise.

Figure 3.17 shows the outcomes in this hypothetical case.[45] The results show that, as expected, the modal paths coincide with the passive fiscal policy results in Figure 3.9. Because agents (wrongly) believe that the probability of a switch to active fiscal policy is zero, the expected paths (solid blue lines) lie on top of the

[45] The solution algorithm allows this case to be analyzed by setting the probability of a switch to active fiscal policy to zero in each period ($p_t = 0, t = 1, \ldots, T$).

Figure 3.17: Debt reduction with zero perceived fiscal risk: baseline model



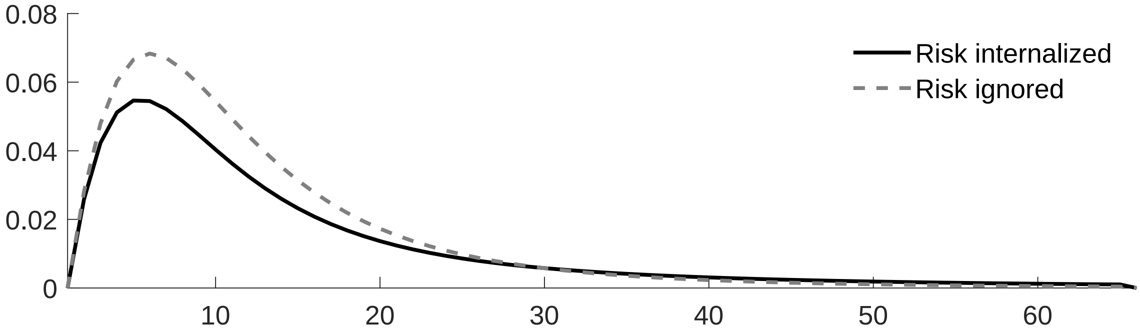
Notes: The panels show the result of a debt reduction scenario with exogenous fiscal risk in the baseline model. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. All agents (including the policymaker) believe that fiscal policy will remain passive with certainty. However, for $t = 2, \ldots, T$, the probability that the fiscal policy rule permanently switches to active, conditional on a switch not having already occurred, is given by (3.33). The solid grey line shows the outcomes conditional on a switch never occurring. The solid blue line shows the expected path. The dashed black lines show outcomes in which a switch in the fiscal rule occurs in periods 27 and 42. All variables are plotted as percentage point deviations from the final steady state.

modal paths. Along the modal path, inflation is stable and the debt stock falls slowly. As a result, the effects of a (completely unanticipated) switch to active fiscal policy (dashed black lines) are larger than in Figures 3.11 and 3.14. Moreover, the probability of a switch to active fiscal policy is generally higher than the case studied in Figure 3.14, because the modal path for the primary surplus is higher. This is illustrated in Figure 3.18.

The net effect of (a) better outcomes along the modal path and (b) worse outcomes in the event of a switch to active fiscal policy can be assessed by computing the expected losses associated with each simulation.[46] This is done in Figure 3.19, which plots conditional expected losses for the experiment in which the effects of

[46] Appendix 3.G.13 describes the necessary computations.

Figure 3.18: *Ex ante* probabilities of switch to active fiscal policy rule



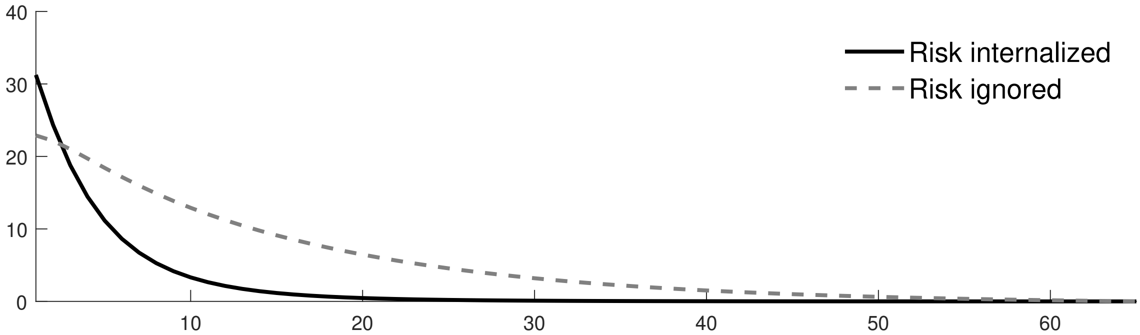
Notes: The solid black line shows the *ex ante* probability of a switch to active fiscal policy for the experiment in which agents and the policymaker internalize the endogeneity of risk (Figure 3.16). The dashed grey line shows the *ex ante* probability of a switch to active fiscal policy for the experiment in which agents and the policymaker completely ignore the risk (Figure 3.17). In both cases the *ex ante* probability is computed as $p_t \prod_{j=1}^{t-1} (1 - p_j)$ where p_t is the conditional probability of a switch to active fiscal policy conditional on such a switch having not already occurred.

endogenous probabilities are internalized (solid black line) and completely ignored (dashed grey line). The expected losses are computed conditional on a switch to active fiscal policy having not already occurred. Given the underlying informational assumptions, the expected loss in period 1 is the unconditional expected loss for the debt reduction scenario. Importantly, for the case in which the risk is ignored, expected losses are computed using the true probabilities of a switch (shown in Figure 3.18).[47]

Figure 3.19 demonstrates that the expected loss in period 1 is *higher* for the case in which the policymaker internalizes the endogeneity of the probability of a switch to active fiscal policy. Expected losses (conditional on a switch not having already occurred) are lower from period 3 onwards when the risk is internalized. This reflects the two factors discussed previously. First, the probability of of a switch to active fiscal policy is lower when the risk is internalized (Figure 3.18). Second, when the risk is internalized, the high inflation in the near term (Figure 3.16, panel(b)) reduces the real value of debt and hence the impact of a switch to active fiscal policy, conditional on a switch having not already occurred.

[47]In contrast, Figure 3.17 plots expectations under the hypothetical case in which agents incorrectly believe the risk to be zero in all periods.

Figure 3.19: Expected losses when effects of endogenous probabilities are internalized and ignored



Notes: The solid black line shows expected losses for the experiment in which agents and the policymaker internalize the endogeneity of risk (Figure 3.16). The dashed grey line shows expected losses for the experiment in which agents and the policymaker completely ignore the risk (Figure 3.17). In both cases expected loss is computed conditional on a switch to active fiscal policy having *not* previously occurred.

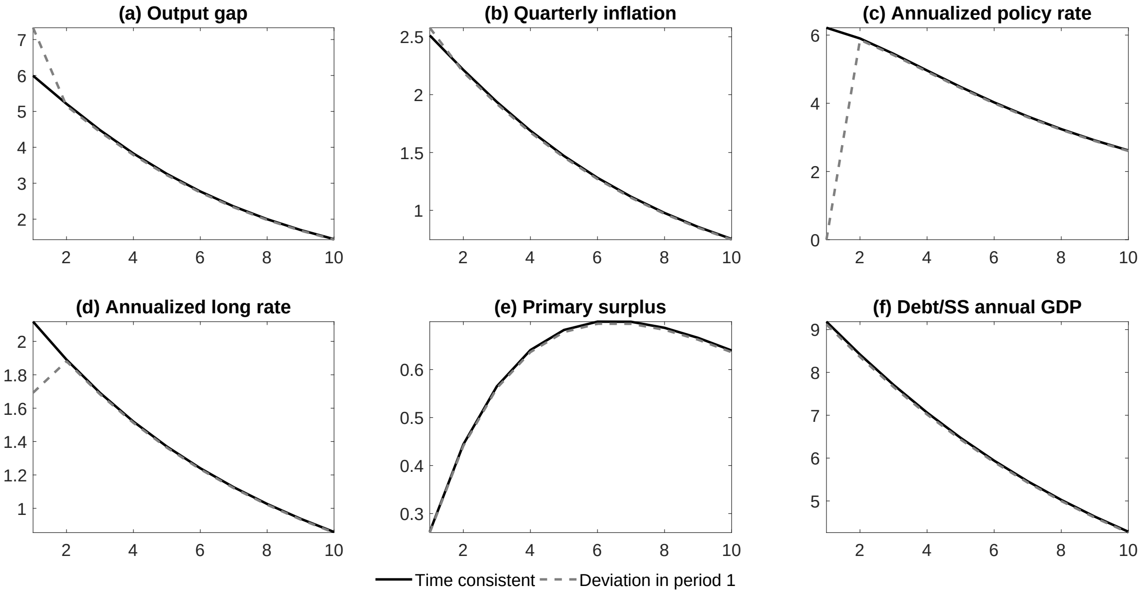
However, lower expected conditional losses from period 3 onwards can only be achieved by higher inflation in the near term. The higher inflation is sufficient to deliver the result that the unconditional expected loss for the debt reduction scenario is higher when the risk is internalized rather than ignored.

At first sight, this result may seem puzzling. It implies that 'ignorance is bliss': lower welfare losses can be achieved if the policymaker and private agents ignore the risk. The explanation is simply that policy behavior is restricted to be time consistent. Recall that this means that the policymaker acts as a Stackelberg leader with respect to the private sector and future policymakers. The policymaker internalizes the fact that future policymakers will also act in a time consistent manner. The private sector internalizes that current and future policymakers will behave in this way. While ignoring the risk may deliver a lower unconditional expected loss, it is not possible to coordinate beliefs on such an equilibrium as it violates time consistency. There is an incentive to deviate from this set of beliefs.

To demonstrate this, Figure 3.20 presents a variant of the scenario in which it is assumed that, in period 1, the policymaker deviates from the time-consistent policy plan. Specifically, in period 1, the instrument is set to the level that is optimal if the risk is ignored by all agents and policymakers in all periods.[48] However, private

[48]From Figure 3.17, panel (c), the policy rate is set to its steady-state value.

Figure 3.20: Debt reduction experiment with deviation from time consistency



Notes: The panels show the result of a debt reduction scenario with exogenous fiscal risk in the baseline model. An announcement is made at the start of period 1 that the debt stock will be permanently reduced by ten percentage points of (steady-state) annual GDP. In period 0, the economy is at steady state. In period 1, fiscal policy is passive. All agents (including the policymaker) believe that fiscal policy will remain passive with certainty. However, for $t = 2, \ldots, T$, the probability that the fiscal policy rule permanently switches to active, conditional on a switch not having already occurred, is given by (3.33). The solid black line shows the outcomes conditional on a switch not occurring under time-consistent monetary policy. The dashed grey lines show outcomes in which the monetary policymaker deviates from time consistent policy in period 1. All variables are plotted as percentage point deviations from the final steady state.

agents believe that future policymakers will behave in a time consistent manner. Appendix 3.G.14 provides details of how this simulation is constructed.

The solid black lines in Figure 3.20 show the modal path outcomes under time consistent monetary policy (as shown in Figure 3.16). The dashed grey lines show the modal path when the policymaker deviates from time consistent policy in period 1, as described above. In this case, agents know that future monetary policy will be time consistent. They therefore recognize that future policymakers have an incentive to create inflation in the near term, for the same reasoning discussed in the description of Figure 3.16. High expected inflation combined with a policy rate set at the steady state level generates even higher inflation (and a large positive output gap) in period 1. The unconditional expected loss when the policymaker deviates in period 1 is therefore higher than the case in which policy is set in a time-consistent manner throughout.[49]

3.7 Conclusion

This chapter has studied the behavior of a simple model with long-term government debt, time consistent monetary policy and active fiscal policy ('fiscal dominance').

The analytical results using the linear quadratic variant of the model show that the duration of government debt plays a key role in determining the extent of the 'debt stabilization bias' (Leith and Wren-Lewis, 2013; Leeper et al., 2019).

The analysis incorporating the possibility that the zero bound may bind indicate that active fiscal policy may support inflation expectations in a liquidity trap, as agents expect higher future inflation to be used to reduce the real value of debt accumulated during the recession. This effect may be sufficient to improve welfare relative to the textbook case in which fiscal policy is passive.

The analysis of fiscal risk shows that, when government debt is materially above its desired level, even small probabilities of a switch to active fiscal policy can generate a rise in inflation expectations that the monetary policymaker accommodates in the form of higher actual inflation. This result is partly driven by the requirement that monetary policy be time consistent.

[49] The unconditional expected loss is 31.65 when the policymaker deviates, compared with 31.24 under fully time consistent monetary policy.

Appendix 3.A The log-linear model

This appendix derives a log-linear representation of the model. For variable X_t, $\hat{X}_t \equiv \ln X_t - \ln X$ defines the log-deviation of X_t from its steady-state value, X. A useful feature of the derivation is that the steady-state level of output is normalized to unity, as described in Appendix 3.A.4.

3.A.1 Households

The first-order conditions for the optimization problem are:

$$\phi_t c_t^{-\frac{1}{\sigma}} = \mu_t P_t \tag{3.34}$$

$$\phi_t n_t^{\psi} = W_t \mu_t \tag{3.35}$$

$$0 = -\mu_t + \beta R_t \mathbb{E}_t \mu_{t+1} \tag{3.36}$$

$$0 = -V_t \mu_t + \beta \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \mu_{t+1} \tag{3.37}$$

where μ is the Lagrange multiplier on the nominal budget constraint (3.4).

Let the real Lagrange multiplier be defined as:

$$\Lambda_t \equiv P_t \mu_t$$

and real short bond holdings and long-term debt as

$$b_t \equiv \frac{B_t}{P_t}$$

$$d_t \equiv \frac{D_t}{P_t}$$

The first order conditions for short-term and long-term bond holdings, (3.36) and (3.37) can be written in terms of real-valued variables as:

$$0 = -\Lambda_t + \beta R_t \mathbb{E}_t \Lambda_{t+1} \pi_{t+1}^{-1} \tag{3.38}$$

$$0 = -\Lambda_t V_t + \beta \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \Lambda_{t+1} \pi_{t+1}^{-1} \tag{3.39}$$

Combining these equations gives:

$$R_t V_t \mathbb{E}_t \Lambda_{t+1} \pi_{t+1}^{-1} = \mathbb{E}_t \left(\varrho + \chi V_{t+1}\right) \Lambda_{t+1} \pi_{t+1}^{-1} \tag{3.40}$$

In steady state this implies that:

$$V = \varrho \left(R - \chi \right)^{-1} = \varrho \left(\beta^{-1} - \chi \right)^{-1}$$

Setting $\varrho = \beta^{-1} - \chi$ therefore implies that $V = 1$, that is, the steady-state price of debt is unity. Adopting this assumption means that the real debt stock d can be treated as the real *par* value of debt.

Log-linearizing (3.40) gives:

$$RV \left(\hat{R}_t + \hat{V}_t \right) = \chi V \hat{V}_{t+1}$$

which implies that:

$$\hat{V}_t = -\hat{R}_t + \chi \beta \hat{V}_{t+1} \tag{3.41}$$

which uses the fact that $R = \beta^{-1}$ in a zero inflation steady state.

Combining (3.34) and (3.38) creates an Euler equation for consumption:

$$\phi_t c_t^{-\frac{1}{\sigma}} = \beta R_t \mathbb{E}_t \phi_{t+1} c_{t+1}^{-\frac{1}{\sigma}} \pi_{t+1}^{-1}$$

which can be log-linearized to give:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma \left[\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} \right] - \sigma \mathbb{E}_t \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right) \tag{3.42}$$

The first order conditions for labor supply (3.35) and consumption (3.34) can be combined and log-linearized to give

$$\psi \hat{n}_t = \hat{w}_t - \sigma^{-1} \hat{c}_t \tag{3.43}$$

3.A.2 Firms

The pricing structure is the same as considered in Chapter 1 and described in detail in Appendix 1.B.2.

The real profit of producer j is:

$$\frac{(1+\Gamma) P_{j,t}}{P_t} y_{j,t} - w_t n_{j,t} = \left((1+\Gamma) \frac{P_{j,t}}{P_t} - \frac{w_t}{A_t} \right) \left(\frac{P_{j,t}}{P_t} \right)^{-\eta_t} y_t$$

where $\Gamma > 0$ is the subsidy that ensures that the steady-state level of output is efficient.

The objective function for a producer that is able to reset prices is:

$$\max \mathbb{E}_t \sum_{k=t}^{\infty} \Lambda_k \left(\beta\alpha\right)^{k-t} \left(\left(1+\Gamma\right) \frac{P_{j,t}}{P_k} - \frac{w_k}{A_k} \right) \left(\frac{P_{j,t}}{P_k} \right)^{-\eta_t} y_k$$

where Λ represents the household's stochastic discount factor and $0 \leq \alpha < 1$ is the probability that the producer is *not* allowed to reset its price each period.

The derivation of the log-linearized pricing equation is identical to that presented in Appendix 1.B.2, with the exception that the productivity-adjusted real wage w_t/A_t appears in place of w_t. So the log-linearized pricing equation is given by:

$$\hat{\pi}_t = \frac{\left(1-\beta\alpha\right)\left(1-\alpha\right)}{\alpha} \left(\hat{w}_t - \hat{A}_t - \frac{\eta}{\eta-1}\hat{\eta}_t \right) + \beta \mathbb{E}_t \hat{\pi}_{t+1} \tag{3.44}$$

3.A.3 Government

Log-linearizing (3.8) gives:

$$\hat{V}_t + \hat{d}_t = \frac{\varrho + \chi V}{V} \left(\hat{d}_{t-1} - \pi_t \right) + \chi \hat{V}_t - \frac{s_t - s}{Vd}$$

Since the steady-state level of output is normalized to unity, $Vd = \zeta$ where ζ is the steady state ratio of government debt to output. Moreover, in the steady state, $\frac{\varrho+\chi V}{V} = R = \beta^{-1}$.

These observations imply that:

$$\hat{d}_t = \beta^{-1} \left(\hat{d}_{t-1} - \hat{\pi}_t \right) - \left(1-\chi\right) \hat{V}_t - \zeta^{-1} \bar{s}_t \tag{3.45}$$

where $\bar{s}_t \equiv s_t - s$ is the linear deviation of the surplus from steady state.

The real surplus is given by:

$$s_t = \tau - g_t$$

since taxes are held fixed ($\tau_t = \tau, \forall t$).

In deviations from steady state, this implies that:

$$\bar{s}_t = -\bar{g}_t \tag{3.46}$$

Using this result in (3.45) gives

$$\hat{d}_t = \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) - (1-\chi)\,\hat{V}_t + \zeta^{-1}\bar{g}_t \tag{3.47}$$

3.A.4 Market clearing and the efficient allocation

Without loss of generality, I specify the steady-state level of productivity, A to ensure that the steady-state level of output y is equal to unity. This implies that the steady-state level of government spending g represents that fraction of output consumed by the government (so $0 \leq g < 1$).

As in Chapter 1, aggregate output satisfies

$$A_t n_t = \mathcal{D}_t y_t \tag{3.48}$$

where

$$\mathcal{D}_t \equiv \int_0^1 \left(\frac{P_{jt}}{P_t}\right)^{-\eta} \mathrm{d}j \tag{3.49}$$

is a measure of price dispersion.

Goods market clearing requires:

$$c_t + g_t = y_t$$

where $\mathcal{D}_t$ is a *second order* price dispersion term, as analyzed in Appendix 1.C.

To a log-linear approximation, this is:

$$(1-g)\,\hat{c}_t + \bar{g}_t = \hat{y}_t \tag{3.50}$$

The subsidy required to make the steady state efficient is:

$$\Gamma = \frac{\eta}{\eta - 1}$$

In a flexible price equilibrium with no distortion from monopolistic competition, the real wage will equal the marginal product of labor, which is equal to A_t. So the efficient allocations, denoted with an asterisk, can be found from the labor supply relation (3.43):

$$\psi \hat{n}_t^* = \hat{A}_t - \sigma^{-1} \hat{c}_t^*$$

Imposing market clearing, $(1-g)\, c_t^* + \bar{g}_t = \hat{A}_t + n_t^* = y_t^*$ implies that potential output is given by:

$$\hat{y}_t^* = \frac{\sigma (1-g)(1+\psi)}{1+\psi\sigma(1-g)} \hat{A}_t + \frac{1}{1+\psi\sigma(1-g)} \bar{g}_t$$

3.A.5 The 'gap' representation

The Phillips curve and Euler equation can be written in terms of the output gap, defined as the deviation between output and the efficient level of output.

Substituting the labor supply equation (3.43) into the log-linearized pricing equation (3.44) gives:

$$\begin{aligned}
\hat{\pi}_t &= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\psi \hat{n}_t + \sigma^{-1} \hat{c}_t - \hat{A}_t \right) + \beta \mathbb{E}_t \hat{\pi}_{t+1} \\
&\quad - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta - 1} \hat{\eta}_t \\
&= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{y}_t - (1+\psi) \hat{A}_t - \sigma^{-1} (1-g)^{-1} \bar{g}_t \right) \\
&\quad + \beta \mathbb{E}_t \hat{\pi}_{t+1} - \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t \\
&= \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{x}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} + u_t \\
&\quad + \frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \left[\left(\psi + \sigma^{-1} (1-g)^{-1} \right) \hat{y}_t^* - (1+\psi) \hat{A}_t - \sigma^{-1} (1-g)^{-1} \bar{g}_t \right]
\end{aligned}$$

where the second line uses market clearing and the third line uses the definition of the output gap $\hat{y}_t - \hat{y}_t^* \equiv \hat{x}_t$ and defines the cost push shock, u, as:

$$u_t \equiv -\frac{(1-\beta\alpha)(1-\alpha)}{\alpha} \frac{\eta}{\eta-1} \hat{\eta}_t$$

Notice that the final term in brackets on the final line is given by:

$$\begin{aligned}
&\left(\psi + \sigma^{-1}(1-g)^{-1}\right)\hat{y}_t^* - (1+\psi)\hat{A}_t - \sigma^{-1}(1-g)^{-1}\bar{g}_t \\
&= \sigma^{-1}(1-g)^{-1}\left[(1+\psi\sigma(1-g))\hat{y}_t^* - (1+\psi)\sigma(1-g)\hat{A}_t - \bar{g}_t\right] \\
&= 0
\end{aligned}$$

The Phillips curve can therefore be written as:

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + u_t \tag{3.51}$$

where

$$\kappa \equiv \frac{(1-\beta\alpha)(1-\alpha)}{\alpha}\left(\psi + \sigma^{-1}(1-g)^{-1}\right)$$

The Euler equation for consumption (3.42) can be written as:

$$\begin{aligned}
(1-g)^{-1}\hat{y}_t - (1-g)^{-1}\bar{g}_t = {} & (1-g)^{-1}\mathbb{E}_t\left[\hat{y}_{t+1} - \bar{g}_{t+1}\right] - \sigma\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}\right] \\
& - \sigma\mathbb{E}_t\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right)
\end{aligned}$$

which incorporates the market clearing condition for output.

Rearranging gives:

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \sigma(1-g)\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}\right] + \mathbb{E}_t\left[\bar{g}_t - \bar{g}_{t+1} - \sigma(1-g)\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right)\right]$$

This implies that:

$$\begin{aligned}
\hat{y}_t - \hat{y}_t^* + \hat{y}_t^* = {} & \mathbb{E}_t\left(\hat{y}_{t+1} - y_{t+1}^* + y_{t+1}^*\right) - \sigma(1-g)\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}\right] \\
& + \mathbb{E}_t\left[\bar{g}_t - \bar{g}_{t+1} - \sigma(1-g)\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right)\right]
\end{aligned}$$

or

$$\hat{x}_t = \mathbb{E}_t\hat{x}_{t+1} - \sigma(1-g)\left[\hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1} - r_t^*\right] \tag{3.52}$$

where the efficient rate of interest r^* satisfies

$$
\begin{aligned}
r_t^* &= \mathbb{E}_t \left[\sigma^{-1} (1-g)^{-1} \left(y_{t+1}^* - y_t^* + \widetilde{g}_t - \widetilde{g}_{t+1} \right) - \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right) \right] \\
&= \mathbb{E}_t \sigma^{-1} (1-g)^{-1} \begin{pmatrix} \frac{\sigma(1-g)(1+\psi)}{1+\psi\sigma(1-g)} \hat{A}_{t+1} + \frac{1}{1+\psi\sigma(1-g)} \bar{g}_{t+1} \\ -y_t^* + \bar{g}_t - \bar{g}_{t+1} \end{pmatrix} \\
&\quad - \mathbb{E}_t \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right) \\
&= \mathbb{E}_t \left[- \left(\hat{\phi}_{t+1} - \hat{\phi}_t \right) + \frac{1+\psi}{1+\psi\sigma(1-g)} \left(\hat{A}_{t+1} - \hat{A}_t \right) \right] \\
&\quad - \frac{\psi}{1+\psi\sigma(1-g)} \mathbb{E}_t \left(\bar{g}_{t+1} - \bar{g}_t \right)
\end{aligned}
$$

Appendix 3.B The utility-based loss function

The model structure is very similar to that analyzed in Chapter 1. As a result, the derivation of the utility-based loss function follows many of the same steps. The derivation here uses key results from Chapter 1 (Appendix 1.C) and focuses on differences arising from the presence of government spending and the absence of portfolio adjustment costs.

Ignoring constants, the period utility function is:

$$
U_t = \phi_t \left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} \right]
$$

As in Appendix 1.C markup shocks are ignored (by setting $\eta_t = \eta, \forall t$) to simplify notation. Since cost push shocks are independent of policy this does not affect the derivation.

Approximating the utility from consumption to second order gives:

$$
\phi_t \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right) - \frac{1}{2\sigma} c^{1-\frac{1}{\sigma}} \left(\frac{c_t - c}{c} \right)^2 + c^{1-\frac{1}{\sigma}} \frac{c_t - c}{c} \frac{\phi_t - \phi}{\phi} + t.i.p. \tag{3.53}
$$

where $t.i.p.$ denotes 'terms independent of policy' (that is, functions of exogenous disturbances) and the fact that $\phi = 1$ in steady state is used to simplify the first two terms.

Using the second order approximation for the percentage changes in consumption implies that:

$$\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \approx c^{1-\frac{1}{\sigma}} \left(\hat{c}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t \right) + h.o.t.$$

where $h.o.t.$ are 'higher order terms'.

The sub-utility function for labor supply is:

$$\begin{aligned}
\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx & \frac{n^{1+\psi}}{1+\psi} + n^{1+\psi}\frac{n_t-n}{n} + \frac{\psi n^{1+\psi}}{2}\left(\frac{n_t-n}{n}\right)^2 + \frac{n^{1+\psi}}{1+\psi}\frac{\phi_t-\phi}{\phi} \\
& + n^{1+\psi}\frac{n_t-n}{n}\frac{\phi_t-\phi}{\phi} \\
\approx & n^{1+\psi}\frac{n_t-n}{n} + \frac{\psi n^{1+\psi}}{2}\left(\frac{n_t-n}{n}\right)^2 + n^{1+\psi}\frac{n_t-n}{n}\frac{\phi_t-\phi}{\phi} + t.i.p.
\end{aligned}$$

Using the mapping from percentage changes to log-deviations, to second order, implies that:

$$\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx n^{1+\psi}\left[\hat{n}_t + \frac{(1+\psi)}{2}\hat{n}_t^2 + \hat{n}_t\hat{\phi}_t\right] + h.o.t.$$

A second order approximation to the aggregate production function (3.48) is:

$$\hat{y}_t + \frac{1}{2}\hat{y}_t^2 = \hat{n}_t + \frac{1}{2}\hat{n}_t^2 + \hat{A}_t\hat{n}_t - \hat{\mathcal{D}}_t + t.i.p.$$

which uses the fact that $\hat{\mathcal{D}}_t$ is a second order term (see Appendix 1.C).

This implies that:

$$\frac{\phi_t n_t^{1+\psi}}{1+\psi} \approx n^{1+\psi}\left[\hat{y}_t + \frac{(1+\psi)}{2}\hat{y}_t^2 - (1+\psi)\,\hat{y}_t\hat{A}_t + \hat{y}_t\hat{\phi}_t - \hat{\mathcal{D}}_t\right] + h.o.t. + t.i.p.$$

The second-order approximation to the utility function is therefore

$$\begin{aligned}
U_t \approx & c^{1-\frac{1}{\sigma}}\left(\hat{c}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t\right) \\
& - n^{1+\psi}\left[\hat{y}_t + \frac{(1+\psi)}{2}\hat{y}_t^2 - (1+\psi)\,\hat{y}_t\hat{A}_t + \hat{y}_t\hat{\phi}_t - \hat{\mathcal{D}}_t\right]
\end{aligned}$$

The steady-state labor supply relationship is

$$n^{\psi} = wc^{-1/\sigma} = Ac^{-1/\sigma}$$

Steady-state market clearing is

$$c + g = y = An$$

since steady-state price dispersion is $\mathcal{D} = 1$.

This implies that

$$n^{1+\psi} = (1-g)^{-1} c^{1-\frac{1}{\sigma}}$$

so that the utility function can be written as

$$U_t \approx c^{1-\frac{1}{\sigma}} \left[\begin{array}{c} \hat{c}_t + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t - (1-g)^{-1}\hat{y}_t - \frac{(1+\psi)}{2(1-g)}\hat{y}_t^2 \\ +\frac{(1+\psi)}{(1-g)}\hat{y}_t\hat{A}_t - (1-g)^{-1}\hat{y}_t\hat{\phi}_t - (1-g)^{-1}\hat{\mathcal{D}}_t \end{array} \right]$$

The goods market clearing condition is:

$$c_t = y_t - g_t$$

A second order approximation to the goods market clearing condition is:

$$(1-g)\hat{c}_t + \frac{1-g}{2}\hat{c}_t^2 = \hat{y}_t + \frac{1}{2}\hat{y}_t^2 + t.i.p.$$

These results can be used to write the approximation to utility in terms of output deviations and price dispersion only.

First substitute for

$$\hat{c}_t = -\frac{1}{2}\hat{c}_t^2 + \frac{\hat{y}_t}{1-g} + \frac{1}{2(1-g)}\hat{y}_t^2$$

to give:

$$\begin{aligned} U_t &\approx c^{1-\frac{1}{\sigma}} \left[\begin{array}{c} -\frac{1}{2}\hat{c}_t^2 + \frac{\hat{y}_t}{1-g} + \frac{1}{2(1-g)}\hat{y}_t^2 + \frac{1}{2}\left(1-\sigma^{-1}\right)\hat{c}_t^2 + \hat{c}_t\hat{\phi}_t \\ -(1-g)^{-1}\hat{y}_t - \frac{(1+\psi)}{2(1-g)}\hat{y}_t^2 + \frac{(1+\psi)}{(1-g)}\hat{y}_t\hat{A}_t \\ -(1-g)^{-1}\hat{y}_t\hat{\phi}_t - (1-g)^{-1}\hat{\mathcal{D}}_t \end{array} \right] \\ &\approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{1-g}{2\sigma}\hat{c}_t^2 - \frac{\psi}{2}\hat{y}_t^2 + (1-g)\hat{c}_t\hat{\phi}_t + (1+\psi)\hat{y}_t\hat{A}_t - \hat{y}_t\hat{\phi}_t \right] \end{aligned}$$

where the second line collects common terms.

Substituting for $\hat{c}_t^2$ gives:

$$U_t \approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{\psi+\widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1-g)\,\hat{c}_t\hat{\phi}_t + (1+\psi)\,\hat{y}_t\hat{A}_t - \hat{y}_t\hat{\phi}_t\right]$$

where $\widetilde{\sigma} \equiv (1-g)\,\sigma$ as in the main text.

Noting that

$$(1-g)\,\hat{c}_t\hat{\phi}_t - \hat{y}_t\hat{\phi}_t = \hat{\phi}_t\left[(1-g)\,\hat{c}_t - \hat{y}_t\right] = \hat{\phi}_t\left[(1-g)\,\hat{c}_t - (1-g)\,\hat{c}_t - \bar{g}_t\right] = -\hat{\phi}_t\bar{g}_t$$

which is independent of policy gives:

$$U_t \approx \frac{c^{1-\frac{1}{\sigma}}}{1-g}\left[-\hat{\mathcal{D}}_t - \frac{\psi+\widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1+\psi)\,\hat{y}_t\hat{A}_t\right]$$

The terms in $\hat{y}_t$ can be written as:

$$\begin{aligned}
-\frac{\psi+\widetilde{\sigma}^{-1}}{2}\hat{y}_t^2 + \frac{1}{\widetilde{\sigma}}\hat{y}_t\bar{g}_t + (1+\psi)\,\hat{y}_t\hat{A}_t &= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\begin{pmatrix} \frac{1}{1+\psi\widetilde{\sigma}}\bar{g}_t \\ +\frac{\widetilde{\sigma}(1+\psi)}{1+\psi\widetilde{\sigma}}\hat{A}_t \end{pmatrix}\right] \\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\hat{y}_t^*\right] \\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}\left[\hat{y}_t^2 - 2\hat{y}_t\hat{y}_t^* + (\hat{y}_t^*)^2 - (\hat{y}_t^*)^2\right] \\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}(\hat{y}_t - \hat{y}_t^*)^2 + \frac{\psi+\widetilde{\sigma}^{-1}}{2}(\hat{y}_t^*)^2 \\
&= -\frac{\psi+\widetilde{\sigma}^{-1}}{2}x_t^2 + t.i.p.
\end{aligned}$$

Define the discounted loss function to be minimized as:

$$\mathcal{L} = -2\,(1-g)\,c^{\frac{1}{\sigma}-1}\sum_{t=0}^{\infty}\beta^t U_t = \sum_{t=0}^{\infty}\beta^t\left[2\hat{\mathcal{D}}_t + \left(\psi+\widetilde{\sigma}^{-1}\right)\hat{x}_t^2\right]$$

The analysis of the price dispersion term $\hat{\mathcal{D}}_t$ is identical to that in Appendix 1.C so that the loss function can be written as:

$$\mathcal{L} = \sum_{t=0}^{\infty}\beta^t\left[\frac{\alpha\eta}{(1-\alpha\beta)\,(1-\alpha)}\hat{\pi}_t^2 + \left(\psi+\widetilde{\sigma}^{-1}\right)\hat{x}_t^2\right]$$

Normalizing the coefficient on inflation to unity implies that the loss function is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\hat{\pi}_t^2 + \frac{(1-\alpha\beta)(1-\alpha)}{\alpha\eta} \left(\psi + \widetilde{\sigma}^{-1} \right) \hat{x}_t^2 \right] = \sum_{t=0}^{\infty} \beta^t \left[\hat{\pi}_t^2 + \frac{\kappa}{\eta} \hat{x}_t^2 \right]$$

Appendix 3.C Time-consistent linear-quadratic policy

This appendix focuses on solving for the coefficients that describe the dependence of endogenous variables on the debt stock (that is, $F_{\hat{\pi}}$,$F_{\hat{x}}$,$F_{\hat{V}}$,$F_{\hat{d}}$). The starting point is the first order conditions derived in Section 3.4.1 of the main text.

Note first that (3.19) and (3.20) imply that:

$$\mu_t^x = (1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d \tag{3.54}$$

and

$$\mu_t^V = -(1-\chi)\,\mu_t^d \tag{3.55}$$

Using the preceding results in the equation for μ_t^d gives:

$$\begin{aligned}
\mu_t^d &= [F_{\hat{x}} + \widetilde{\sigma} F_{\hat{\pi}}](1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d + \beta F_{\hat{\pi}}\mu_t^{\pi} - \chi\beta F_{\hat{V}}(1-\chi)\,\mu_t^d + \mathbb{E}_t\mu_{t+1}^d \\
&= [F_{\hat{x}} + \widetilde{\sigma} F_{\hat{\pi}}](1-\chi)\,\widetilde{\sigma}^{-1}\mu_t^d + \beta F_{\hat{\pi}}\left(\hat{\pi}_t - \beta^{-1}\mu_t^d\right) - \chi\beta F_{\hat{V}}(1-\chi)\,\mu_t^d \\
&\quad + \mathbb{E}_t\mu_{t+1}^d
\end{aligned}$$

Collecting terms and rearranging gives:

$$\left[1 + \chi\beta(1-\chi)F_{\hat{V}} + \chi F_{\pi} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right]\mu_t^d = \beta F_{\hat{\pi}}\hat{\pi}_t + \mathbb{E}_t\mu_{t+1}^d \tag{3.56}$$

Combining (3.16) and (3.17) gives:

$$0 = \omega\hat{x}_t - \mu_t^x + \kappa\hat{\pi}_t - \kappa\beta^{-1}\mu_t^d$$

which implies that:

$$\omega\hat{x}_t + \kappa\hat{\pi}_t = \left((1-\chi)\,\widetilde{\sigma}^{-1} + \kappa\beta^{-1}\right)\mu_t^d \tag{3.57}$$

or

$$\mu_t^d = \frac{\omega}{\Xi}\hat{x}_t + \frac{\kappa}{\Xi}\hat{\pi}_t \tag{3.58}$$

where

$$\Xi \equiv (1-\chi)\,\widetilde{\sigma}^{-1} + \kappa\beta^{-1} > 0 \tag{3.59}$$

Then:

$$\begin{aligned}&\left[1+\chi\beta\,(1-\chi)\,F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right]\left(\frac{\omega}{\Xi}\hat{x}_t + \frac{\kappa}{\Xi}\hat{\pi}_t\right)\\ &= \beta F_{\hat{\pi}}\hat{\pi}_t + \mathbb{E}_t\left(\frac{\omega}{\Xi}\hat{x}_{t+1} + \frac{\kappa}{\Xi}\hat{\pi}_{t+1}\right)\end{aligned} \tag{3.60}$$

To solve for the 'F' coefficients, ignore exogenous terms and substitute out for expectations (so, for example, $\mathbb{E}_t x_{t+1} = F_{\hat{x}}d_t = F_{\hat{x}}F_{\hat{d}}d_{t-1}$):

$$\begin{aligned}\beta F_{\hat{\pi}}F_{\hat{\pi}} + \left(\frac{\omega}{\Xi}F_{\hat{x}}F_{\hat{d}} + \frac{\kappa}{\Xi}F_{\hat{\pi}}F_{\hat{d}}\right) &= \left[1+\chi\beta\,(1-\chi)\,F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}}\right]\\ &\quad\times\left(\frac{\omega}{\Xi}F_{\hat{x}} + \frac{\kappa}{\Xi}F_{\hat{\pi}}\right)\end{aligned}$$

which implies that

$$\left[1+\chi\beta\,(1-\chi)\,F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\,\widetilde{\sigma}^{-1}F_{\hat{x}} - F_{\hat{d}}\right]\left(\frac{\omega}{\Xi}F_{\hat{x}} + \frac{\kappa}{\Xi}F_{\hat{\pi}}\right) = \beta F_{\hat{\pi}}^2 \tag{3.61}$$

Now apply the same approach for the structural model equations, which gives the following.

$$\begin{aligned}\hat{R}_t &= -\hat{V}_t + \chi\beta F_{\hat{V}}\hat{d}_t\\ \hat{x}_t &= F_{\hat{x}}\hat{d}_t - \widetilde{\sigma}\hat{R}_t + \widetilde{\sigma}F_{\hat{\pi}}\hat{d}_t\end{aligned} \tag{3.62}$$

which implies that:

$$\hat{x}_t = F_{\hat{x}}\hat{d}_t + \widetilde{\sigma}\hat{V}_t - \widetilde{\sigma}\chi\beta F_{\hat{V}}\hat{d}_t + \widetilde{\sigma}F_{\hat{\pi}}\hat{d}_t$$

and hence:

$$F_{\hat{x}} = F_{\hat{x}}F_{\hat{d}} + \widetilde{\sigma}F_{\hat{V}} - \widetilde{\sigma}\chi\beta F_{\hat{V}}F_{\hat{d}} + \widetilde{\sigma}F_{\hat{\pi}}F_{\hat{d}} \tag{3.63}$$

The Phillips curve implies that:

$$\hat{\pi}_t = \kappa \hat{x}_t + \beta F_{\hat{\pi}} \hat{d}_t$$

so that:

$$F_{\hat{\pi}} = \kappa F_{\hat{x}} + \beta F_{\hat{\pi}} F_{\hat{d}} \tag{3.64}$$

The government debt accumulation equation gives:

$$\hat{d}_t = \beta^{-1} \hat{d}_{t-1} - \beta^{-1} \hat{\pi}_t - (1-\chi)\hat{V}_t$$

which implies that:

$$F_{\hat{d}} = \beta^{-1} - \beta^{-1} F_{\hat{\pi}} - (1-\chi) F_{\hat{V}} \tag{3.65}$$

The preceding steps have delivered four equations in four unknowns, repeated here for convenience:

$$\begin{aligned} \beta F_{\hat{\pi}}^2 = & \left[1 + \chi\beta(1-\chi) F_{\hat{V}} + \chi F_{\hat{\pi}} - (1-\chi)\widetilde{\sigma}^{-1} F_{\hat{x}} - F_{\hat{d}}\right] \\ & \times \left(\frac{\omega}{\Xi} F_{\hat{x}} + \frac{\kappa}{\Xi} F_{\hat{\pi}}\right) \end{aligned} \tag{3.66}$$

$$F_{\hat{x}} = F_{\hat{x}} F_{\hat{d}} + \widetilde{\sigma} F_{\hat{V}} - \widetilde{\sigma}\chi\beta F_{\hat{V}} F_{\hat{d}} + \widetilde{\sigma} F_{\hat{\pi}} F_{\hat{d}} \tag{3.67}$$

$$F_{\hat{\pi}} = \kappa F_{\hat{x}} + \beta F_{\hat{\pi}} F_{\hat{d}} \tag{3.68}$$

$$F_{\hat{d}} = \beta^{-1} - \beta^{-1} F_{\hat{\pi}} - (1-\chi) F_{\hat{V}} \tag{3.69}$$

The final two equations can be used to express $F_{\hat{x}}$ and $F_{\hat{V}}$ as functions of $F_{\hat{\pi}}$ and $F_{\hat{d}}$:

$$F_{\hat{x}} = \kappa^{-1} F_{\hat{\pi}} - \kappa^{-1} \beta F_{\hat{\pi}} F_{\hat{d}} \tag{3.70}$$

$$F_{\hat{V}} = (1-\chi)^{-1} \beta^{-1} - (1-\chi)^{-1} \beta^{-1} F_{\hat{\pi}} - (1-\chi)^{-1} F_{\hat{d}} \tag{3.71}$$

Using these expressions in the first two equations gives:

$$\begin{aligned} \beta F_{\hat{\pi}}^2 = & \left[\begin{array}{c} 1 + \chi\beta(1-\chi)(1-\chi)^{-1}\left(\beta^{-1} - \beta^{-1} F_{\hat{\pi}} - F_{\hat{d}}\right) + \chi F_{\hat{\pi}} \\ -(1-\chi)\widetilde{\sigma}^{-1}\left(\kappa^{-1} F_{\hat{\pi}} - \kappa^{-1}\beta F_{\hat{\pi}} F_{\hat{d}}\right) - F_{\hat{d}} \end{array}\right] \\ & \times \left(\frac{\omega}{\Xi}\left(\kappa^{-1} F_{\hat{\pi}} - \kappa^{-1}\beta F_{\hat{\pi}} F_{\hat{d}}\right) + \frac{\kappa}{\Xi} F_{\hat{\pi}}\right) \\ = & \left[1 + \chi - \chi\beta F_{\hat{d}} - (1-\chi)(\kappa\widetilde{\sigma})^{-1}\left(1 - \beta F_{\hat{d}}\right) F_{\hat{\pi}} - F_{\hat{d}}\right] \\ & \times \left(\frac{\omega}{\Xi\kappa}\left(1 - \beta F_{\hat{d}}\right) + \frac{\kappa}{\Xi}\right) F_{\hat{\pi}} \end{aligned}$$

and

$$\begin{aligned}\kappa^{-1}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right) = \kappa^{-1}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right)F_{\hat{d}} + \widetilde{\sigma}F_{\hat{\pi}}F_{\hat{d}} \\ + \widetilde{\sigma}\left((1-\chi)^{-1}\beta^{-1} - (1-\chi)^{-1}\beta^{-1}F_{\hat{\pi}} - (1-\chi)^{-1}F_{\hat{d}}\right) \\ \times\left(1-\beta\chi F_{\hat{d}}\right)\end{aligned}$$

The second equation implies a quadratic equation for $F_{\hat{d}}$ conditional on a solution (or conjecture) for $F_{\hat{\pi}}$. The first equation can be used to solve for $F_{\hat{\pi}}$ conditional on a solution (or conjecture) for $F_{\hat{d}}$.

Specifically, conditional on a solution for $F_{\hat{d}}$, $F_{\hat{\pi}}$ satisfies:

$$F_{\hat{\pi}} = m\left(F_{\hat{d}}\right) \equiv \frac{1+\chi-(1+\beta\chi)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1} + (1-\chi)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)} \tag{3.72}$$

The quadratic equation for $F_{\hat{d}}$ is given by:

$$\begin{aligned}0 = &\left[\frac{\beta\chi\kappa\widetilde{\sigma}}{1-\chi} - \beta F_{\hat{\pi}}\right]F_{\hat{d}}^2 + \left[F_{\hat{\pi}}\left(1+\beta+\kappa\widetilde{\sigma}\right) - \frac{\kappa\widetilde{\sigma}}{1-\chi} - \frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}\left(1-F_{\hat{\pi}}\right)\right]F_{\hat{d}} \\ &+ \left[\frac{\kappa\widetilde{\sigma}}{\beta(1-\chi)}\left(1-F_{\hat{\pi}}\right) - F_{\hat{\pi}}\right]\end{aligned}$$

which can be rearranged to give:

$$\begin{aligned}0 = &\left[\frac{\chi\kappa\widetilde{\sigma}}{1-\chi} - F_{\hat{\pi}}\right]\beta F_{\hat{d}}^2 + \left[F_{\hat{\pi}} - \frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}\right]F_{\hat{d}} \\ &+ \left[F_{\hat{\pi}}\left(\beta+\kappa\widetilde{\sigma}\right) - \frac{\kappa\widetilde{\sigma}}{1-\chi} + \frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}F_{\hat{\pi}}\right]F_{\hat{d}} \\ &+ \left[\frac{\kappa\widetilde{\sigma}}{\beta(1-\chi)}\left(1-F_{\hat{\pi}}\right) - F_{\hat{\pi}}\right]\end{aligned}$$

The third term can be written as:

$$\begin{aligned}&\left[F_{\hat{\pi}}\left(\beta+\kappa\widetilde{\sigma}\right) - \frac{\kappa\widetilde{\sigma}}{1-\chi} + \frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}F_{\hat{\pi}}\right]F_{\hat{d}} \\ &= \left[\beta F_{\hat{\pi}} + \frac{(1-\chi)\kappa\widetilde{\sigma}}{1-\chi}F_{\hat{\pi}} - \frac{\kappa\widetilde{\sigma}}{1-\chi} + \frac{\chi\kappa\widetilde{\sigma}}{(1-\chi)}F_{\hat{\pi}}\right]F_{\hat{d}} \\ &= \left[\beta F_{\hat{\pi}} + \frac{\kappa\widetilde{\sigma}}{1-\chi}\left(F_{\hat{\pi}}-1\right)\right]F_{\hat{d}} \\ &= -\left[\frac{\kappa\widetilde{\sigma}}{1-\chi}\left(1-F_{\hat{\pi}}\right) - F_{\hat{\pi}}\right]\beta F_{\hat{d}}\end{aligned} \tag{3.73}$$

This implies that the quadratic equation can be factorized as:

$$\begin{aligned}
0 =& \left[F_{\hat{\pi}} - \frac{\chi\kappa\widetilde{\sigma}}{1-\chi}\right] F_{\hat{d}}\left(1-\beta F_{\hat{d}}\right) + \left[\frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right) - F_{\hat{\pi}}\right]\left(1-\beta F_{\hat{d}}\right) \\
=& \left(1-\beta F_{\hat{d}}\right)\left[\left(F_{\hat{\pi}} - \frac{\chi\kappa\widetilde{\sigma}}{1-\chi}\right) F_{\hat{d}} + \frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\left(1-F_{\hat{\pi}}\right) - F_{\hat{\pi}}\right]
\end{aligned}$$

So one solution is $F_{\hat{d}} = \beta^{-1}$. That implies that $F_{\hat{\pi}} = m\left(\beta^{-1}\right) = \frac{\kappa}{\Xi\beta}\left(1-\beta^{-1}\right) < 0$.

The other solution for $F_{\hat{d}}$ satisfies:

$$F_{\hat{d}} \equiv h\left(F_{\hat{\pi}}\right) = \frac{\left(1+\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\right) F_{\hat{\pi}} - \kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}}{F_{\hat{\pi}} - \chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}} \tag{3.74}$$

Note that h can be written as:

$$h\left(\cdot\right) = 1 + \frac{\kappa\widetilde{\sigma}\left[\beta\left(1-\chi\right)\right]^{-1}\left(F_{\hat{\pi}} - 1 + \beta\chi\right)}{F_{\hat{\pi}} - \chi\kappa\widetilde{\sigma}\left(1-\chi\right)^{-1}}$$

The previous results can be used to re-write the targeting rule in terms of the output gap and inflation.

3.C.1 Model properties under time-consistent policy

The coefficient in brackets on the left side of (3.60) is:

$$\begin{aligned}
\Omega \equiv& 1 + \chi\beta\left(1-\chi\right) F_{\hat{V}} + \chi F_{\hat{\pi}} - \left(1-\chi\right)\widetilde{\sigma}^{-1} F_{\hat{x}} \\
=& 1 + \chi\left(1 - F_{\hat{\pi}} - \beta F_{\hat{d}}\right) + \chi F_{\hat{\pi}} - \left(1-\chi\right)\widetilde{\sigma}^{-1}\left(\kappa^{-1} F_{\hat{\pi}} - \kappa^{-1}\beta F_{\hat{\pi}} F_{\hat{d}}\right) \\
=& 1 + \chi - \chi\beta F_{\hat{d}} - \frac{1-\chi}{\kappa\widetilde{\sigma}} F_{\hat{\pi}} + \frac{1-\chi}{\kappa\widetilde{\sigma}}\beta F_{\hat{\pi}} F_{\hat{d}}
\end{aligned}$$

Rearranging (3.74) reveals that:

$$F_{\hat{d}} F_{\hat{\pi}} = \frac{\chi\kappa\widetilde{\sigma}}{1-\chi} F_{\hat{d}} + \left(1 + \frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}\right) F_{\hat{\pi}} - \frac{\kappa\widetilde{\sigma}}{\beta\left(1-\chi\right)}$$

so that

$$\frac{1-\chi}{\kappa\widetilde{\sigma}}\beta F_{\hat{\pi}} F_{\hat{d}} = \chi\beta F_{\hat{d}} + \left(1 + \beta\frac{1-\chi}{\kappa\widetilde{\sigma}}\right) F_{\hat{\pi}} - 1$$

and plugging this into the equation for Ω gives:

$$\Omega = \chi + F_{\hat{\pi}} \left(1 - (1-\beta)\frac{1-\chi}{\kappa\widetilde{\sigma}}\right)$$

This allows (3.56) to be written as

$$\mu_t^d = \beta F_{\hat{\pi}} \Omega^{-1} \hat{\pi}_t + \Omega^{-1} \mathbb{E}_t \mu_{t+1}^d$$

and (3.60) to be written as:

$$\Omega\left(\omega \hat{x}_t + \kappa \hat{\pi}_t\right) = \beta \Xi F_{\hat{\pi}} \hat{\pi}_t + \mathbb{E}_t\left(\omega \hat{x}_{t+1} + \kappa \hat{\pi}_{t+1}\right)$$

or

$$\omega \hat{x}_t + \left(\kappa - \beta \Xi F_{\hat{\pi}} \Omega^{-1}\right) \hat{\pi}_t = \Omega^{-1} \mathbb{E}_t\left(\omega \hat{x}_{t+1} + \kappa \hat{\pi}_{t+1}\right)$$

When cost-push shocks are zero, the Phillips curve implies that the output gap satisfies:

$$\hat{x}_t = \kappa^{-1}\left(\hat{\pi}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1}\right)$$

Which implies that the targeting rule can be written as:

$$\Omega\left(\omega\kappa^{-1}\left(\hat{\pi}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1}\right) + \kappa \hat{\pi}_t\right) = \beta \Xi F_{\hat{\pi}} \hat{\pi}_t + \mathbb{E}_t\left(\omega\kappa^{-1}\left(\hat{\pi}_{t+1} - \beta \mathbb{E}_t \hat{\pi}_{t+2}\right) + \kappa \hat{\pi}_{t+1}\right)$$

which uses the law of iterated conditional expectations.

Collecting terms and rearranging gives a second order difference equation for inflation:

$$\left[\Omega\left(\frac{\omega}{\kappa} + \kappa\right) - \beta \Xi F_{\pi}\right] \hat{\pi}_t - \left[\frac{\omega}{\kappa}(1+\beta\Omega) + \kappa\right] \mathbb{E}_t \hat{\pi}_{t+1} + \frac{\beta\omega}{\kappa} \mathbb{E}_t \hat{\pi}_{t+2} = 0$$

For an impulse response, in which an unexpected shock is revealed in period 1 and no additional information arrives thereafter, the path for inflation must satisfy the difference equation. That is

$$\left[\Omega\left(\frac{\omega}{\kappa} + \kappa\right) - \beta \Xi F_{\pi}\right] \hat{\pi}_t - \left[\frac{\omega}{\kappa}(1+\beta\Omega) + \kappa\right] \hat{\pi}_{t+1} + \frac{\beta\omega}{\kappa} \hat{\pi}_{t+2} = 0$$

where the expectation operator has been removed.

We seek a solution of the form $\pi_{t+1} = G\pi_t$ which implies

$$\left(\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\pi}\right]-\left[\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa\right]G+\frac{\beta\omega}{\kappa}G^2\right)\hat{\pi}_t = 0$$

The roots of the characteristic polynominal satisfy:

$$G = \frac{\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa \quad \pm \quad \left(\left[\frac{\omega}{\kappa}\left(1+\beta\Omega\right)+\kappa\right]^2-4\left[\Omega\left(\frac{\omega}{\kappa}+\kappa\right)-\beta\Xi F_{\hat{\pi}}\right]\frac{\beta\omega}{\kappa}\right)^{\frac{1}{2}}}{2\frac{\beta\omega}{\kappa}} \tag{3.75}$$

For the parameter values used in both the baseline and long-duration variants, the two solutions for G are real. In both cases, the larger root exceeds 1 and the smaller root is less than 1.

The preceding analysis has shown that, in the absence of cost push shocks, the path for inflation satisfies:

$$\hat{\pi}_{t+1} = G\hat{\pi}_t, \qquad t \geq 1$$

The Phillips curve implies that:

$$\hat{x}_{t+1} = \kappa^{-1}\left(\hat{\pi}_{t+1}-\beta\hat{\pi}_{t+2}\right) = \kappa^{-1}\left(1-\beta G\right)\hat{\pi}_{t+1}$$

and substituting this into (3.58) evaluated at $t+1$ gives:

$$\mu^d_{t+1} = \left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]\hat{\pi}_{t+1}$$

Also note that, for $t > 1$:

$$\begin{aligned}\hat{x}_{t+2} =& \kappa^{-1}\left(1-\beta G\right)\hat{\pi}_{t+2} = \kappa^{-1}\left(1-\beta G\right)G\hat{\pi}_{t+1} = G\hat{x}_{t+1}\\ \mu^d_{t+2} =& \left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]\hat{\pi}_{t+2} = \left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]G\hat{\pi}_{t+1} = G\mu^d_{t+1}\end{aligned}$$

The preceding results can be used in (3.56) evaluated at $t+1$ to give:

$$\begin{aligned}\begin{bmatrix} 1+\chi\beta\left(1-\chi\right)F_{\hat{V}} \\ +\chi F_{\pi}-\left(1-\chi\right)\widetilde{\sigma}^{-1}F_{\hat{x}}\end{bmatrix}\mu^d_{t+1} =& \beta F_{\hat{\pi}}\hat{\pi}_{t+1}+\mu^d_{t+2}\\ =& \beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}\mu^d_{t+1}+G\mu^d_{t+1}\end{aligned}$$

which implies that the coefficients satisfy:

$$1+\chi\beta\left(1-\chi\right)F_{\hat{V}}+\chi F_{\pi}-\left(1-\chi\right)\widetilde{\sigma}^{-1}F_{\hat{x}}=\beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}+G$$

Using the solutions for $F_{\hat{x}}$ and $F_{\hat{V}}$ ((3.70) and (3.71)) in the left hand side and collecting terms gives:

$$1+\chi\left(1-\beta F_{\hat{d}}\right)-\frac{1-\chi}{\widetilde{\sigma}\kappa}F_{\hat{\pi}}\left(1-\beta F_{\hat{d}}\right)=\beta F_{\hat{\pi}}\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}+G$$

which implies that:

$$F_{\hat{\pi}}=\frac{1+\chi\left(1-\beta F_{\hat{d}}\right)-G}{\beta\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}+\frac{1-\chi}{\widetilde{\sigma}\kappa}\left(1-\beta F_{\hat{d}}\right)}$$

Equating this to the solution for F_{π} in (3.72) gives:

$$\begin{aligned}&\frac{1+\chi\left(1-\beta F_{\hat{d}}\right)-G}{\beta\left[\frac{\omega}{\kappa\Xi}\left(1-\beta G\right)+\frac{\kappa}{\Xi}\right]^{-1}+\frac{1-\chi}{\widetilde{\sigma}\kappa}\left(1-\beta F_{\hat{d}}\right)}\\&=\frac{1+\chi-\left(1+\beta\chi\right)F_{\hat{d}}}{\beta\left(\frac{\omega}{\kappa\Xi}\left(1-\beta F_{\hat{d}}\right)+\frac{\kappa}{\Xi}\right)^{-1}+\left(1-\chi\right)\left(\kappa\widetilde{\sigma}\right)^{-1}\left(1-\beta F_{\hat{d}}\right)}\end{aligned}$$

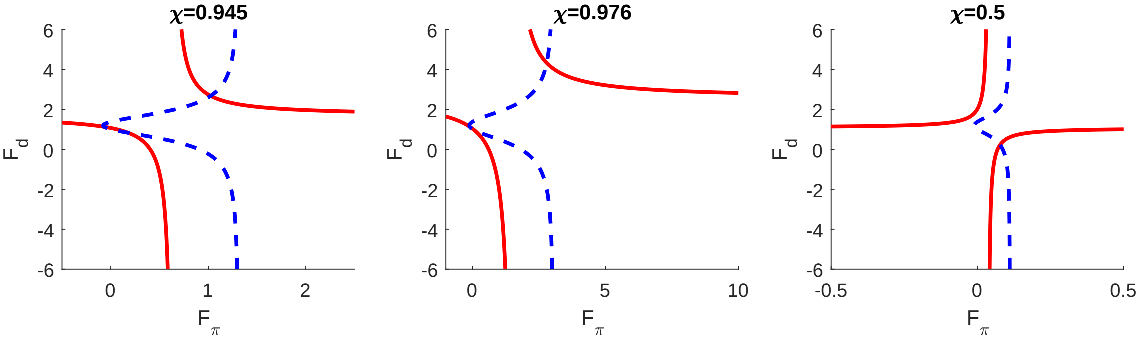
which reveals that

$$G=F_{\hat{d}}$$

Appendix 3.D Global analysis of stable roots under time-consistent policy

Figure 3.21 presents an analysis of the h and m functions (solid red and dashed blue lines respectively) over a broader range for $F_{\hat{\pi}}$ and $F_{\hat{d}}$ than considered in the main text. This demonstrates that, conditional on the values of the other model parameters, the stable Markov perfect equilibria are unique for the baseline and long duration values of χ as well as for a case in which the duration of bonds is very short ($\chi = 0.5$). In the very short duration case, $F_{\hat{d}} \approx 0$ which suggests that the 'debt stabilization bias' is significant in this case, consistent with the results of Leith and Wren-Lewis (2013) who analyze a model with one period debt ($\chi = 0$).

Figure 3.21: Global analysis of $F_{\hat{\pi}}$ and $F_{\hat{d}}$



Notes: Each panel plots the functions m and h defined by equations (3.22) (dashed blue line) and (3.23) (solid red line) respectively. Each panel examines a variant of the model for alternative values f χ: the baseline model ($\chi = 0.945$), the long-duration debt variant ($\chi = 0.976$) and a variant with very short debt duration ($\chi = 0.5$).

Appendix 3.E Optimal policy under commitment

The optimal policy problem is:

$$\min \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} \frac{1}{2}\left[\hat{\pi}_t^2 + \omega \hat{x}_t^2\right] - \lambda_t^{\pi}\left[\hat{\pi}_t - \kappa \hat{x}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1} - u_t\right] \\ -\lambda_t^x \left[\hat{x}_t - \mathbb{E}_t \hat{x}_{t+1} + \widetilde{\sigma}\left(\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - r_t^*\right)\right] \\ -\lambda_t^d \left[\hat{d}_t - \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) + (1-\chi)\hat{V}_t - \zeta^{-1}\bar{g}_t\right] \\ -\lambda_t^V \left[\hat{V}_t + \hat{R}_t - \chi \beta \hat{V}_{t+1}\right] \end{array} \right\} \tag{3.76}$$

The first order conditions with respect to $\hat{\pi}_t$, $\hat{x}_t$, $\hat{d}_t$, $\hat{V}_t$ and $\hat{R}_t$, for $t \geq 1$ are:

$$0 = \hat{\pi}_t - \lambda_t^{\pi} - \beta^{-1}\lambda_t^d + \beta^{-1}\beta\lambda_{t-1}^{\pi} + \beta^{-1}\widetilde{\sigma}\lambda_{t-1}^x \tag{3.77}$$

$$0 = \omega \hat{x}_t + \kappa \lambda_t^{\pi} - \lambda_t^x + \beta^{-1}\lambda_{t-1}^x \tag{3.78}$$

$$0 = -\lambda_t^d + \beta\beta^{-1}\mathbb{E}_t \lambda_{t+1}^d \tag{3.79}$$

$$0 = -\lambda_t^V + \beta^{-1}\chi\beta\lambda_{t-1}^V \tag{3.80}$$

$$0 = -\widetilde{\sigma}\lambda_t^x \tag{3.81}$$

and the corresponding first order conditions for $t = 0$ are:

$$0 = \hat{\pi}_0 - \lambda_0^\pi - \beta^{-1}\lambda_0^d \tag{3.82}$$

$$0 = \omega\hat{x}_0 + \kappa\lambda_0^\pi - \lambda_0^x \tag{3.83}$$

$$0 = -\lambda_0^d + \beta\beta^{-1}\mathbb{E}_0\lambda_1^d \tag{3.84}$$

$$0 = -\lambda_0^V \tag{3.85}$$

$$0 = -\widetilde{\sigma}\lambda_0^x \tag{3.86}$$

From (3.81) and (3.86), we see that the IS curve is not a binding constraint ($\lambda_t^x = 0, \forall t$), so that we can write the system of first order conditions as:

$$0 = \hat{\pi}_t - \lambda_t^\pi - \beta^{-1}\lambda_t^d + \lambda_{t-1}^\pi \tag{3.87}$$

$$0 = \omega\hat{x}_t + \kappa\lambda_t^\pi \tag{3.88}$$

$$0 = -\lambda_t^d + \mathbb{E}_t\lambda_{t+1}^d \tag{3.89}$$

$$0 = -\lambda_t^V + \chi\lambda_{t-1}^V \tag{3.90}$$

Note also that λ_t^V is decoupled from the rest of the system of equations, so the solution is $\lambda_t^V = \chi^t\lambda_0 = 0$ where the final equality follows from (3.85). This implies that the first order conditions can be written as:

$$\hat{\pi}_t = -\frac{\omega}{\kappa}\left(\hat{x}_t - \hat{x}_{t-1}\right) + \beta^{-1}\lambda_t^d \tag{3.91}$$

$$\lambda_t^d = \mathbb{E}_t\lambda_{t+1}^d \tag{3.92}$$

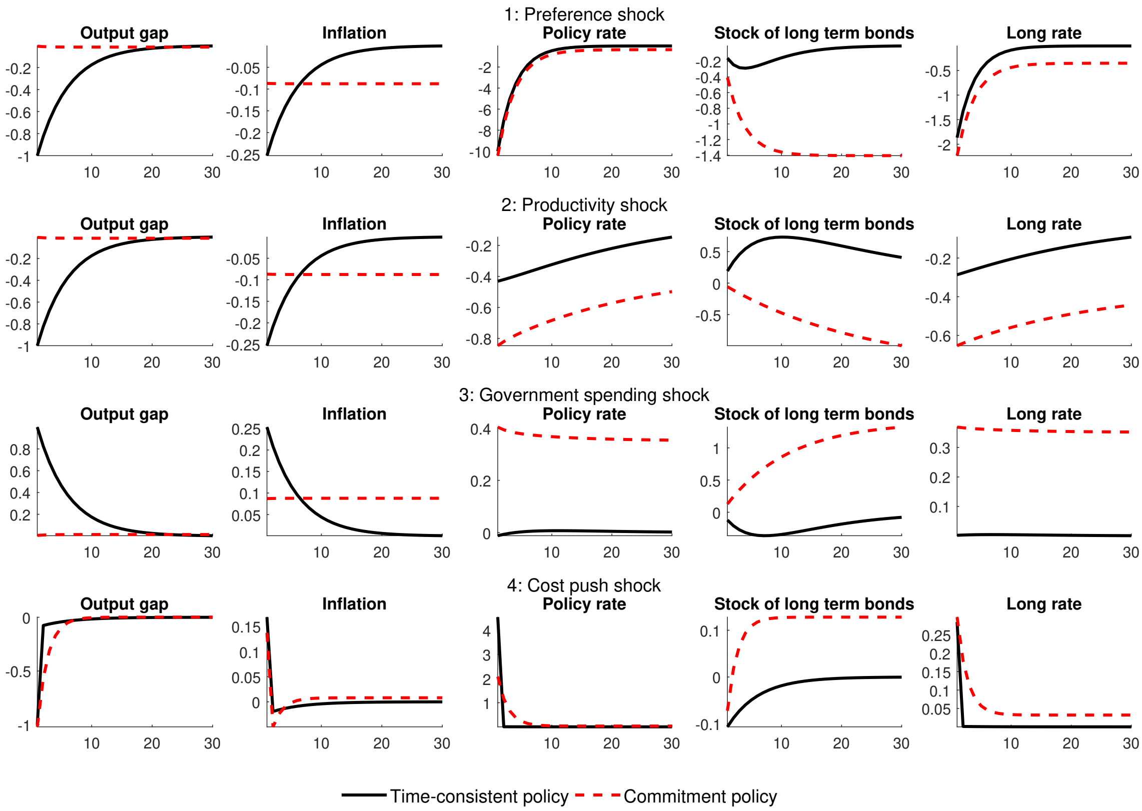
for $t \geq 1$ and the first order condition at $t = 0$ is

$$\hat{\pi}_0 = -\frac{\omega}{\kappa}\hat{x}_0 + \beta^{-1}\lambda_0^d \tag{3.93}$$

For the standard New Keynesian model, the evolution of government debt is not a constraint on policy, so that $\lambda_t^d = 0, \forall t$ and (3.91) reduces to the familiar textbook condition: $\hat{\pi}_t = -\frac{\omega}{\kappa}\left(\hat{x}_t - \hat{x}_{t-1}\right)$.

Figure 3.22 shows the responses of the model under optimal commitment (dashed red lines) alongside the baseline assumption of time-consistent policy (solid black lines). Under optimal commitment, the policymaker finds it optimal to allow the stock of long-term bonds (fourth column) to move permanently, so that the debt stock is a random walk. This result stems from the fact that the first order condition for the debt stock implies that $\lambda_t^d = \mathbb{E}_t \lambda_{t+1}^d$ (Leith and Wren-Lewis, 2013).

Figure 3.22: Responses to shocks in baseline model: time-consistent policy and commitment



Notes: Impulse responses to shocks to the baseline model with time-consistent monetary policy (solid black lines) and optimal commitment policy (dashed red lines). The scale of all shocks is normalized to deliver a 1% response of the output gap in the baseline variant. Policy rate and long rate plotted in annualized units. All variables are shown in percentage point deviations from steady state.

These permanent movements in the stock of debt require permanent movements in the short-term nominal interest rate, the output gap and inflation. This creates a mechanism through which the policymaker operating under commitment can dampen near term fluctuations in the output gap and inflation relative to the outcomes achievable under time-consistent policy (first two columns). However, the permanent components of the output gap and inflation responses create persistent welfare losses in the longer term. For example, it is clear that inflation deviations are larger under commitment than time-consistent policy, creating an incentive to deviate from the commitment policy. This illustrates that the commitment policy in the model is particularly time inconsistent.[50]

Appendix 3.F Solution of the model accounting for the lower bound

This appendix details the solution of the model when the presence of the zero lower bound is accounted for. The algorithm is presented in subsection 3.F.6, with preceding subsections defining notation and deriving key ingredients required for the solution.

3.F.1 First order conditions

The constrained loss minimization problem is:

$$\begin{aligned}
&\min \frac{1}{2}\left[\hat{\pi}_t^2 + \omega \hat{x}_t^2\right] \\
&- \mu_t^x \left[\hat{x}_t - \mathbb{E}_t \hat{x}_{t+1} + \sigma\left(1-g\right)\left(\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} - r_t^*\right)\right] \\
&- \mu_t^\pi \left[\hat{\pi}_t - \kappa \hat{x}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1} - u_t\right] \\
&- \mu_t^d \left[\hat{d}_t - \beta^{-1}\left(\hat{d}_{t-1} - \hat{\pi}_t\right) + \left(1-\chi\right)\hat{V}_t - \zeta^{-1}\bar{g}_t\right] \\
&- \mu_t^V \left[\hat{V}_t + \hat{R}_t - \chi\beta \hat{V}_{t+1}\right] \\
&- \mu_t^Z \left[\hat{R}_t - \beta^{-1} + 1\right] \\
&+ \beta \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}
\end{aligned}$$

where μ^Z is the Lagrange multiplier on the zero bound constraint.

[50] The extent of the time inconsistency depends on the relative losses from pursuing the commitment policy relative to the time-consistent policy. Those losses also depend on the output gap (with a relatively small weight). Though it is difficult to discern by eye given the scale of the charts in Figure 3.22, there is a permanent movement in the output gap, which exceeds the response under time-consistent policy from around period 25 onwards.

The first order conditions are:

$$0 = \hat{\pi}_t - \mu_t^\pi - \beta^{-1}\mu_t^d \tag{3.94}$$

$$0 = \omega\hat{x}_t - \mu_t^x + \kappa\mu_t^\pi \tag{3.95}$$

$$\begin{aligned} 0 = {} & \mu_t^x \left[\frac{\partial \mathbb{E}_t \hat{x}_{t+1}}{\partial \hat{d}_t} + \sigma(1-g)\frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t}\right] + \beta\mu_t^\pi \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \\ & - \mu_t^d + \chi\beta\mu_t^V \frac{\partial \mathbb{E}_t \hat{V}_{t+1}}{\partial \hat{d}_t} + \beta\frac{\partial \mathbb{E}_t \widetilde{\mathcal{L}}_{t+1}}{\partial \hat{d}_t} \end{aligned} \tag{3.96}$$

$$0 = -(1-\chi)\mu_t^d - \mu_t^V \tag{3.97}$$

$$0 = -\sigma(1-g)\mu_t^x - \mu_t^V - \mu_t^Z \tag{3.98}$$

$$0 = \mu_t^Z\left(\hat{R}_t - \beta^{-1} + 1\right) \tag{3.99}$$

where (3.99) is the contemporary slackness condition.

Applying the envelope condition implies that (3.96) can be written as:

$$\begin{aligned} 0 = {} & \mu_t^x \left[\frac{\partial \mathbb{E}_t \hat{x}_{t+1}}{\partial \hat{d}_t} + \sigma(1-g)\frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t}\right] + \beta\mu_t^\pi \frac{\partial \mathbb{E}_t \hat{\pi}_{t+1}}{\partial \hat{d}_t} \\ & - \mu_t^d + \chi\beta\mu_t^V \frac{\partial \mathbb{E}_t \hat{V}_{t+1}}{\partial \hat{d}_t} + \mathbb{E}_t\mu_{t+1}^d \end{aligned}$$

The policy function iteration technique has the following basic structure (the full algorithm is described below). First, outcomes for each element of the state space are solved, conditional on guesses for expectations and the derivatives of expectations with respect to debt. Then these outcomes are used to form a guess for the policy functions. Those guesses are then used to update the estimates of expectations and the derivatives of those expectations with respect to government debt. This process continues until the policy functions converge.

3.F.2 Conditional solutions

To simplify notation, the following conventions are adopted. Time subscripts are removed, with a prime used to denote outcomes in the following period. Then For variable z, let partial derivatives be represented as:

$$\mathcal{D}_z \equiv \frac{\partial \mathbb{E}z'}{\partial \hat{d}} \tag{3.100}$$

Consider first the solution under the assumption that the lower bound does not bind. We can stack the equations characterizing the equilibrium to give:

$$\underbrace{\begin{bmatrix} 0 & 1 & \widetilde{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta^{-1} & 0 & 0 & 1-\chi & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\beta^{-1} & -1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & \kappa & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \beta\mathcal{D}_{\hat{\pi}} & \mathcal{D}_{\hat{x}} + \widetilde{\sigma}\mathcal{D}_{\hat{\pi}} & \chi\beta\mathcal{D}_{\hat{V}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-\chi) & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{\sigma} & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^d \\ \mu^\pi \\ \mu^x \\ \mu^V \\ \mu^Z \end{bmatrix}}_{z}$$

$$= \underbrace{\begin{bmatrix} \mathbb{E}\hat{x}' + \widetilde{\sigma}\mathbb{E}\hat{\pi}' + \widetilde{\sigma}r^* \\ \beta\mathbb{E}\hat{\pi}' + u \\ \beta^{-1}\hat{d} \\ \chi\beta\mathbb{E}\hat{V}' \\ 0 \\ 0 \\ -\mathbb{E}\mu^{d\prime} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{C} \tag{3.101}$$

which can be solved for the vector of endogenous variables and Lagrange multipliers as:

$$z = M^{-1}C$$

In the case in which the zero bound does bind we have:

$$\underbrace{\begin{bmatrix} 0 & 1 & \widetilde{\sigma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta^{-1} & 0 & 0 & 1-\chi & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\beta^{-1} & -1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & \kappa & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \beta\mathcal{D}_{\hat{\pi}} & \mathcal{D}_{\hat{x}} + \widetilde{\sigma}\mathcal{D}_{\hat{\pi}} & \chi\beta\mathcal{D}_{\hat{V}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-\chi) & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\widetilde{\sigma} & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\widetilde{M}} \underbrace{\begin{bmatrix} \hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^d \\ \mu^\pi \\ \mu^x \\ \mu^V \\ \mu^Z \end{bmatrix}}_{z}$$

$$= \underbrace{\begin{bmatrix} \mathbb{E}\hat{x}' + \widetilde{\sigma}\mathbb{E}\hat{\pi}' + \widetilde{\sigma} r^* \\ \beta\mathbb{E}\hat{\pi}' + u \\ \beta^{-1}\hat{d} \\ \chi\beta\mathbb{E}\hat{V}' \\ 0 \\ 0 \\ -\mathbb{E}\mu^{d\prime} \\ 0 \\ 0 \\ 1-\beta^{-1} \end{bmatrix}}_{\widetilde{C}} \tag{3.102}$$

which can be solved for the vector of endogenous variables and Lagrange multipliers as:

$$z = \widetilde{M}^{-1}\widetilde{C}$$

Note that the differences between M and $\widetilde{M}$ and between C and $\widetilde{C}$ are isolated to the bottom row of each matrix.

3.F.3 State space and policy functions: notation

The description of the algorithm can be simplified by introducing some notation for the key objects that will be solved for.

The vector of endogenous variables are denoted by z, defined implicitly above, but explicitly here:

$$z \equiv \begin{bmatrix} \hat{\pi} \\ \hat{x} \\ \hat{R} \\ \hat{V} \\ \hat{d}' \\ \mu^d \\ \mu^\pi \\ \mu^x \\ \mu^V \\ \mu^Z \end{bmatrix}$$

The exogenous states are denoted s:

$$s \equiv \begin{bmatrix} r^* \\ u \end{bmatrix}$$

and full state vector for relevant policy functions is, $\tilde{s}$:

$$\tilde{s} \equiv \begin{bmatrix} s \\ \hat{d} \end{bmatrix}$$

The exogenous state is defined as a set of fixed values for the cost push shock and natural rate. Specifically, $S_r \equiv \left\{ r_1^* \quad \ldots \quad r_{n_r}^* \right\}$ and $S_u \equiv \{ u_1 \quad \ldots \quad u_{n_u} \}$. The transition matrices for the Markov processes are Ω_r and Ω_u.

The combined (exogenous) state-space is given by $S = S_u \times S_r$ with transition matrix $\Omega = \Omega_r \otimes \Omega_u$.[51] The endogenous state is $\hat{d}$, which is discretized on a grid

[51]The first n_u elements of the state space are $\{u_1, r_1^*\}, \ldots, \{u_{n_u}, r_1^*\}$, followed by $\{u_1, r_2^*\}, \ldots, \{u_{n_u}, r_2^*\}$ and so on.

$S_d \equiv \left\{ \hat{d}_1 \quad \dots \quad \hat{d}_{n_d} \right\}$, with $\hat{d}_i > \hat{d}_{i-1}$, $i = 2, \dots, n_d$. The endogenous state is assumed to be ordered last. So the full state space is given by $\tilde{S} = S \times S_d$.[52] Thus, $\tilde{S}$ is a $n_{\tilde{s}} \times 3$ matrix, where $n_{\tilde{s}} \equiv n_s \times n_d$. The index of the element $\left\{ u_i, r_j^*, \hat{d}_k \right\} \in \tilde{S}$ is $(k-1) \times n_s + (j-1) \times n_u + i$.

This implies that the combined state can be written as:

$$\tilde{S} = \begin{bmatrix} S & \hat{d}_1 \mathbf{1}_{n_s} \\ \vdots & \vdots \\ S & \hat{d}_{n_d} \mathbf{1}_{n_s} \end{bmatrix}$$

where $\mathbf{1}_{n_s}$ is a $n_s \times 1$ unit vector.

This representation of the state space is useful for subsequent computations since approximation of expectations requires interpolation between grid points for the endogenous state, while integrating across the exogenous state S. Similar methods are used for the estimation of derivatives of expectations.

The objects of interest are policy functions. These are $n_{\tilde{s}} \times n_z$ matrices. Let a generic policy function be denoted $\mathbf{f}$.

3.F.4 Expectations

It is useful to define an 'expectation' operator that integrates out exogenous state uncertainty but holds the endogenous state vector constant:

$$\mathbb{E}_S \mathbf{f} \equiv \overline{\mathbf{f}}^{\mathbf{S}} \equiv \begin{bmatrix} \Omega & 0 & \dots & 0 & 0 \\ 0 & \Omega & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega & 0 \\ 0 & 0 & \dots & 0 & \Omega \end{bmatrix} \mathbf{f}$$

so that the 'bar' is used as a summary notation for expectations and the S superscript indicates that the expectation is computed with respect to the exogenous state variables only.

[52] Thus the first $n_s \equiv n_u \times n_r$ elements are given by the triples $\left\{ u_1, r_1^*, \hat{d}_1 \right\}, \dots, \left\{ u_{n_u}, r_{n_r}^*, \hat{d}_1 \right\}$, the next n_s elements are $\left\{ u_1, r_1^*, \hat{d}_2 \right\}, \dots, \left\{ u_{n_u}, r_{n_r}^*, \hat{d}_2 \right\}$ and so on.

To compute the actual expectation requires conditioning on the solution for $\hat{d}'$ at the particular point in the state space. This can be done as follows.

- Extract the relevant column of $\mathbf{f}$ that corresponds to debt. Let this column vector be denoted $\mathbf{d}$. The elements of this vector denote the solutions $\hat{d}' \in z$ for each state $1, \ldots, n_{\tilde{s}}$.
- Let the elements of $\mathbf{d}$ be denoted $\mathbf{d}_k, k = 1, \ldots, n_{\tilde{s}}$. Let the exogenous state corresponding to this solution be $S_{<k>}$. For each k, perform the following:
  - Compute the indices in S_d that bracket this element.[53] This gives two indices $i_1, i_2 \in S_d$ with $1 \leq i_1 < i_2 \, (= i_1 + 1) \leq n_d$.
  - Compute the weights that should apply to each of these gridpoints (using linear interpolation). This gives $\phi_1 = \frac{\mathbf{d}_k - S_d(i_1)}{S_d(i_2) - S_d(i_1)}$ and $\phi_2 = 1 - \phi_1$.
  - Compute the indices of the elements of $\tilde{S}$ corresponding to the elements in $\tilde{S}$ for which (a) $S = S_{<k>}$ and (b) $S_d = S_d\,(i_1)$ and $S_d = S_d\,(i_2)$. Denote these indices as $\tilde{i}_1$ and $\tilde{i}_2$.
  - Estimate the expectation using linear interpolation as:

    $$\bar{\mathbf{f}}_{\mathbf{k},\cdot} = \phi_1 \bar{\mathbf{f}}^{\mathbf{S}}_{\tilde{\mathbf{i}}_1,\cdot} + \phi_2 \bar{\mathbf{f}}^{\mathbf{S}}_{\tilde{\mathbf{i}}_2,\cdot}$$

    where the subscript '$j, \cdot$' denotes the j-th row of a matrix.

The penultimate step (finding $\tilde{i}_1$ and $\tilde{i}_2$) can be aided by a pre-computation operation. To see this, recall that for each $k \in \{1, \ldots, n_{\tilde{s}}\}$, there is an exogenous state, $S_{<k>}$. The indices corresponding to different values of $\hat{d}$ for the same value of $S_{<k>}$ are multiples of n_s away from from k. This allows us to form a $n_{\tilde{s}} \times n_d$ matrix of indices – a 'lookup matrix', denoted Λ – as follows.

For each $k \in \{1, \ldots, n_{\tilde{s}}\}$

- Compute j, defined as the index of the grid point $\hat{d}'_k$ within S_p. Recall that $\hat{d}'$ is the final (third) state.
- For $m = 1, \ldots, n_s$, form the k-th row of Λ as:

  $$\Lambda_{k,m} = k - (j - m)\, n_s$$

[53] Extrapolation is conceptually identical, but for the purposes of exposition, I assume that interpolation is required.

Then, in the computation of expectations, for each k the indices are found by setting $\tilde{i}_1 = \Lambda_{k,i_1}$ and $\tilde{i}_2 = \Lambda_{k,i_2}$.

3.F.5 Derivatives

The first order conditions depend on derivatives of expectations of the policy functions. To approximate these derivatives, a two-sided finite difference approach is used. The derivatives are computed in two steps. In the first step, two-sided finite difference derivatives of the static expectations are computed, using adjacent gridpoints for $\hat{d}$. In the second step, linear interpolation is used to approximate the derivatives at the relevant values of $\mathbf{d}$.

The first step is to approximate the derivative of the static expectation function $\overline{\mathbf{f}}^{\mathbf{S}}$. We seek the finite difference approximation to the derivative of $\overline{\mathbf{f}}^{\mathbf{S}}$ for each row $m = 1, \ldots, n_{\tilde{s}}$. First note that S_d is assumed to be formed of an evenly-spaced grid of values: $S_d = \left\{ \hat{d}_1 \quad \ldots \quad \hat{d}_{n_d} \right\}$, with $\hat{d}_{i+1} = \hat{d}_i + h_d$. So the difference between each grid point is h_d.

Consider an m for which the corresponding element of S_d is $\hat{d}_i$ with $1 < i < n_d$, that is, an interior point. Then the 'static derivative' at point m is given by:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{2h_d} \left(\overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}+\mathbf{n_s},\cdot} - \overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}-\mathbf{n_s},\cdot} \right)$$

Now consider the endpoints. For $1 \leq m \leq n_s$, $i = 1$ and a one-sided difference is used:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{h_d} \left(\overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}+\mathbf{n_s},\cdot} - \overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m},\cdot} \right)$$

Similarly, for $n_{\tilde{s}} - n_s + 1 \leq m \leq n_{\tilde{s}}$, $i = n_d$ and the one-sided approximation is given by:

$$\mathbf{D}^{\mathbf{S}}_{\mathbf{m},\cdot} = \frac{1}{h_d} \left(\overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m},\cdot} - \overline{\mathbf{f}}^{\mathbf{S}}_{\mathbf{m}-\mathbf{n_s},\cdot} \right)$$

The second step is to form an estimate of the derivative at $\mathbf{d}$ using linear interpolation. This step is set out in the description of the algorithm below.

3.F.6 Algorithm

The objective of the algorithm is to solve for the policy function $\mathbf{f}$ by iterating directly on it.

1. Initialize a guess, $\mathbf{f}^{<0>}$, for the policy function $\mathbf{f}$.
2. Build the 'lookup matrix', Λ, as described above.
3. For each iteration $j = 1, \dots$

   Update expectations

   a) Update the guess for 'static' expectations. As described above, this integrates out exogenous state uncertainty but holds the endogenous state vector constant:

$$\bar{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>} = \begin{bmatrix} \Omega & 0 & \dots & 0 & 0 \\ 0 & \Omega & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega & 0 \\ 0 & 0 & \dots & 0 & \Omega \end{bmatrix} \mathbf{f}^{<\mathbf{j}-\mathbf{1}>}$$

   b) Extract the vector of $\tilde{d}'$ values, $\mathbf{d}^{<\mathbf{j}-\mathbf{1}>}$ as the relevant column of $\mathbf{f}^{<\mathbf{j}-\mathbf{1}>}$.

   c) Compute the indices and weights of the elements of S_p that bracket the values in $\mathbf{d}^{<\mathbf{j}-\mathbf{1}>}$. Use the lookup matrix Λ to convert these into $n_{\tilde{s}} \times 2$ matrices of indicators and interpolation/extrapolation weights, denoted Υ and Φ respectively.

   d) For each $m = 1, \dots, n_{\tilde{s}}$: Compute expectations by extracting interpolation indices $[\iota_1 \;\; \iota_2] = \Upsilon_m$ and weights $[\phi_1 \;\; \phi_2] = \Phi_m$. Translate the interpolation weights into $\tilde{S}$ space by setting $\tilde{\iota}_1 = \Lambda_{m,\iota_1}$ and $\tilde{\iota}_2 = \Lambda_{m,\iota_2}$. Now set

$$\bar{\mathbf{f}}_{\mathbf{m},\cdot} = \phi_1 \bar{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>}_{\tilde{\iota}_1,\cdot} + \phi_2 \bar{\mathbf{f}}^{\mathbf{S}<\mathbf{j}>}_{\tilde{\iota}_2,\cdot}$$

   Update the estimate of the derivative of expectations

   e) Update the estimate of the 'static' derivatives, $\mathbf{D}^{\mathbf{S}}$, as described in 3.F.5.

f) Compute the derivatives prevailing at $\mathbf{d}$ by linear interpolation. For each $m = 1, \ldots, n_{\tilde{s}}$, set:

$$\mathbf{D}_{\mathbf{m},\cdot} = \phi_1 \mathbf{D}^{\mathbf{S}}_{\tilde{\imath}_1,\cdot} + \phi_2 \mathbf{D}^{\mathbf{S}}_{\tilde{\imath}_2,\cdot}$$

where the indices $\tilde{\imath}_1, \tilde{\imath}_2$ and weights ϕ_1, ϕ_2 are the same as in step 3d.

Update the guess for the policy function

g) For each $m = 1, \ldots, n_{\tilde{s}}$:

   i. Extract latest guesses for expectations and their derivatives:

   $$\mathbb{E}z' = \bar{\mathbf{f}}_{\mathbf{m},\cdot} \qquad \mathcal{D} = \mathbf{D}_{\mathbf{m},\cdot}$$

   ii. Assume that the zero bound does not bind. Form M and C using $\mathbb{E}z'$ and $\mathcal{D}$ and solve the system (3.101) as $z = M^{-1}C$.

   iii. Check whether this solution is indeed consistent with a positive interest rate by checking whether the relevant element of z exceeds the zero bound. If it does, proceed to step 3(g)v, otherwise proceed to step 3(g)iv.

   iv. Compute the solution imposing the lower bound. Form $\widetilde{M}$ and $\widetilde{C}$ using $\mathbb{E}z'$ and $\mathcal{D}$ and solving (3.102) as $z = \widetilde{M}^{-1}\widetilde{C}$.

   v. Load the solutions into the latest guess for the policy function:

   $$\mathbf{f}^{<\mathbf{j}>}_{\mathbf{m},\cdot} = z$$

4. Check for convergence. If $\left|\mathbf{f}^{<\mathbf{j}>} - \mathbf{f}^{<\mathbf{j-1}>}\right| < \varepsilon$, set $\mathbf{f} = \mathbf{f}^{<\mathbf{j}>}$ and stop, otherwise set $j = j + 1$ and return to step 3.

3.F.7 Practical implementation

Solutions for the policy functions were found using a heuristic iterative procedure. Before finding the solution, the equilibrium distribution of the endogenous state variable is unknown. So some iterative experimentation with the end points of the grid (ie $\hat{d}_1$ and $\hat{d}_{n_d}$) was required to ensure that the policy functions did not require extrapolation beyond these points.

Similarly, some experimentation was required to choose the increment h_d between grid points for $\hat{d}$ in way that provided a balance between computational efficiency and accurate computation of the derivatives of expected policy functions. In practice, the model was solved on a coarse grid for $\hat{d}$ and the resulting solution used to produce a guess (using linear interpolation) for the policy function on a finer grid.

Appendix 3.G Optimal monetary policy in the presence of fiscal uncertainty

This Appendix provides details of the solution algorithm used to analyze scenarios under fiscal uncertainty.

3.G.1 Structure of uncertainty

There is a state s_t that may take one of $K+1$ distinct values at each date $t = 1, \ldots, T$: $s_t \in \mathcal{S} \equiv \{s_{<0>}, s_{<1>}, \ldots, s_{<K>}\}$. The subscript $< j >$ is used to denote state j to clarify that the subscript indicates the state rather than a time period.

State zero ($s_{<0>}$) is the initial state, or status quo. The process starts in this state in period 1, so that $s_1 = s_{<0>}$. The remaining K states, $\{s_{<j>}\}_{j=1}^{K}$ are called 'exit states'.

The transition law between states varies over time. At date t, the transition matrix is:

$$\mathcal{M}_t = \begin{bmatrix} 1 - \sum_{j=1}^{K} p_{j,t} & p_{1,t} & p_{2,t} & \ldots & p_{K,t} \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

for $p_{j,t} \geq 0, \forall j, t$ and $\sum_{j=1}^{K} p_{j,t} \leq 1, \forall t > 1$.

This implies that, each of the exit states $s_{<j>}, j = 1, \ldots, K$ is an absorbing state. Once the economy has transitioned to that state, it never leaves it.

The fact that the transition matrix is time-varying makes it possible to incorporate differences in the likelihood of switches at different moments in time. For example, it may be be (credibly) announced that a switch will definitely occur between periods $t_1(> 0)$ and $t_2(> t_1)$ in which cases the probabilities of switching outside this date range could be set to zero.

All uncertainty is assumed to be resolved at date $T + 1$. That requires that $\sum_{j=1}^{K} p_{j,T+1} = 1$, so that in period $T + 1$ the economy will be in one of the exit states $s_{<j>}, j = 1, \ldots, K$ and will remain in that state forever more. Delivering a unique date at which uncertainty is resolved requires $\sum_{j=1}^{K} p_{j,t} < 1, t = 1, \ldots, T$.

3.G.2 Timing

The timing of events within each period is as follows. At the start of period t, the state s_t is revealed to all agents. Then any other shocks that hit the economy are revealed to all agents. Finally, all agents (the private sector and the policymaker) make their decisions.

3.G.3 Model structure

The model equations describe the evolution of the endogenous variables (x) and the policy instruments (r):[54]

$$\begin{aligned} & F(s_t)\left[\mathbb{E}_t x_{t+1} - \bar{x}(s_t)\right] + G(s_t)\left[x_t - \bar{x}(s_t)\right] \\ & + H(s_t)\left[x_{t-1} - \bar{x}(s_t)\right] + M(s_t)\left[r_t - \bar{r}(s_t)\right] = \Psi(s_t)\, z_t \end{aligned}$$

where the coefficient matrices (F, G, H, M, Ψ) are functions of the state s_t as are the steady state values of the endogenous variables ($\bar{x}$) and the policy instrument(s) ($\bar{r}$).[55] The model is perturbed by exogenous shocks z_t which are iid with mean zero and unit variance.

[54] It is assumed that r is a vector, so that multiple instruments can be considered.

[55] It is possible that the structural changes implied by some or all of the final states $(\{s_j\}_{j=2}^{K})$ imply changes in the levels of policy instruments consistent with the policy objectives. For example, a structural changes that permanently change the efficient rate of interest would likely change the level of the short-term policy rate consistent with hitting a particular inflation target on average.

3.G.4 Policymaker behavior

The monetary policymaker sets its policy instrument(s) in a time-consistent manner. Specifically, the policymaker minimizes a loss function that is assumed to be a quadratic function of the model variables:

$$\mathcal{L}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[\begin{array}{c} (x_{t+i} - \bar{x}(s_{t+i}))' W (x_{t+i} - \bar{x}(s_{t+i})) \\ + (r_{t+i} - \bar{r}(s_{t+i}))' Q (r_{t+i} - \bar{r}(s_{t+i})) \end{array} \right]$$

where W is a positive semi-definite weighting matrix and a discount factor $0 < \beta < 1$. The loss function is assumed to depend on the deviations of the endogenous variables from their long-run levels, which may change over time in response to the underlying state s_t. Policy preferences (as encoded in the W matrix and the discount factor β) are assumed to be invariant to the state s_t.

3.G.5 Shocks and uncertainty

The object of interest is the set of possible paths that the economy may follow over the periods $t = 1, \ldots, T$. It is assumed that this object is computed under the assumption that date-1 uncertainty has been resolved. That is, the solution is conditional on the following outcomes. First, it has been revealed at the start of period 1 that a structural change has not occurred. Second, the shocks hitting the economy in period 1 (z_1) have been revealed. So the solutions are conditional on information (and associated expectations) at date 1.

These assumptions mean that shocks in period 1 are fully observed $\mathbb{E}_1 z_1 = z_1$ and expectations of future shocks are all zero: $\mathbb{E}_1 z_t = 0, \forall t > 1$. This precludes analysis of certain experiments (for example, the revelation in period 1 that a certain shock will take a particular value in period 10), though some of these experiments could be undertaken by an appropriate augmentation of the vector of endogenous variables x_t.

The benefit of this approach, of course, is simplicity. For notational convenience, the expectations operator is omitted unless doing so is likely to cause confusion.

3.G.6 Solution algorithm

Three of the maintained assumptions mean that equilibrium outcomes can be found by backward induction.

First, the fact that the policymaker acts under discretion implies that they take the behavior of future policymakers as given. The second assumption is that each of the final states is an absorbing state. Once the economy has reached one of these states, there are no further structural changes. This means that the future evolution of the endogenous variables in each of these states can be computed using standard linear quadratic procedures and policy will be characterized by a time-invariant feedback rule. Finally, we assume that all uncertainty is resolved at a known date, $T+1$. This means that from period $T+1$ onward, there are only K possible states of nature. Again, each of these states is characterized by a time-invariant model and, in period T, expected outcomes can be computed by weighting the outcomes from each state by the respective probabilities of entering that state in period $T+1$.[56]

These observations imply that the object of interest is the equilibrium path of the economy (and policy) for the case in which the economy stays in state 0 (status quo) until period T. All other outcomes (that is, structural change at any date $u = 1, \ldots, T+1$) can be computed by combining the 'non-exit' trajectory for periods $t = 1, \ldots, u$ and the relevant exit path from period $u+1$ onwards. Expected paths can be computed by probability weighting these paths.

Solutions in each exit state

Suppose that in period $1 < u \leq T$, state $s_{<j>}, j \in \{1, \ldots, K\}$ is revealed.[57] This implies that the policymaker at date u solves:

$$\min_{r_u} \sum_{i=0}^{\infty} \beta^i \left[(x_{u+i} - \bar{x}_{<j>})' W (x_{u+i} - \bar{x}_{<j>}) + (r_{u+i} - \bar{r}_{<j>})' Q (r_{u+i} - \bar{r}_{<j>}) \right] \tag{3.103}$$

[56]That is, $\{p_{j,T+1}\}_{j=1}^{K}$.

[57]Note that u is strictly greater than 1 because outcomes are computed conditional on no structural change in period 1.

subject to:

$$\begin{aligned}&F_{<j>}\left[\mathbb{E}_1 x_{t+1}-\bar{x}_{<j>}\right]+G_{<j>}\left[x_t-\bar{x}_{<j>}\right]\\&+H_{<j>}\left[x_{t-1}-\bar{x}_{<j>}\right]+M_{<j>}\left[r_t-\bar{r}_{<j>}\right]=0\end{aligned}$$

where for notational simplicity we use the $< j >$ subscript to index state-dependent objects to state $< j >$, for example:

$$\begin{aligned}&\bar{x}_{<j>}\equiv\bar{x}\left(s_{<j>}\right)\\&F_{<j>}\equiv F\left(s_{<j>}\right)\end{aligned}$$

and so on. Note that the expectations operator is conditional on date-1 information and the fact that $\mathbb{E}_1 z_t=0$ has been used.

This is a standard linear quadratic policy problem and can be solved using a variety of methods. I use the algorithm developed by Dennis (2007). The solution has the form:

$$x_t-\bar{x}_{<j>}=B_{<j>}\left[x_{t-1}-\bar{x}_{<j>}\right] \tag{3.104}$$

$$r_t-\bar{r}_{<j>}=D_{<j>}\left[x_{t-1}-\bar{x}_{<j>}\right] \tag{3.105}$$

Computing expectations

Recall that the objective is to compute the equilibrium outcomes for the case in which exit does not occur until period $T+1$. For this case, in each period $t=1,\ldots,T$, all agents understand that the economy may move to the K exit states with some probability.[58]

The expected outcome for the endogenous variables conditional on transition to state $s_{<j>}$ is:

$$\mathbb{E}_1\left[x_{t+1}|s_{t+1}=s_{<j>}\right]=\bar{x}_{<j>}+B_{<j>}\left[x_t-\bar{x}_{<j>}\right]$$

because $\mathbb{E}_1 z_{t+1}=0$.

[58]The probability of exit may be zero in some periods t ($p_{j,t}=0, j=1,\ldots,K$) without affecting the approach to computing the equilibrium.

This means that the expected outcome is given by:

$$\mathbb{E}_1 x_{t+1} = \sum_{j=1}^{K} p_{j,t+1} \left(\bar{x}_{<j>} + B_{<j>} \left[\widetilde{x}_t - \bar{x}_{<j>} \right] \right) + \left(1 - \mathbf{p}_{t+1}\right) \widetilde{x}_{t+1}$$

where the 'tilde' is used to denote outcomes in the case where exit does not occur in a particular period. So $\widetilde{x}_t$ denotes the outcome in period t conditional on exit not having occurred by date t. The notation $\mathbf{p_{t+1}}$ is used to denote the probability that exit occurs (to any of the K exit states) in period $t+1$:

$$\mathbf{p}_{t+1} \equiv \sum_{j=1}^{K} p_{j,t+1}$$

Expectations can therefore be written as:

$$\mathbb{E}_1 x_{t+1} = B_{t+1} \widetilde{x}_t + C_{t+1} + \left(1 - \mathbf{p}_{t+1}\right) \widetilde{x}_{t+1} \tag{3.106}$$

where

$$B_{t+1} \equiv \sum_{j=1}^{K} p_{j,t+1} B_{<j>}$$

$$C_{t+1} \equiv \sum_{j=1}^{K} p_{j,t+1} \left(\mathbb{I} - B_{<j>} \right) \bar{x}_{<j>}$$

The policy problem

As noted above, the policy problem can be solved recursively by backward induction.

Period T

Consider first the policymaker in date T. The policymaker realizes that a structural change will occur with certainty in period $T+1$ (though will, in general, be uncertain about which of the states will be reached). That is, $\mathbf{p}_{T+1} = 1$.

The policy problem is then:

$$\min_{r_T} \mathbb{E}_T \sum_{i=0}^{\infty} \beta^i \left[\begin{array}{c} \left(x_{T+i} - \bar{x}\left(s_{T+i}\right)\right)' W \left(x_{T+i} - \bar{x}\left(s_{T+i}\right)\right) \\ + \left(r_{T+i} - \bar{r}\left(s_{T+i}\right)\right)' Q \left(r_{T+i} - \bar{r}\left(s_{T+i}\right)\right) \end{array} \right] \tag{3.107}$$

subject to:

$$
\begin{aligned}
&F_{<0>}\left[B_{T+1}\widetilde{x}_T + C_{T+1} - \bar{x}_{<0>}\right] + G_{<0>}\left[\widetilde{x}_T - \bar{x}_{<0>}\right] \\
&+ H_{<0>}\left[\widetilde{x}_{T-1} - \bar{x}_{<0>}\right] + M_{<0>}\left[\widetilde{r}_T - \bar{r}_{<0>}\right] = 0
\end{aligned}
$$

where expectations have been eliminated using (3.106) (incorporating the fact that $\mathbf{p}_{T+1} = 1$) and denoted the current choice of the policy instruments as $\widetilde{r}_T$ (because they correspond to choices along the 'non-exit' path).

Rearranging gives:

$$
\begin{aligned}
&\left[F_{<0>}B_{T+1} + G_{<0>}\right]\widetilde{x}_T + F_{<0>}\left[C_{T+1} - \bar{x}_{<0>}\right] \\
&- G_{<0>}\bar{x}_{<0>} + H_{<0>}\left[\widetilde{x}_{T-1} - \bar{x}_{<0>}\right] + M_{<0>}\left[\widetilde{r}_T - \bar{r}_{<0>}\right] = 0
\end{aligned}
$$

If $G_{<0>}$ has full rank, then $F_{<0>}B_{T+1} + G_{<0>}$ should also have full rank. This implies that:

$$
\begin{aligned}
\widetilde{x}_T = &- \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} F_{<0>}\left[C_{T+1} - \bar{x}_{<0>}\right] \\
&+ \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} G_{<0>}\bar{x}_{<0>} \\
&- \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} H_{<0>}\left[\widetilde{x}_{T-1} - \bar{x}_{<0>}\right] \\
&- \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} M_{<0>}\left[\widetilde{r}_T - \bar{r}_{<0>}\right]
\end{aligned}
$$

Collecting terms gives:

$$
\widetilde{x}_T = \mathcal{A}_T + \mathcal{B}_T\widetilde{x}_{T-1} + \mathcal{C}_T\widetilde{r}_T
$$

where:

$$
\begin{aligned}
\mathcal{A}_T \equiv &\left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} \\
&\times \left[M_{<0>}\bar{r}_{<0>} + \left(F_{<0>} + G_{<0>} + H_{<0>}\right)\bar{x}_{<0>} - F_{<0>}C_{T+1}\right] \\
\mathcal{B}_T \equiv &- \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} H_{<0>} \\
\mathcal{C}_T \equiv &- \left[F_{<0>}B_{T+1} + G_{<0>}\right]^{-1} M_{<0>}
\end{aligned}
$$

The loss function can be expanded as:

$$\begin{aligned}\mathcal{L}_T &\equiv \sum_{i=0}^{\infty} \beta^i \left[\begin{array}{c} (x_{T+i} - \bar{x}(s_{T+i}))' W (x_{T+i} - \bar{x}(s_{T+i})) \\ + (r_{T+i} - \bar{r}(s_{T+i}))' Q (r_{T+i} - \bar{r}(s_{T+i})) \end{array} \right] \\ &= (\widetilde{x}_T - \bar{x}_{<0>})' W (\widetilde{x}_T - \bar{x}_{<0>}) + (\widetilde{r}_T - \bar{r}_{<0>})' Q (\widetilde{r}_T - \bar{r}_{<0>}) \\ &\quad + \sum_{j=1}^{K} p_{j,T+1} \sum_{i=1}^{\infty} \beta^i \left(x_{T+i}^{<j>} - \bar{x}_{<j>}\right)' W \left(x_{T+i}^{<j>} - \bar{x}_{<j>}\right) \\ &\quad + \sum_{j=1}^{K} p_{j,T+1} \sum_{i=1}^{\infty} \beta^i \left(r_{T+i}^{<j>} - \bar{r}_{<j>}\right)' Q \left(r_{T+i}^{<j>} - \bar{r}_{<j>}\right) \end{aligned} \tag{3.108}$$

Appendix 3.G.8 demonstrates that the loss function (3.108) can be written as:

$$\begin{aligned}\mathcal{L}_T &= (\widetilde{x}_T - \bar{x}_{<0>})' W (\widetilde{x}_T - \bar{x}_{<0>}) + (\widetilde{r}_T - \bar{r}_{<0>})' Q (\widetilde{r}_T - \bar{r}_{<0>}) \\ &\quad + \beta \sum_{j=1}^{K} p_{j,T+1} (\widetilde{x}_T - \bar{x}_{<j>})' V_{<j>} (\widetilde{x}_T - \bar{x}_{<j>})\end{aligned}$$

where

$$V_{<j>} = B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>} + \beta B'_{<j>} V_{<j>} B_{<j>}$$

Putting the preceding results together implies that the policy problem can be written as:

$$\begin{aligned}\min_{\widetilde{r}_T} &\left(\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<0>}\right)' W \left(\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<0>}\right) \\ &+ (\widetilde{r}_T - \bar{r}_{<0>})' Q (\widetilde{r}_T - \bar{r}_{<0>}) \\ &+ \beta \sum_{j=1}^{K} p_{j,T+1} \left[\begin{array}{c} (\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<j>})' \\ \times V_{<j>} \\ \times (\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<j>}) \end{array} \right]\end{aligned}$$

The first order condition with respect to $\widetilde{r}_T$ is:

$$\begin{aligned}0 =& \mathcal{C}'_T W \left(\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<0>}\right) + Q (\widetilde{r}_T - \bar{r}_{<0>}) \\ &+ \beta \sum_{j=1}^{K} p_{j,T+1} \mathcal{C}'_T V_{<j>} \left(\mathcal{A}_T + \mathcal{B}_T \widetilde{x}_{T-1} + \mathcal{C}_T \widetilde{r}_T - \bar{x}_{<j>}\right)\end{aligned}$$

The first order condition can be rearranged to give:

$$\begin{aligned}\widetilde{r}_T =& \mathcal{F}_T\left[\mathcal{C}_T'W\left[\mathcal{A}_T+\mathcal{B}_T\widetilde{x}_{T-1}-\bar{x}_{<0>}\right]-Q\bar{r}_{<0>}\right]\\&+\beta\mathcal{F}_T\sum_{j=1}^{K}p_{j,T+1}\mathcal{C}_T'V_{<j>}\left(\mathcal{A}_T+\mathcal{B}_T\widetilde{x}_{T-1}-\bar{x}_{<j>}\right)\end{aligned}$$

where

$$\mathcal{F}_T \equiv -\left(Q+\mathcal{C}_T'W\mathcal{C}_T+\beta\sum_{j=1}^{K}p_{j,T+1}\mathcal{C}_T'V_{<j>}\mathcal{C}_T\right)^{-1}$$

This can be expressed more compactly as:

$$\widetilde{r}_T = \mathcal{R}_T^* + \mathcal{D}_T^*\widetilde{x}_{T-1}$$

where

$$\begin{aligned}\mathcal{R}_T^* \equiv& \mathcal{F}_T\left[\mathcal{C}_T'W\left(\mathcal{A}_T-\bar{x}_{<0>}\right)-Q\bar{r}_{<0>}+\beta\sum_{j=1}^{K}p_{j,T+1}\mathcal{C}_T'V_{<j>}\left(\mathcal{A}_T-\bar{x}_{<j>}\right)\right]\\ \mathcal{D}_T^* =& \mathcal{F}_T\left[\mathcal{C}_T'W\mathcal{B}_T+\beta\sum_{j=1}^{K}p_{j,T+1}\mathcal{C}_T'V_{<j>}\mathcal{B}_T\right]\end{aligned}$$

and the asterisk superscript indicates that the coefficients embody the optimal policy responses in period T.

Plugging the solution for the instrument back into the expression for $\widetilde{x}_T$ gives:

$$\begin{aligned}\widetilde{x}_T =& \mathcal{A}_T+\mathcal{B}_T\widetilde{x}_{T-1}+\mathcal{C}_T\left[\mathcal{R}_T^*+\mathcal{D}_T^*\widetilde{x}_{T-1}\right]\\=&\mathcal{A}_T+\mathcal{C}_T\mathcal{R}_T^*+\mathcal{B}_T\widetilde{x}_{T-1}+\mathcal{C}_T\mathcal{D}_T^*\widetilde{x}_{T-1}\\=&\mathcal{A}_T^*+\mathcal{B}_T^*\widetilde{x}_{T-1}\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_T^* \equiv& \mathcal{A}_T+\mathcal{C}_T\mathcal{R}_T^*\\ \mathcal{B}_T^* \equiv& \mathcal{B}_T+\mathcal{C}_T\mathcal{D}_T^*\end{aligned}$$

and, again, the asterisk indicates that the coefficients embody the optimal policy responses in period T.

Period $T-1$

Now consider the policy problem at date $T-1$. The model equations that constrain the policymaker at date $T-1$ can be written as:

$$\begin{aligned}
&F_{<0>}\left[B_T\widetilde{x}_{T-1}+C_T+(1-\mathbf{p}_T)\,\widetilde{x}_T-\bar{x}_{<0>}\right]+G_{<0>}\left[\widetilde{x}_{T-1}-\bar{x}_{<0>}\right]\\
&+H_{<0>}\left[\widetilde{x}_{T-2}-\bar{x}_{<0>}\right]+M_{<0>}\left[\widetilde{r}_{T-1}-\bar{r}_{<0>}\right]=0
\end{aligned}$$

which includes outcomes at date T. The results from the analysis of period T can be applied, so that:

$$\begin{aligned}
&F_{<0>}\left[B_T\widetilde{x}_{T-1}+C_T+(1-\mathbf{p}_T)\left(\mathcal{A}_T^*+\mathcal{B}_T^*\widetilde{x}_{T-1}\right)-\bar{x}_{<0>}\right]+G_{<0>}\left[\widetilde{x}_{T-1}-\bar{x}_{<0>}\right]\\
&+H_{<0>}\left[\widetilde{x}_{T-2}-\bar{x}_{<0>}\right]+M_{<0>}\left[\widetilde{r}_{T-1}-\bar{r}_{<0>}\right]=0
\end{aligned}$$

Collecting terms gives:

$$\begin{aligned}
&\left[F_{<0>}\left(B_T+(1-\mathbf{p}_T)\,\mathcal{B}_T^*\right)+G_{<0>}\right]\widetilde{x}_{T-1}+F_{<0>}\left[C_T+(1-\mathbf{p}_T)\,\mathcal{A}_T^*\right]\\
&-\left[F_{<0>}+G_{<0>}+H_{<0>}\right]\bar{x}_{<0>}-M_{<0>}\bar{r}_{<0>}\\
&+H_{<0>}\widetilde{x}_{T-2}+M_{<0>}\widetilde{r}_{T-1}=0
\end{aligned}$$

which can be written as:

$$\widetilde{x}_{T-1}=\mathcal{A}_{T-1}+\mathcal{B}_{T-1}\widetilde{x}_{T-2}+\mathcal{C}_{T-1}\widetilde{r}_{T-1}$$

where:

$$\begin{aligned}
\mathcal{A}_{T-1}&\equiv-\left[F_{<0>}\left(B_T+(1-\mathbf{p}_T)\,\mathcal{B}_T^*\right)+G_{<0>}\right]^{-1}\\
&\qquad\times\left[\begin{array}{l}F_{<0>}\left[C_T+(1-\mathbf{p}_T)\,\mathcal{A}_T^*\right]-M_{<0>}\bar{r}_{<0>}\\-\left[F_{<0>}+G_{<0>}+H_{<0>}\right]\bar{x}_{<0>}\end{array}\right]\\
\mathcal{B}_{T-1}&\equiv-\left[F_{<0>}\left(B_T+(1-\mathbf{p}_T)\,\mathcal{B}_T^*\right)+G_{<0>}\right]^{-1}H_{<0>}\\
\mathcal{C}_{T-1}&\equiv-\left[F_{<0>}\left(B_T+(1-\mathbf{p}_T)\,\mathcal{B}_T^*\right)+G_{<0>}\right]^{-1}M_{<0>}
\end{aligned}$$

The loss function is:

$$\mathcal{L}_{T-1}\equiv\sum_{i=0}^{\infty}\beta^i\left[\begin{array}{l}\left(x_{T-1+i}-\bar{x}\left(s_{T-1+i}\right)\right)'W\left(x_{T-1+i}-\bar{x}\left(s_{T-1+i}\right)\right)\\+\left(r_{T-1+i}-\bar{r}\left(s_{T-1+i}\right)\right)'Q\left(r_{T-1+i}-\bar{r}\left(s_{T-1+i}\right)\right)\end{array}\right]$$

which can be expanded to give:

$$\begin{aligned}\mathcal{L}_{T-1} =& \left(\widetilde{x}_{T-1} - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_{T-1} - \bar{x}_{<0>}\right) + \left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right) \\ &+ \sum_{j=1}^{K} p_{j,T} \sum_{i=1}^{\infty} \beta^i \left(x_{T-1+i}^{<j>} - \bar{x}_{<j>}\right)' W \left(x_{T-1+i}^{<j>} - \bar{x}_{<j>}\right) \\ &+ \sum_{j=1}^{K} p_{j,T} \sum_{i=1}^{\infty} \beta^i \left(r_{T-1+i}^{<j>} - \bar{r}_{<j>}\right)' Q \left(r_{T-1+i}^{<j>} - \bar{r}_{<j>}\right) \\ &+ \beta \left(1 - \mathbf{p}_T\right) \mathcal{L}_T\end{aligned}$$

The first three terms of the loss function are isomorphic to those in the loss function at date T, which means that:

$$\begin{aligned}\mathcal{L}_{T-1} =& \left(\widetilde{x}_{T-1} - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_{T-1} - \bar{x}_{<0>}\right) + \left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right) \\ &+ \beta \sum_{j=1}^{K} p_{j,T} \left(\widetilde{x}_{T-1} - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_{T-1} - \bar{x}_{<j>}\right) \\ &+ \beta \left(1 - \mathbf{p}_T\right) \mathcal{L}_T\end{aligned}$$

In comparison to period T, the only difference in the loss function is therefore the fact that the policymaker must account for outcomes in the event that exit does not occur in period T. The probability of non-exit in period T is $(1 - \mathbf{p}_T)$ and is strictly positive (given the assumed structure of uncertainty).

Incorporating optimal responses into the loss function gives:

$$\begin{aligned}\mathcal{L}_T =& \left(\widetilde{x}_T - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_T - \bar{x}_{<0>}\right) + \left(\widetilde{r}_T - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_T - \bar{r}_{<0>}\right) \\ &+ \beta \sum_{j=1}^{K} p_{j,T+1} \left(\widetilde{x}_T - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_T - \bar{x}_{<j>}\right) \\ =& \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<0>}\right)' W \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<0>}\right) \\ &+ \left(\mathcal{R}_T^* + \mathcal{D}_T^* \widetilde{x}_{T-1} - \bar{r}_{<0>}\right)' Q \left(\mathcal{R}_T^* + \mathcal{D}_T^* \widetilde{x}_{T-1} - \bar{r}_{<0>}\right) \\ &+ \beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<j>}\right)' V_{<j>} \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<j>}\right)\end{aligned}$$

With this expression in hand $\mathcal{L}_{T-1}$ can be differentiated with respect to $\widetilde{r}_{T-1}$, taking into account the mapping from $\widetilde{r}_{T-1}$ to $\widetilde{x}_{T-1}$ implied by the structural equations (that is $\widetilde{x}_{T-1} = \mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1}$). Taking the derivative and setting it equal to zero gives:

$$\begin{aligned}
0 =& \mathcal{C}_{T-1}' W \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1} - \bar{x}_{<0>}\right) + Q\left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right) \\
&+ \beta \sum_{j=1}^{K} p_{j,T} \mathcal{C}_{T-1}' V_{<j>} \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1} - \bar{x}_{<j>}\right) \\
&+ \beta \left(1 - \mathbf{p}_T\right) \begin{bmatrix} \left(\mathcal{B}_T^* \mathcal{C}_{T-1}\right)' W \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<0>}\right) \\ + \left(\mathcal{D}_T^* \mathcal{B}_T^* \mathcal{C}_{T-1}\right)' Q \left(\mathcal{R}_T^* + \mathcal{D}_T^* \widetilde{x}_{T-1} - \bar{r}_{<0>}\right) \\ + \beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}_T^* \mathcal{C}_{T-1}\right)' V_{<j>} \left(\mathcal{A}_T^* + \mathcal{B}_T^* \widetilde{x}_{T-1} - \bar{x}_{<j>}\right) \end{bmatrix}
\end{aligned}$$

The term $\widetilde{x}_{T-1}$ can be eliminated using $\widetilde{x}_{T-1} = \mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1}$ to give:

$$\begin{aligned}
0 =& \mathcal{C}_{T-1}' W \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1} - \bar{x}_{<0>}\right) + Q\left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right) \\
&+ \beta \sum_{j=1}^{K} p_{j,T} \mathcal{C}_{T-1}' V_{<j>} \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1} - \bar{x}_{<j>}\right) \\
&+ \beta \left(1 - \mathbf{p}_T\right) \\
&\times \begin{bmatrix} \left(\mathcal{B}_T^* \mathcal{C}_{T-1}\right)' W \left(\mathcal{A}_T^* + \mathcal{B}_T^* \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1}\right) - \bar{x}_{<0>}\right) \\ + \left(\mathcal{D}_T^* \mathcal{B}_T^* \mathcal{C}_{T-1}\right)' Q \begin{pmatrix} \mathcal{D}_T^* \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1}\right) \\ + \mathcal{R}_T^* - \bar{r}_{<0>} \end{pmatrix} \\ + \beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}_T^* \mathcal{C}_{T-1}\right)' V_{<j>} \\ \times \left(\mathcal{A}_T^* + \mathcal{B}_T^* \left(\mathcal{A}_{T-1} + \mathcal{B}_{T-1}\widetilde{x}_{T-2} + \mathcal{C}_{T-1}\widetilde{r}_{T-1}\right) - \bar{x}_{<j>}\right) \end{bmatrix}
\end{aligned}$$

Collecting terms delivers an optimal response function for the instrument:

$$\widetilde{r}_{T-1} = \mathcal{R}_{T-1}^* + \mathcal{D}_{T-1}^* \widetilde{x}_{T-2}$$

where

$$\mathcal{R}^*_{T-1} = \mathcal{F}_{T-1} \begin{bmatrix} \mathcal{C}'_{T-1} W \left(\mathcal{A}_{T-1} - \bar{x}_{<0>}\right) - Q\bar{r}_{<0>} \\ +\beta \sum_{j=1}^{K} p_{j,T} \mathcal{C}'_{T-1} V_{<j>} \left(\mathcal{A}_{T-1} - \bar{x}_{<j>}\right) \\ +\beta \left(1 - \mathbf{p}_T\right) \times \\ \begin{bmatrix} \left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' W \left(\mathcal{A}^*_T + \mathcal{B}^*_T \mathcal{A}_{T-1} - \bar{x}_{<0>}\right) \\ - \left(\mathcal{D}^*_T \mathcal{B}^*_T \mathcal{C}_{T-1}\right)' Q\bar{r}_{<0>} \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' V_{<j>} \left(\mathcal{A}^*_T + \mathcal{B}^*_T \mathcal{A}_{T-1} - \bar{x}_{<j>}\right) \end{bmatrix} \end{bmatrix}$$

$$\mathcal{D}^*_{T-1} = \mathcal{F}_{T-1} \times$$
$$\begin{bmatrix} \mathcal{C}'_{T-1} W \mathcal{B}_{T-1} + \beta \sum_{j=1}^{K} p_{j,T} \mathcal{C}'_{T-1} V_{<j>} \mathcal{B}_{T-1} \\ +\beta \left(1 - \mathbf{p}_T\right) \\ \times \left(\left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' W \mathcal{B}^*_T \mathcal{B}_{T-1} + \beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' V_{<j>} \mathcal{B}^*_T \mathcal{B}_{T-1} \right) \end{bmatrix}$$

$$\mathcal{F}_{T-1} = -\begin{bmatrix} Q + \mathcal{C}'_{T-1} W \mathcal{C}_{T-1} + \beta \sum_{j=1}^{K} p_{j,T} \mathcal{C}'_{T-1} V_{<j>} \mathcal{C}_{T-1} \\ +\beta \left(1 - \mathbf{p}_T\right) \times \\ \begin{bmatrix} \left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' W \mathcal{B}^*_T \mathcal{C}_{T-1} + \left(\mathcal{D}^*_T \mathcal{B}^*_T \mathcal{C}_{T-1}\right)' Q \mathcal{D}^*_T \mathcal{C}_{T-1} \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}^*_T \mathcal{C}_{T-1}\right)' V_{<j>} \mathcal{B}^*_T \mathcal{C}_{T-1} \end{bmatrix} \end{bmatrix}^{-1}$$

With the optimal response function for the instrument in hand, the optimal evolution of x is given by:

$$\widetilde{x}_{T-1} = \mathcal{A}^*_{T-1} + \mathcal{B}^*_{T-1} \widetilde{x}_{T-2}$$

where

$$\mathcal{A}^*_{T-1} \equiv \mathcal{A}_{T-1} + \mathcal{C}_{T-1} \mathcal{R}^*_{T-1}$$
$$\mathcal{B}^*_{T-1} \equiv \mathcal{B}_{T-1} + \mathcal{C}_{T-1} \mathcal{D}^*_{T-1}$$

using the same manipulations as in period T.

Period $T-2$

Eliminating expectations from the model equations in period $T-2$ gives:

$$\widetilde{x}_{T-2} = \mathcal{A}_{T-2} + \mathcal{B}_{T-2} \widetilde{x}_{T-3} + \mathcal{C}_{T-2} \widetilde{r}_{T-2}$$

where:

$$
\begin{aligned}
\mathcal{A}_{T-2} \equiv & -\left[F_{<0>}\left(B_{T-1}+\left(1-\mathbf{p}_{T-1}\right)\mathcal{B}_{T-1}^{*}\right)+G_{<0>}\right]^{-1} \\
& \times\left[\begin{array}{l}
F_{<0>}\left[C_{T-1}+\left(1-\mathbf{p}_{T-1}\right)\mathcal{A}_{T-1}^{*}\right]-M_{<0>}\bar{r}_{<0>} \\
-\left[F_{<0>}+G_{<0>}+H_{<0>}\right]\bar{x}_{<0>}
\end{array}\right] \\
\mathcal{B}_{T-2} \equiv & -\left[F_{<0>}\left(B_{T-1}+\left(1-\mathbf{p}_{T-1}\right)\mathcal{B}_{T-1}^{*}\right)+G_{<0>}\right]^{-1} H_{<0>} \\
\mathcal{C}_{T-2} \equiv & -\left[F_{<0>}\left(B_{T-1}+\left(1-\mathbf{p}_{T-1}\right)\mathcal{B}_{T-1}^{*}\right)+G_{<0>}\right]^{-1} M_{<0>}
\end{aligned}
$$

The loss function is:

$$
\begin{aligned}
\mathcal{L}_{T-2} = & \left(\widetilde{x}_{T-2}-\bar{x}_{<0>}\right)' W\left(\widetilde{x}_{T-2}-\bar{x}_{<0>}\right)+\left(\widetilde{r}_{T-2}-\bar{r}_{<0>}\right)' Q\left(\widetilde{r}_{T-2}-\bar{r}_{<0>}\right) \\
& +\beta \sum_{j=1}^{K} p_{j,T-1}\left(\widetilde{x}_{T-2}-\bar{x}_{<j>}\right)' V_{<j>}\left(\widetilde{x}_{T-2}-\bar{x}_{<j>}\right) \\
& +\beta\left(1-\mathbf{p}_{T-1}\right)\mathcal{L}_{T-1}
\end{aligned}
$$

Substituting for $\mathcal{L}_{T-1}$ and $\mathcal{L}_{T}$ and the optimal policy responses in future periods gives:

$$
\begin{aligned}
\mathcal{L}_{T-2} = & \left(\mathcal{A}_{T-2}+\mathcal{B}_{T-2}\widetilde{x}_{T-3}+\mathcal{C}_{T-2}\widetilde{r}_{T-2}-\bar{x}_{<0>}\right)' W \times \\
& \left(\mathcal{A}_{T-2}+\mathcal{B}_{T-2}\widetilde{x}_{T-3}+\mathcal{C}_{T-2}\widetilde{r}_{T-2}-\bar{x}_{<0>}\right) \\
& + \left(\widetilde{r}_{T-2}-\bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{T-2}-\bar{r}_{<0>}\right) \\
& + \beta \sum_{j=1}^{K} p_{j,T-1} \begin{bmatrix} \left(\mathcal{A}_{T-2}+\mathcal{B}_{T-2}\widetilde{x}_{T-3}+\mathcal{C}_{T-2}\widetilde{r}_{T-2}-\bar{x}_{<j>}\right)' \\ \times V_{<j>} \times \\ \left(\mathcal{A}_{T-2}+\mathcal{B}_{T-2}\widetilde{x}_{T-3}+\mathcal{C}_{T-2}\widetilde{r}_{T-2}-\bar{x}_{<j>}\right) \end{bmatrix} \\
& + \beta \left(1-\mathbf{p}_{T-1}\right) \begin{bmatrix} \left(\mathcal{A}^*_{T-1}+\mathcal{B}^*_{T-1}\widetilde{x}_{T-2}-\bar{x}_{<0>}\right)' \\ \times W \times \\ \left(\mathcal{A}^*_{T-1}+\mathcal{B}^*_{T-1}\widetilde{x}_{T-2}-\bar{x}_{<0>}\right) \\ + \left(\widetilde{r}_{T-1}-\bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{T-1}-\bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,T} \begin{bmatrix} \left(\mathcal{A}^*_{T-1}+\mathcal{B}^*_{T-1}\widetilde{x}_{T-2}-\bar{x}_{<j>}\right)' \\ \times V_{<j>} \times \\ \left(\mathcal{A}^*_{T-1}+\mathcal{B}^*_{T-1}\widetilde{x}_{T-2}-\bar{x}_{<j>}\right) \end{bmatrix} \end{bmatrix} \\
& + \beta^2 \left(1-\mathbf{p}_{T-1}\right)\left(1-\mathbf{p}_{T}\right) \times \\
& \begin{bmatrix} \left(\mathcal{A}^*_{T}+\mathcal{B}^*_{T}\widetilde{x}_{T-1}-\bar{x}_{<0>}\right)' W \left(\mathcal{A}^*_{T}+\mathcal{B}^*_{T}\widetilde{x}_{T-1}-\bar{x}_{<0>}\right) \\ + \left(\mathcal{R}^*_{T}+\mathcal{D}^*_{T}\widetilde{x}_{T-1}-\bar{r}_{<0>}\right)' Q \left(\mathcal{R}^*_{T}+\mathcal{D}^*_{T}\widetilde{x}_{T-1}-\bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \begin{bmatrix} \left(\mathcal{A}^*_{T}+\mathcal{B}^*_{T}\widetilde{x}_{T-1}-\bar{x}_{<j>}\right)' \\ \times V_{<j>} \times \\ \left(\mathcal{A}^*_{T}+\mathcal{B}^*_{T}\widetilde{x}_{T-1}-\bar{x}_{<j>}\right) \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

To differentiate this expression with respect to the instrument $\widetilde{r}_{T-2}$, note that the chain rule and the form of the optimal response functions implies that:

$$
\begin{aligned}
\frac{\partial \widetilde{x}_{T-2}}{\partial \widetilde{r}_{T-2}} &= \mathcal{C}'_{T-2} \\
\frac{\partial \widetilde{x}_{T-1}}{\partial \widetilde{r}_{T-2}} &= \left(\mathcal{B}^*_{T-1}\mathcal{C}_{T-2}\right)' \\
\frac{\partial \widetilde{r}_{T-1}}{\partial \widetilde{r}_{T-2}} &= \left(\mathcal{D}^*_{T-1}\mathcal{C}_{T-2}\right)'
\end{aligned}
$$

The first order condition with respect to $\widetilde{r}_{T-2}$ is:

$$
\begin{aligned}
0 =\ & \mathcal{C}_{T-2}' W \left(\mathcal{A}_{T-2} + \mathcal{B}_{T-2}\widetilde{x}_{T-3} + \mathcal{C}_{T-2}\widetilde{r}_{T-2} - \bar{x}_{<0>}\right) + Q\left(\widetilde{r}_{T-2} - \bar{r}_{<0>}\right) \\
& + \beta \sum_{j=1}^{K} p_{j,T-1}\mathcal{C}_{T-2}' V_{<j>}\left(\mathcal{A}_{T-2} + \mathcal{B}_{T-2}\widetilde{x}_{T-3} + \mathcal{C}_{T-2}\widetilde{r}_{T-2} - \bar{x}_{<j>}\right) \\
& + \beta\left(1 - \mathbf{p}_{T-1}\right) \\
& \times \left[\begin{array}{c}
\left(\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' W\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\widetilde{x}_{T-2} - \bar{x}_{<0>}\right) \\
+\left(\mathcal{D}_{T-1}^{*}\mathcal{C}_{T-2}\right)' Q\left(\widetilde{r}_{T-1} - \bar{r}_{<0>}\right) \\
+\beta\sum_{j=1}^{K} p_{j,T}\left(\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' V_{<j>}\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\widetilde{x}_{T-2} - \bar{x}_{<j>}\right)
\end{array}\right] \\
& + \beta^{2}\left(1 - \mathbf{p}_{T-1}\right)\left(1 - \mathbf{p}_{T}\right) \\
& \times \left[\begin{array}{c}
\left(\mathcal{B}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' W\left(\mathcal{A}_{T}^{*} + \mathcal{B}_{T}^{*}\widetilde{x}_{T-1} - \bar{x}_{<0>}\right) \\
+\left(\mathcal{D}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' Q\left(\mathcal{R}_{T}^{*} + \mathcal{D}_{T}^{*}\widetilde{x}_{T-1} - \bar{r}_{<0>}\right) \\
+\beta\sum_{j=1}^{K} p_{j,T+1}\left(\mathcal{B}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' V_{<j>}\left(\mathcal{A}_{T}^{*} + \mathcal{B}_{T}^{*}\widetilde{x}_{T-1} - \bar{x}_{<j>}\right)
\end{array}\right]
\end{aligned}
$$

Collecting terms gives

$$
r_{T-2} = \mathcal{R}_{T-2}^{*} + \mathcal{D}_{T-2}^{*} x_{T-3} \tag{3.109}
$$

where

$$
\begin{aligned}
\mathcal{R}_{T-2}^{*} =\ & \mathcal{F}_{T-2}\left[\begin{array}{c}
\mathcal{C}_{T-2}' W\left(\mathcal{A}_{T-2} - \bar{x}_{<0>}\right) - Q\bar{r}_{<0>} \\
+\beta\sum_{j=1}^{K} p_{j,T-1}\mathcal{C}_{T-2}' V_{<j>}\left(\mathcal{A}_{T-2} - \bar{x}_{<j>}\right)
\end{array}\right] \\
& + \beta\left(1 - \mathbf{p}_{T-1}\right)\mathcal{F}_{T-2} \\
& \times \left[\begin{array}{c}
\left(\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' W\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\mathcal{A}_{T-2} - \bar{x}_{<0>}\right) \\
+\left(\mathcal{D}_{T-1}^{*}\mathcal{C}_{T-2}\right)' Q\left(\mathcal{R}_{T-1}^{*} + \mathcal{D}_{T-1}^{*}\mathcal{A}_{T-2} - \bar{r}_{<0>}\right) \\
+\beta\sum_{j=1}^{K} p_{j,T}\left(\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' V_{<j>}\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\mathcal{A}_{T-2} - \bar{x}_{<j>}\right)
\end{array}\right] \\
& + \beta^{2}\left(1 - \mathbf{p}_{T-1}\right)\left(1 - \mathbf{p}_{T}\right)\mathcal{F}_{T-2} \\
& \times \left[\begin{array}{cc}
\multicolumn{2}{c}{\left(\mathcal{B}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' W\left(\mathcal{A}_{T}^{*} + \mathcal{B}_{T}^{*}\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\mathcal{A}_{T-2}\right) - \bar{x}_{<0>}\right)} \\
\multicolumn{2}{c}{+\left(\mathcal{D}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' Q\left(\mathcal{R}_{T}^{*} + \mathcal{D}_{T}^{*}\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\mathcal{A}_{T-2}\right) - \bar{r}_{<0>}\right)} \\
+\beta\sum_{j=1}^{K} p_{j,T+1} & \left[\begin{array}{c}
\left(\mathcal{B}_{T}^{*}\mathcal{B}_{T-1}^{*}\mathcal{C}_{T-2}\right)' \\
\times V_{<j>} \times \\
\left(\mathcal{A}_{T}^{*} + \mathcal{B}_{T}^{*}\left(\mathcal{A}_{T-1}^{*} + \mathcal{B}_{T-1}^{*}\mathcal{A}_{T-2}\right) - \bar{x}_{<j>}\right)
\end{array}\right]
\end{array}\right]
\end{aligned}
$$

$$\mathcal{D}^*_{T-2} = \mathcal{F}_{T-2} \left[\begin{array}{l} \mathcal{C}'_{T-2} W \mathcal{B}_{T-2} + \beta \sum_{j=1}^{K} p_{j,T-1} \mathcal{C}'_{T-2} V_{<j>} \mathcal{B}_{T-2} \\ +\beta \left(1 - \mathbf{p}_{T-1}\right) \\ \times \left[\begin{array}{c} \left(\mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' W \mathcal{B}^*_{T-1} \mathcal{B}_{T-2} \\ + \left(\mathcal{D}^*_{T-1} \mathcal{C}_{T-2}\right)' Q \mathcal{D}^*_{T-1} \mathcal{B}_{T-2} \\ +\beta \sum_{j=1}^{K} p_{j,T} \left(\mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' V_{<j>} \mathcal{B}^*_{T-1} \mathcal{B}_{T-2} \end{array} \right] \\ +\beta^2 \left(1 - \mathbf{p}_{T-1}\right) \left(1 - \mathbf{p}_{T}\right) \\ \times \left[\begin{array}{c} \left(\mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' W \mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{B}_{T-2} \\ + \left(\mathcal{D}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' Q \mathcal{D}^*_{T} \mathcal{B}^*_{T-1} \mathcal{B}_{T-2} \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' V_{<j>} \mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{B}_{T-2} \end{array} \right] \end{array} \right]$$

$$\mathcal{F}_{T-2} = - \left[\begin{array}{c} \mathcal{C}'_{T-2} W \mathcal{C}_{T-2} + Q + \beta \sum_{j=1}^{K} p_{j,T-1} \mathcal{C}'_{T-2} V_{<j>} \mathcal{C}_{T-2} \\ +\beta \left(1 - \mathbf{p}_{T-1}\right) \left[\begin{array}{c} \left(\mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' W \mathcal{B}^*_{T-1} \mathcal{C}_{T-2} \\ + \left(\mathcal{D}^*_{T-1} \mathcal{C}_{T-2}\right)' Q \mathcal{D}^*_{T-1} \mathcal{C}_{T-2} \\ +\beta \sum_{j=1}^{K} p_{j,T} \left(\mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' V_{<j>} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2} \end{array} \right] \\ +\beta^2 \left(1 - \mathbf{p}_{T-1}\right) \left(1 - \mathbf{p}_{T}\right) \\ \times \left[\begin{array}{c} \left(\mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' W \mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2} \\ + \left(\mathcal{D}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' Q \mathcal{D}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2} \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \left(\mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2}\right)' V_{<j>} \mathcal{B}^*_{T} \mathcal{B}^*_{T-1} \mathcal{C}_{T-2} \end{array} \right] \end{array} \right]^{-1}$$

Period $1 < t < T$

The preceding steps reveal that it is possible to specify an iterative scheme for computing the optimal response functions at an arbitrary period t. Note that the matrices characterizing optimal decisions in all future periods $s = t+1, \ldots, T$ $(\mathcal{A}^*_s, \mathcal{B}^*_s, \mathcal{D}^*_s, \mathcal{R}^*_s)$ will have already been computed in previous steps.

First, construct a variant of the model with expectations substituted out:

$$\widetilde{x}_t = \mathcal{A}_t + \mathcal{B}_t \widetilde{x}_{t-1} + \mathcal{C}_t \widetilde{r}_t$$

where:

$$
\begin{aligned}
\mathcal{A}_t &= -\left[F_{<0>}\left(B_{t+1} + (1-\mathbf{p}_{t+1})\,\mathcal{B}^*_{t+1}\right) + G_{<0>}\right]^{-1} \\
&\quad \times \left[\begin{array}{l} F_{<0>}\left[C_{t+1} + (1-\mathbf{p}_{t+1})\,\mathcal{A}^*_{t+1}\right] - M_{<0>}\bar{r}_{<0>} \\ -\left[F_{<0>} + G_{<0>} + H_{<0>}\right]\bar{x}_{<0>} \end{array}\right] \\
\mathcal{B}_t &= -\left[F_{<0>}\left(B_{t+1} + (1-\mathbf{p}_{t+1})\,\mathcal{B}^*_{t+1}\right) + G_{<0>}\right]^{-1} H_{<0>} \\
\mathcal{C}_t &= -\left[F_{<0>}\left(B_{t+1} + (1-\mathbf{p}_{t+1})\,\mathcal{B}^*_{t+1}\right) + G_{<0>}\right]^{-1} M_{<0>} \\
B_{t+1} &= \sum_{j=1}^{K} p_{j,t+1} B_{<j>} \\
C_{t+1} &= \sum_{j=1}^{K} p_{j,t+1}\left(\mathbb{I} - B_{<j>}\right)\bar{x}_{<j>} \\
\mathbf{p}_{t+1} &= \sum_{j=1}^{K} p_{j,t+1}
\end{aligned}
$$

With the aid of the following recursive variables:

$$
\begin{aligned}
\Theta_u &= \begin{cases} \mathcal{A}^*_u + \mathcal{B}^*_u\Theta_{u-1} & u > t \\ \mathcal{A}_u & u = t \end{cases} \\
\Gamma_u &= \begin{cases} \mathcal{B}^*_u\Gamma_{u-1} & u > t \\ \mathbb{I} & u = t \end{cases} \\
\mathcal{P}_u &= \begin{cases} \beta\,(1-\mathbf{p}_u)\,\mathcal{P}_{u-1} & u > t \\ 1 & u = t \end{cases}
\end{aligned}
$$

the optimal response function for the instrument(s) can be written as:

$$
\widetilde{r}_t = \mathcal{R}^*_t + \mathcal{D}^*_t\widetilde{x}_{t-1}
$$

where

$$\mathcal{R}_t^* = \mathcal{F}_t \left[\begin{array}{c} \mathcal{C}_t' W \left(\mathcal{A}_t - \bar{x}_{<0>}\right) - Q\bar{r}_{<0>} + \beta \sum_{j=1}^K p_{j,t+1} \mathcal{C}_t' V_{<j>} \left(\mathcal{A}_t - \bar{x}_{<j>}\right) \\ + \sum_{u=t+1}^T \mathcal{P}_u \left[\begin{array}{c} \left(\Gamma_u \mathcal{C}_t\right)' W \left(\Theta_u - \bar{x}_{<0>}\right) \\ + \left(\mathcal{D}_u^* \Gamma_{u-1} \mathcal{C}_t\right)' Q \left(\mathcal{R}_u^* + \mathcal{D}_u^* \Theta_{u-1} - \bar{r}_{<0>}\right) \\ + \beta \sum_{j=1}^K p_{j,u+1} \left(\Gamma_u \mathcal{C}_t\right)' V_{<j>} \left(\Theta_u - \bar{x}_{<j>}\right) \end{array} \right] \end{array} \right]$$

$$\mathcal{D}_t^* = \mathcal{F}_t \left[\begin{array}{c} \mathcal{C}_t' W \mathcal{B}_t + \beta \sum_{j=1}^K p_{j,t+1} \mathcal{C}_t' V_{<j>} \mathcal{B}_t \\ + \sum_{u=t+1}^T \mathcal{P}_u \left[\begin{array}{c} \left(\Gamma_u \mathcal{C}_t\right)' W \Gamma_u \mathcal{B}_t + \left(\mathcal{D}_u^* \Gamma_{u-1} \mathcal{C}_t\right)' Q \mathcal{D}_u^* \Gamma_{u-1} \mathcal{B}_t \\ + \beta \sum_{j=1}^K p_{j,u+1} \left(\Gamma_u \mathcal{C}_t\right)' V_{<j>} \Gamma_u \mathcal{B}_t \end{array} \right] \end{array} \right]$$

and

$$\mathcal{F}_t = - \left[\begin{array}{c} \mathcal{C}_t' W \mathcal{C}_t + Q + \beta \sum_{j=1}^K p_{j,t+1} \mathcal{C}_t' V_{<j>} \mathcal{C}_t \\ + \sum_{u=t+1}^T \mathcal{P}_u \left[\begin{array}{c} \left(\Gamma_u \mathcal{C}_t\right)' W \Gamma_u \mathcal{C}_t + \left(\mathcal{D}_u^* \Gamma_{u-1} \mathcal{C}_t\right)' Q \mathcal{D}_u^* \Gamma_{u-1} \mathcal{C}_t \\ + \beta \sum_{j=1}^K p_{j,u+1} \left(\Gamma_u \mathcal{C}_t\right)' V_{<j>} \Gamma_u \mathcal{C}_t \end{array} \right] \end{array} \right]^{-1}$$

With the optimal response function for the instrument in hand, the law of motion for $\widetilde{x}$ is:

$$\widetilde{x}_t = \mathcal{A}_t^* + \mathcal{B}_t^* \widetilde{x}_{t-1}$$

where

$$\mathcal{A}_t^* \equiv \mathcal{A}_t + \mathcal{C}_t \mathcal{R}_t^*$$
$$\mathcal{B}_t^* \equiv \mathcal{B}_t + \mathcal{C}_t \mathcal{D}_t^*$$

Once these optimal response matrices have been computed, the algorithm moves back to period $t-1$ and applies the same steps again.

Period 1

The general approach for computing the optimal responses in period 1 is the same as for subsequent periods. However, in this case the values of the contemporaneous shocks may be non-zero. Recall that all solutions are conditional on period 1 information. That information set contains the fact that a structural change does not occur in period 1 and the values of the shocks in period 1.

These considerations imply that the model equations can be written as:

$$\widetilde{x}_1 = \mathcal{A}_1 + \mathcal{B}_1\widetilde{x}_0 + \mathcal{C}_1\widetilde{r}_1 + \widetilde{\Psi} z_1$$

where

$$\widetilde{\Psi} = \left[F_{<0>}\left(B_{t+1} + (1-\mathbf{p}_{t+1})\,\mathcal{B}^*_{t+1}\right) + G_{<0>}\right]^{-1}\Psi_{<0>}$$

and that the optimal policy response function satisfies:

$$\widetilde{r}_1 = \mathcal{R}^*_1 + \mathcal{D}^*_1\widetilde{x}_0 + \Omega z_1$$

The shocks appear in the structural model equations in the same way as x_0 which implies that:

$$\Omega = \mathcal{F}_1 \left[\begin{array}{c} \mathcal{C}_1' W \widetilde{\Psi}_1 + \beta \sum_{j=1}^{K} p_{j,2} \mathcal{C}_1' V_{<j>} \widetilde{\Psi}_1 \\ + \sum_{u=2}^{T} \mathcal{P}_u \left[\begin{array}{c} (\Gamma_u \mathcal{C}_1)' W \Gamma_u \widetilde{\Psi}_1 + (\mathcal{D}^*_u \Gamma_{u-1} \mathcal{C}_1)' Q \mathcal{D}^*_u \Gamma_{u-1} \widetilde{\Psi}_1 \\ + \beta \sum_{j=1}^{K} p_{j,u+1} (\Gamma_u \mathcal{C}_1)' V_{<j>} \Gamma_u \widetilde{\Psi}_1 \end{array} \right] \end{array} \right]$$

This means that the optimal outcomes in period 1 are given by:

$$\widetilde{x}_1 = \mathcal{A}^*_1 + \mathcal{B}^*_1\widetilde{x}_0 + \Phi z_1$$

where

$$\Phi = \widetilde{\Psi} + \mathcal{C}_1\Omega$$

3.G.7 Computing expected paths

The procedure outlined above demonstrates how to compute the sequence of optimal feedback matrices $\{\mathcal{A}^*_t, \mathcal{B}^*_t, \mathcal{R}^*_t, \mathcal{D}^*_t\}_{t=1,\ldots,T}$ as well as the optimal responses to shocks in period 1: Ω and Φ.

With these matrices in hand the path of the economy along the ‘no exit’ case ($\{\widetilde{x}_t, \widetilde{r}_t\}_{t=1,\ldots,T}$) can be computed using the following recursions:

$$\widetilde{x}_t = \begin{cases} \mathcal{A}^*_1 + \mathcal{B}^*_1\widetilde{x}_0 + \Phi z_1, & t = 1 \\ \mathcal{A}^*_t + \mathcal{B}^*_t\widetilde{x}_{t-1}, & t = 2, \ldots, T \end{cases}$$

$$\widetilde{r}_t = \begin{cases} \mathcal{R}^*_1 + \mathcal{D}^*_1\widetilde{x}_0 + \Omega z_1, & t = 1 \\ \mathcal{R}^*_t + \mathcal{D}^*_t\widetilde{x}_{t-1}, & t = 2, \ldots, T \end{cases}$$

The trajectory associated with exit to state j at date $1 < t \leq T$ is given by:

$$\widetilde{x}_u = \begin{cases} \widetilde{x}_u, & u = 1, \ldots, t-1 \\ \bar{x}_{<j>} + B_{<j>}^{u-t+1} \left(\widetilde{x}_{u-1} - \bar{x}_{<j>}\right), & u = t, \ldots \end{cases}$$

$$\widetilde{r}_u = \begin{cases} \widetilde{r}_u, & u = 1, \ldots, t-1 \\ \bar{r}_{<j>} + D_{<j>} B_{<j>}^{u-t} \left(\widetilde{x}_{u-1} - \bar{x}_{<j>}\right), & u = t, \ldots \end{cases}$$

The probability of observing exit to state j in period t is given by:

$$q_{j,t} = p_{j,t} \prod_{u=2}^{t-1} (1 - \mathbf{p}_u)$$

The trajectories for exit at to each state at each date t can therefore be weighted by their respective probabilities to construct the outcomes expected conditional on the date 1 information set.

3.G.8 Loss function manipulations

This appendix details some loss function manipulations that work with discounted sums of future losses. The focus is cases in which exit to a particular state $j \in \{1, \ldots, K\}$ has occurred. Note that exit can only occur in a period t strictly greater than 1, since the solution is conditional on it being revealed that exit has not occurred in period 1. Moreover, the solution is conditioned on date-1 expectations, so the expected values of future shocks are zero.

As noted above, in this case optimal discretionary policy delivers the following laws of motion for the endogenous variables and instruments:

$$\begin{aligned} x_t - \bar{x}_{<j>} &= B_{<j>} \left[x_{t-1} - \bar{x}_{<j>}\right] \\ r_t - \bar{r}_{<j>} &= D_{<j>} \left[x_{t-1} - \bar{x}_{<j>}\right] \end{aligned}$$

for $t > 1$. Note again that these expressions embody the fact that $\mathbb{E}_1 z_t = 0$.

This implies that, for $i \geq 1$:

$$x_{t+i} - \bar{x}_{<j>} = B_{<j>}^{i} \left[x_t - \bar{x}_{<j>}\right]$$

Similarly,

$$r_{t+i} - \bar{r}_{<j>} = D_{<j>} B_{<j>}^{i-1} \left[x_t - \bar{x}_{<j>}\right]$$

Period T loss

The starting point is (3.108):

$$\mathcal{L}_T \equiv \left(\widetilde{x}_T - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_T - \bar{x}_{<0>}\right) + \left(\widetilde{r}_T - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_T - \bar{r}_{<0>}\right)$$
$$+ \sum_{j=1}^{K} p_{j,T+1} \sum_{i=1}^{\infty} \beta^i \left(x_{T+i}^{<j>} - \bar{x}_{<j>}\right)' W \left(x_{T+i}^{<j>} - \bar{x}_{<j>}\right)$$
$$+ \sum_{j=1}^{K} p_{j,T+1} \sum_{i=1}^{\infty} \beta^i \left(r_{T+i}^{<j>} - \bar{r}_{<j>}\right)' Q \left(r_{T+i}^{<j>} - \bar{r}_{<j>}\right)$$

Consider the component of the loss associated with exit to state $j \in \{1, \ldots, K\}$ in period $T + 1$.

This is given by.

$$
\begin{aligned}
\mathcal{L}_{T+1}^{<j>} &\equiv \sum_{i=1}^{\infty}\beta^i \begin{bmatrix} \left(x_{T+i}^{<j>}-\bar{x}_{<j>}\right)' W \left(x_{T+i}^{<j>}-\bar{x}_{<j>}\right) \\ +\left(r_{T+i}^{<j>}-\bar{r}_{<j>}\right)' Q \left(r_{T+i}^{<j>}-\bar{r}_{<j>}\right) \end{bmatrix} \\
&= \sum_{i=1}^{\infty}\beta^i \left(x_{T+i}^{<j>}-\bar{x}_{<j>}\right)' W \left(x_{T+i}^{<j>}-\bar{x}_{<j>}\right) \\
&\quad + \sum_{i=1}^{\infty}\beta^i \left(r_{T+i}^{<j>}-\bar{r}_{<j>}\right)' Q \left(r_{T+i}^{<j>}-\bar{r}_{<j>}\right) \\
&= \sum_{i=1}^{\infty}\beta^i \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right)' W \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right) \\
&\quad + \sum_{i=1}^{\infty}\beta^i \left(D_{<j>}B_{<j>}^{i-1}\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right)' Q \left(D_{<j>}B_{<j>}^{i-1}\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right) \\
&= \beta\sum_{i=0}^{\infty}\beta^i \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right)' \left[B_{<j>}'WB_{<j>}\right] \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right) \\
&\quad + \beta\sum_{i=0}^{\infty}\beta^i \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right)' \left[D_{<j>}'QD_{<j>}\right] \left(B_{<j>}^i\left[\widetilde{x}_T-\bar{x}_{<j>}\right]\right) \\
&= \beta\left[\widetilde{x}_T-\bar{x}_{<j>}\right]' \left(\sum_{i=0}^{\infty}\beta^i\left(B_{<j>}^i\right)'\left[B_{<j>}'WB_{<j>}\right]B_{<j>}^i\right)\left[\widetilde{x}_T-\bar{x}_{<j>}\right] \\
&\quad + \beta\left[\widetilde{x}_T-\bar{x}_{<j>}\right]' \left(\sum_{i=0}^{\infty}\beta^i\left(B_{<j>}^i\right)'\left[D_{<j>}'QD_{<j>}\right]B_{<j>}^i\right)\left[\widetilde{x}_T-\bar{x}_{<j>}\right] \\
&= \beta\left[\widetilde{x}_T-\bar{x}_{<j>}\right]' V_{<j>}\left[\widetilde{x}_T-\bar{x}_{<j>}\right]
\end{aligned}
$$

where

$$
V_{<j>} \equiv \sum_{i=0}^{\infty}\beta^i\left(B_{<j>}^i\right)'\left[B_{<j>}'WB_{<j>}\right]B_{<j>}^i + \sum_{i=0}^{\infty}\beta^i\left(B_{<j>}^i\right)'\left[D_{<j>}'QD_{<j>}\right]B_{<j>}^i
$$

The manipulations to derive the final expression are straightforward. The first equality splits the expression into two infinite sums. The second uses the laws of motion for $x_{T+i}^{<j>}$ and $r_{T+i}^{<j>}$ to express them in terms of $\widetilde{x}_T$. The third equality rewrites the expressions in terms of infinite geometric sums starting from $i=0$. The fourth equality collects terms and the final equality implicitly defines $V_{<j>}$.

Note that $V_{<j>}$ can be written as follows:

$$\begin{aligned}
V_{<j>} =& \sum_{i=0}^{\infty} \beta^i \left(B^i_{<j>}\right)' \left[B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>}\right] B^i_{<j>} \\
=& \left[B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>}\right] \\
& + \sum_{i=1}^{\infty} \beta^i \left(B^i_{<j>}\right)' \left[B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>}\right] B^i_{<j>} \\
=& \left[B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>}\right] \\
& + \beta B'_{<j>} \left\{ \sum_{i=0}^{\infty} \beta^i \left(B^i_{<j>}\right)' \left[B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>}\right] B^i_{<j>} \right\} B_{<j>} \\
=& B'_{<j>} W B_{<j>} + D'_{<j>} Q D_{<j>} + \beta B'_{<j>} V_{<j>} B_{<j>}
\end{aligned}$$

and a fixed point for $V_{<j>}$ can be found numerically, using a doubling algorithm.[59]

3.G.9 Internalizing the effects of endogenous probabilities

If the policymaker internalizes the effects of endogenous probabilities then the first order condition at date t is the sum of two terms:

$$\underbrace{\sum_{s=t}^{T} \frac{\partial \mathcal{L}_t}{\partial \widetilde{x}_{t+s}} \frac{\partial \widetilde{x}_{t+s}}{\partial r_t}\Bigg|_{p_{j,s}, j=1,\ldots,K, s=t,\ldots,T+1}}_{\text{Exogenous probabilities}} + \underbrace{\sum_{s=t}^{T+1} \sum_{j=1}^{K} \frac{\partial \mathcal{L}_t}{\partial p_{j,s}} \frac{\partial p_{j,s}}{\partial r_t}\Bigg|_{\widetilde{x}_s, s=t,\ldots,T+1}}_{\text{Probability effect}} = 0$$

When probabilities are treated as exogenous by the policymaker the second term is ignored. So the solution developed in the algorithm presented in Appendix 3.G.6 solves

$$\sum_{s=t}^{T} \frac{\partial \mathcal{L}_t}{\partial \widetilde{x}_{t+s}} \frac{\partial \widetilde{x}_{t+s}}{\partial r_t}\Bigg|_{p_{j,s}, j=1,\ldots,K, s=t,\ldots,T+1} = 0$$

In this Appendix, I show how the solution algorithm from Appendix 3.G.6 can be extended to include the effects of the second 'probability effect' term. To do so I

[59] Validity of the derivation requires that all eigenvalues of $B_{<j>}$ lie within the unit disk, so that the equilibrium under optimal discretion in each exit state is unique and non-explosive.

conjecture that the extended solution algorithm will generate a solution of the same *form* as that of Appendix 3.G.6, namely:

$$\begin{aligned}\widetilde{r}_t &= \mathcal{R}_t^* + \mathcal{D}_t^* \widetilde{x}_{t-1} \\ \widetilde{x}_t &= \mathcal{A}_t^* + \mathcal{B}_t^* \widetilde{x}_{t-1}\end{aligned}$$

but with different expressions for the coefficient matrices $\mathcal{R}_t^*, \mathcal{D}_t^*, \mathcal{A}_t^*, \mathcal{B}_t^*$. This conjecture is verified below and the updated solutions for the coefficient matrices are derived.

The first objective is to compute the derivative of the loss function with respect to the effect of the instrument on the switching probabilities, holding the path of the endogenous variables $\{\widetilde{x}\}_{s=t}^{T}$ fixed. The starting point is to characterize the loss function.

The loss function

From Appendix 3.G.6, the loss function at date t can be written as:

$$\begin{aligned}
\mathcal{L}_t =& \left(\widetilde{x}_t - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_t - \bar{x}_{<0>}\right) + \left(\widetilde{r}_t - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_t - \bar{r}_{<0>}\right) \\
&+ \beta \sum_{j=1}^{K} p_{j,t+1} \left(\widetilde{x}_t - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_t - \bar{x}_{<j>}\right) \\
&+ \beta \left(1 - \mathbf{p}_{t+1}\right) \left[\begin{array}{c} \left(\widetilde{x}_{t+1} - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_{t+1} - \bar{x}_{<0>}\right) \\ + \left(\widetilde{r}_{t+1} - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{t+1} - \bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,t+2} \left(\widetilde{x}_{t+1} - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_{t+1} - \bar{x}_{<j>}\right) \end{array} \right] \\
&+ \beta^2 \left(1 - \mathbf{p}_{t+1}\right) \left(1 - \mathbf{p}_{t+2}\right) \times \\
&\left[\begin{array}{c} \left(\widetilde{x}_{t+2} - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_{t+2} - \bar{x}_{<0>}\right) + \left(\widetilde{r}_{t+2} - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_{t+2} - \bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,T+3} \left(\widetilde{x}_{t+2} - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_{t+2} - \bar{x}_{<j>}\right) \end{array} \right] \\
&+ \ldots \\
&+ \beta^{T-t} \prod_{s=t+1}^{T} \left(1 - \mathbf{p}_s\right) \times \\
&\left[\begin{array}{c} \left(\widetilde{x}_T - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_T - \bar{x}_{<0>}\right) + \left(\widetilde{r}_T - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_T - \bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,T+1} \left(\widetilde{x}_T - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_T - \bar{x}_{<j>}\right) \end{array} \right]
\end{aligned}$$

Since the derivative holds $\{\widetilde{x}_s, \widetilde{r}_s\}_{s=t}^{T}$ fixed, the loss function can be simplified by defining:

$$\begin{aligned}
\mathcal{W}_t &\equiv \left(\widetilde{x}_t - \bar{x}_{<0>}\right)' W \left(\widetilde{x}_t - \bar{x}_{<0>}\right) + \left(\widetilde{r}_t - \bar{r}_{<0>}\right)' Q \left(\widetilde{r}_t - \bar{r}_{<0>}\right) \\
\mathcal{V}_{j,t} &\equiv \left(\widetilde{x}_t - \bar{x}_{<j>}\right)' V_{<j>} \left(\widetilde{x}_t - \bar{x}_{<j>}\right)
\end{aligned}$$

Using these definitions gives:

$$\begin{aligned}
\mathcal{L}_t = {} & \mathcal{W}_t + \beta \sum_{j=1}^{K} p_{j,t+1} \mathcal{V}_{j,t} + \beta \left(1 - \mathbf{p}_{t+1}\right) \left[\mathcal{W}_{t+1} + \beta \sum_{j=1}^{K} p_{j,t+2} \mathcal{V}_{j,t+1} \right] \\
& + \beta^2 \left(1 - \mathbf{p}_{t+1}\right)\left(1 - \mathbf{p}_{t+2}\right) \left[\mathcal{W}_{t+2} + \beta \sum_{j=1}^{K} p_{j,T+3} \mathcal{V}_{j,t+2} \right] \\
& + \dots \\
& + \beta^{T-t} \prod_{s=t+1}^{T} \left(1 - \mathbf{p}_s\right) \left[\mathcal{W}_T + \beta \sum_{j=1}^{K} p_{j,T+1} \mathcal{V}_{j,T} \right] \\
= {} & \mathcal{W}_t + \beta \sum_{j=1}^{K} p_{j,t+1} \mathcal{V}_{j,t} + \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+s} \left[\mathcal{W}_{t+i} + \beta \sum_{j=1}^{K} p_{j,t+i+1} \mathcal{V}_{j,t+i} \right]
\end{aligned}$$

where the definition $\mathbf{q}_{t+s} \equiv \left(1 - \mathbf{p}_{t+s}\right)$ is used to simplify notation.

3.G.10 Derivatives

The derivative capturing the 'probability effect' is:

$$\begin{aligned}
\mathcal{T}_t \equiv \left.\frac{\partial \mathcal{L}_t}{\partial r_t}\right|_{\{\widetilde{x}_s\}_{s=t}^{T}} = {} & \beta \sum_{j=1}^{K} \frac{\partial p_{j,t+1}}{\partial r_t} \mathcal{V}_{j,t} \\
& + \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \frac{\partial \mathbf{q}_{t+s}}{\partial r_t} \left[\mathcal{W}_{t+i} + \beta \sum_{j=1}^{K} p_{j,t+i+1} \mathcal{V}_{j,t+i} \right] \\
& + \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+s} \beta \sum_{j=1}^{K} \frac{\partial p_{j,t+i+1}}{\partial r_t} \mathcal{V}_{j,t+i}
\end{aligned}$$

which can be written as:

$$\mathcal{T}_t = \beta \sum_{j=1}^{K} \frac{\partial p_{j,t+1}}{\partial r_t} \mathcal{V}_{j,t} + \hat{\mathcal{T}}_t + \beta \sum_{j=1}^{K} \widetilde{\mathcal{T}}_{j,t}$$

where

$$\hat{\mathcal{T}}_t \equiv \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \frac{\partial \mathbf{q}_{t+s}}{\partial r_t} \left[\mathcal{W}_{t+i} + \beta \sum_{j=1}^{K} p_{j,t+i+1} \mathcal{V}_{j,t+i} \right]$$

$$\widetilde{\mathcal{T}}_{j,t} \equiv \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+s} \frac{\partial p_{j,t+i+1}}{\partial r_t} \mathcal{V}_{j,t+i}$$

These components can each be written recursively, since

$$\begin{aligned}
\hat{\mathcal{T}}_t &= \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \frac{\partial \mathbf{q}_{t+s}}{\partial r_t} \left[\mathcal{W}_{t+i} + \beta \sum_{j=1}^{K} p_{j,t+i+1} \mathcal{V}_{j,t+i} \right] \\
&= \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \left[\mathcal{W}_{t+1} + \beta \sum_{j=1}^{K} p_{j,t+2} \mathcal{V}_{j,t+1} \right] \\
&\quad + \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \sum_{i=1}^{T-(t+1)} \beta^i \prod_{s=1}^{i} \frac{\partial \mathbf{q}_{t+1+s}}{\partial r_t} \left[\mathcal{W}_{t+1+i} + \beta \sum_{j=1}^{K} p_{j,t+1+i+1} \mathcal{V}_{j,t+1+i} \right] \\
&= \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \left[\mathcal{W}_{t+1} + \beta \sum_{j=1}^{K} p_{j,t+2} \mathcal{V}_{j,t+1} \right] \\
&\quad + \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \frac{\partial r_{t+1}}{\partial r_t} \sum_{i=1}^{T-(t+1)} \beta^i \prod_{s=1}^{i} \frac{\partial \mathbf{q}_{t+1+s}}{\partial r_{t+1}} \left[\mathcal{W}_{t+1+i} + \beta \sum_{j=1}^{K} p_{j,t+1+i+1} \mathcal{V}_{j,t+1+i} \right] \\
&= \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \left[\mathcal{W}_{t+1} + \beta \sum_{j=1}^{K} p_{j,t+2} \mathcal{V}_{j,t+1} \right] + \beta \frac{\partial \mathbf{q}_{t+1}}{\partial r_t} \frac{\partial r_{t+1}}{\partial r_t} \hat{\mathcal{T}}_{t+1}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{T}}_{j,t} &= \sum_{i=1}^{T-t} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+s} \frac{\partial p_{j,t+i+1}}{\partial r_t} \mathcal{V}_{j,t+i} \\
&= \beta \mathbf{q}_{t+1} \frac{\partial p_{j,t+1}}{\partial r_t} \mathcal{V}_{j,t+1} + \beta \mathbf{q}_{t+1} \sum_{i=1}^{T-(t+1)} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+1+s} \frac{\partial p_{j,t+1+i+1}}{\partial r_t} \mathcal{V}_{j,t+1+i} \\
&= \beta \mathbf{q}_{t+1} \frac{\partial p_{j,t+1}}{\partial r_t} \mathcal{V}_{j,t+1} + \beta \mathbf{q}_{t+1} \frac{\partial r_{t+1}}{\partial r_t} \sum_{i=1}^{T-(t+1)} \beta^i \prod_{s=1}^{i} \mathbf{q}_{t+1+s} \frac{\partial p_{j,t+1+i+1}}{\partial r_{t+1}} \mathcal{V}_{j,t+1+i} \\
&= \beta \mathbf{q}_{t+1} \frac{\partial p_{j,t+1}}{\partial r_t} \mathcal{V}_{j,t+1} + \beta \mathbf{q}_{t+1} \frac{\partial r_{t+1}}{\partial r_t} \widetilde{\mathcal{T}}_{j,t+1}
\end{aligned}$$

To compute the derivatives, first note that the functions mapping endogenous outcomes to probabilities can be written as:

$$p_{j,t+1} = f_j\left(\mathcal{S}\widetilde{x}_t\right)$$

where $\mathcal{S}$ is a $1 \times n_{\widetilde{x}}$ vector that extracts a linear combination of endogenous variables. In the experiment considered in Section 3.6.6, for example, it simply selects the element of $\widetilde{x}$ corresponding to the primary surplus.

This means that

$$\frac{\partial p_{j,t+1}}{\partial r_t} = f_j'\left(\mathcal{S}\widetilde{x}_t\right)\mathcal{S}\frac{\partial \widetilde{x}_t}{\partial r_t} = f_j'\left(\mathcal{S}\widetilde{x}_t\right)\left(\mathcal{S}\mathcal{C}_t\right)'$$

which embodies the equilibrium mapping between $\widetilde{x}_t$ and r_t in the same way as the derivation in Appendix 3.G.6.

As in the derivation in Appendix 3.G.6, the best response functions can be used to show that:

$$\frac{\partial r_{t+1}}{\partial r_t} = \left(\mathcal{D}^*_{t+1}\mathcal{C}_t\right)'$$

These results mean that the recursive solutions can be written as:

$$\begin{aligned}
\hat{\mathcal{T}}_t &= \beta\frac{\partial \mathbf{q}_{t+1}}{\partial r_t}\left[\mathcal{W}_{t+1} + \beta\sum_{j=1}^{K} p_{j,t+2}\mathcal{V}_{j,t+1}\right] + \beta\frac{\partial \mathbf{q}_{t+1}}{\partial r_t}\left(\mathcal{D}^*_{t+1}\mathcal{C}_t\right)'\hat{\mathcal{T}}_{t+1} \\
\widetilde{\mathcal{T}}_{j,t} &= \beta\mathbf{q}_{t+1}f_j'\left(\mathcal{S}\widetilde{x}_t\right)\left(\mathcal{S}\mathcal{C}_t\right)'\mathcal{V}_{j,t+1} + \beta\mathbf{q}_{t+1}\left(\mathcal{D}^*_{t+1}\mathcal{C}_t\right)'\widetilde{\mathcal{T}}_{j,t+1}
\end{aligned}$$

where all of the terms on the right hand sides of these expressions can be pre-computed, including

$$\frac{\partial \mathbf{q}_{t+1}}{\partial r_t} = -\sum_{j=1}^{K}\frac{\partial p_{j,t+1}}{\partial r_t} = -\sum_{j=1}^{K} f_j'\left(\mathcal{S}\widetilde{x}_t\right)\left(\mathcal{S}\mathcal{C}_t\right)'$$

Similarly, $\mathcal{T}_t$ is given by:

$$\mathcal{T}_t = \hat{\mathcal{T}}_t + \beta\sum_{j=1}^{K}\left[f_j'\left(\mathcal{S}\widetilde{x}_t\right)\left(\mathcal{S}\mathcal{C}_t\right)'\mathcal{V}_{j,t} + \widetilde{\mathcal{T}}_{j,t}\right]$$

The steps above show that it is possible to compute $\mathcal{T}_t$ by a backward recursion starting from $\mathcal{T}_T$. Since all uncertainty is resolved in period $T+1$, $\mathbf{q}_{T+1} = 0$, which implies that $\mathcal{T}_{j,T} = 0, j = 1, \ldots, K$. The same observation implies that $\frac{\partial \mathbf{q}_{T+1}}{\partial r_T} = 0$, so that $\hat{\mathcal{T}}_T = 0$.[60]

3.G.11 Adjusting the solution algorithm to incorporate $\mathcal{T}_t$

The previous steps have shown how to compute $\mathcal{T}_t$ recursively using expressions that can be easily evaluated, conditional on sequences for $\{\widetilde{x}_t, \widetilde{r}_t\}_{t=1}^T$, probabilities $\{p_{j,t}\}_{t=1}^T, j = 1, \ldots, K$ and solution matrices $\{\mathcal{D}_t^*, \mathcal{C}_t\}_{t=1}^T$.

The solution algorithm in Appendix 3.G.6 solves for optimal response functions by setting the 'exogenous probabilities' derivative to zero, ignoring the probability effect ($\mathcal{T}_t$). The effect of the probability effect can be incorporated by adjusting the first order condition as follows:

$$\sum_{s=t}^{T} \frac{\partial \mathcal{L}_t}{\partial \widetilde{x}_{t+s}} \frac{\partial \widetilde{x}_{t+s}}{\partial r_t}\bigg|_{p_{j,s}, j=1,\ldots,K, s=t,\ldots,T+1} + \mathcal{T}_t = 0$$

Appendix 3.G.6 derives the first term on the left hand side, so the probability effect can be incorporated by setting that expression plus $\mathcal{T}_t$ equal to zero. Inspection of the first order conditions in Appendix 3.G.6 reveals that doing this leads to $\mathcal{T}_t$ being incorporated into the 'levels' adjustment coefficient $\mathcal{R}_t^*$. The intuition for this is that the probability effect is computed conditional on the future solution trajectory and is therefore not a function of $\widetilde{x}_{t-1}$ (which is also the case for the other components of $\mathcal{R}_t^*$).

Specifically, to incorporate the probability effect, the solution for $\mathcal{R}_t^*$ must be

[60]The fact that $\mathbf{q}_{T+1} = 0$ means that the sum of the marginal effects of r_T on the probabilities $p_{j,T+1}$ must be zero to ensure that $\mathbf{q}_{T+1} = 0$ is satisfied for all possible realizations of r_T.

adjusted to:

$$\mathcal{R}_t^* = \mathcal{F}_t \left[\begin{array}{c} \mathcal{T}_t + \mathcal{C}_t' W \left(\mathcal{A}_t - \bar{x}_{<0>}\right) - Q\bar{r}_{<0>} \\ +\beta \sum_{j=1}^{K} p_{j,t+1} \mathcal{C}_t' V_{<j>} \left(\mathcal{A}_t - \bar{x}_{<j>}\right) \\ + \sum_{u=t+1}^{T} \mathcal{P}_u \times \\ \left(\begin{array}{c} \left(\Gamma_u \mathcal{C}_t\right)' W \left(\Theta_u - \bar{x}_{<0>}\right) \\ + \left(\mathcal{D}_u^* \Gamma_{u-1} \mathcal{C}_t\right)' Q \left(\mathcal{R}_u^* + \mathcal{D}_u^* \Theta_{u-1} - \bar{r}_{<0>}\right) \\ +\beta \sum_{j=1}^{K} p_{j,u+1} \left(\Gamma_u \mathcal{C}_t\right)' V_{<j>} \left(\Theta_u - \bar{x}_{<j>}\right) \end{array} \right) \end{array} \right] \tag{3.110}$$

3.G.12 The modified solution algorithm

The preceding steps have shown how to characterize the solution in which the policymaker internalizes the effects of their actions on future switching probabilities. However, the non-linearity of the relationships required to compute $\mathcal{T}_t$ complicate simultaneous solution of equation (3.110) and $\mathcal{T}_t$. That is, equation (3.110) involves non-linear functions of the solution trajectories $\{\widetilde{x}_t, \widetilde{r}_t\}_{t=1}^{T}$ which are themselves functions of $\mathcal{R}_t^*$.

For this reason an iterative approach is adopted. The algorithm as follows:

1. Initialize a guess for $\{\mathcal{T}_t\}_{t=1}^{T}$ and probabilities $\{p_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, K$.
2. Solve for best response matrices $\{\mathcal{R}_t^*, \mathcal{D}_t^*, \mathcal{A}_t^*, \mathcal{B}_t^*\}_{t=1}^{T}$ using the algorithm in Appendix 3.G.6, but with $\mathcal{R}_t^*$ computed using (3.110).
3. Compute the solution trajectories $\{\widetilde{x}_t, \widetilde{r}_t\}_{t=1}^{T}$ using the method in Appendix 3.G.7.
4. Update the guess for $\{\mathcal{T}_t\}_{t=1}^{T}$ using the steps in Appendix 3.G.10.
5. Update the sequence of probabilities, $\{p_{j,t} = f_j\left(\mathcal{S}\widetilde{x}_{t-1}\right)\}_{t=1}^{T}$, $j = 1, \ldots, K$.
6. If the guesses for $\{\mathcal{T}_t\}_{t=1}^{T}$ and probabilities $\{p_{j,t}\}_{t=1}^{T}$, $j = 1, \ldots, K$ are sufficiently close to the previous guesses, stop. Otherwise return to step 2.

In practice, the feedback between the updating the probabilities $\{p_{j,t}\}_{t=1}^{T}$ and $\{\mathcal{T}_t\}_{t=1}^{T}$ can be powerful, so applying damping to the updating steps may be required.

3.G.13 Expected loss computations

The loss function manipulations in Appendix 3.G.9 can be used to construct a recursive representation for the expected loss in period t conditional on not having switched to the active fiscal rule by that date.

Let this loss be denoted $\widetilde{\mathcal{L}}_t$. The results from Appendix 3.G.9 reveal that:

$$\widetilde{\mathcal{L}}_t = \mathcal{W}_t + \beta \sum_{j=1}^{K} p_{j,t+1} \mathcal{V}_{j,t} + \beta \left(1 - \mathbf{p}_{t+1}\right) \widetilde{\mathcal{L}}_{t+1}$$

which can be computed as a backward recursion from T noting that

$$\widetilde{\mathcal{L}}_{T+1} = \sum_{j=1}^{K} p_{j,T+1} \mathcal{V}_{j,T}$$

Note that, since there is no uncertainty over the fiscal regime in period 1, $\widetilde{\mathcal{L}}_1$ represents the unconditional expected loss in period 1.

3.G.14 Simulation to study time consistency

This Appendix provides details of the simulation used to explore the effects of time inconsistency in Section 3.6.7.

I use the same notation as above. However, to simplify the exposition, I ignore the '$< 0 >$' subscript on the model solution matrices for the baseline model (with passive fiscal policy). Since the steady states $\bar{x}$ and $\bar{r}$ are the same for all model variants, these are normalized to zero.

The model structure is given by:

$$F \mathbb{E}_t x_{t+1} + G x_t + H x_{t-1} + M r_t = \Psi z_t$$

In period 1, we have:

$$F \mathbb{E}_1 x_2 + G x_1 + H x_0 + M r_1 = 0$$

since there are no shocks in period 1.

If in period 1 the policymaker sets the policy rate at the level that would be appropriate if all agents ignored the risk, then $r_1 = 0$. If we further suppose that the private sector recognizes that future policymakers will be have in a time consistent manner, then expectations are given by:

$$\mathbb{E}_1 x_2 = \sum_{j=1}^{K} p_{j,2} B_{<j>} x_1 + (1 - \mathbf{p}_{t+1}) \widetilde{x}_2$$

Putting these observations together implies that:

$$F \sum_{j=1}^{K} p_{j,2} B_{<j>} x_1 + (1 - \mathbf{p}_{t+1}) F \widetilde{x}_2 + G x_1 + H x_0 = 0$$

so that:

$$x_1 = - \left[G + F \sum_{j=1}^{K} p_{j,2} B_{<j>} \right]^{-1} \left[(1 - \mathbf{p}_2) F \widetilde{x}_2 + H x_0 \right] \tag{3.111}$$

The solution for $\widetilde{x}_2$ is obtained by using the solution algorithm for time-consistent monetary policy, with initial conditions given by x_1. Thus, since $\widetilde{x}_2$ depends on x_1, an iterative procedure is required.

The solution algorithm therefore follows these steps:

1. Construct a guess for x_1.
2. Solve for $\widetilde{x}_t, t = 2, \ldots, T$ using the algorithm in Appendix 3.G.11, with initial conditions given by x_1.
3. Update the guess for x_1 by:
   a) Computing $\{p_{j,2}\}_{j=1}^{K}$ using the endogenous mapping from x_1 to a probability of a switch to active fiscal policy in period 2.
   b) Solve for x_1 using (3.111).
4. If the latest guess for x_1 is sufficiently close to the previous guess, stop. Otherwise return to Step 2.

Conclusion

This thesis has explored a number of questions regarding the effects of monetary and fiscal policies, and their optimal conduct, in the presence of a lower bound on the short-term interest rate. Here I bring together the key contributions of my work, consider the implications for policy and discuss possible extensions that may be considered in future work.

Key contributions

My research has uncovered several new results.

Chapter 1 contains the first analysis of optimal quantitative easing (QE) in a stochastic model containing the portfolio balance mechanism, through which most monetary policymakers believe QE works. The predictions of the model broadly match the actual behavior of monetary policymakers in the United States and the United Kingdom in response to the financial crisis.

However, the model predicts that QE should start to be unwound before the short-term policy rate rises from the lower bound. In practice, policymakers have preferred to delay QE unwind until after the short-term interest rate has risen above the lower bound. Possible reasons for this discrepancy are discussed in Chapter 1 and prompt some ideas for further research, discussed below.

Chapter 2 demonstrates that recent findings that money-financed government spending has large macroeconomic effects (Galí, 2014a) are driven by the implications of monetary financing for the implicit rule for the short-term nominal interest rate, rather than the nature of financing *per se*. While other research (English et al., 2017, undertaken concurrently with Chapter 2) finds the same result, I further demonstrate that a rule that finances government spending using money creation may perform very poorly in response to other (non-fiscal) shocks.

These insights prompt an analysis of money-financed transfers in a setting that breaks the conventional duality between interest rate rules and money supply rules. To my knowledge, this is the first attempt to analyze monetary financing in a framework in which monetary transfers can be regarded as an independent policy instrument. That is achieved by assuming that money may earn interest and that a simple financial friction causes households to regard government liabilities as net wealth. In this setting, simulations suggest that money-financed transfers could have sizable expansionary effects at the zero bound. However, the scale of the transfer required to deliver a substantial increase in spending may need to be extremely large. Moreover, robustness analysis shows that the responses to money-financed transfers are highly sensitive to the nature of the frictions giving rise to a demand for money.

Chapter 3 contains several insights into the interplay between fiscal policy and optimal time-consistent monetary policy. The analytical results deepen our understanding of the 'debt stabilization bias' (Leith and Wren-Lewis, 2013; Leeper and Leith, 2016). While similar results have been found in concurrent research using richer models (for example Leeper et al., 2019), the simplicity of the model used in Chapter 3 highlights the crucial role of the duration of government debt.

Another insight is that the constraints placed on monetary policy behavior by active fiscal policy may deliver welfare benefits in some situations. In particular, active fiscal policy generates the expectation of higher future inflation following a recessionary shock that raises the real value of debt. This effect can help to stabilize the economy when it encounters the zero bound. If the duration of government debt is long enough, the improved stabilization at the zero bound may offset the welfare costs of imperfect stabilization in normal times, a new and striking result.

Chapter 3 also develops a new solution algorithm to analyze the implications of a risk of a permanent switch in fiscal policy behavior. The results of that analysis suggest that a time-consistent monetary policymaker may optimally accommodate the inflationary effects of the risk of a switch to active fiscal policy. More broadly, the solution algorithm can be easily applied to a range of other applications in which policymakers face the risk of a permanent (or highly persistent) structural change (for example, changes in trade policies, the arrival of a financial crisis).

Policy implications

Even though the models used in this thesis are stylized, the results provide some potential lessons for policymakers. At the very least, they raise possible questions for further research (some of which are discussed below).

The results of Chapter 1 suggest that policymakers should pay close attention to the evolution of the government debt stock (and its maturity structure) if QE operates through a portfolio balance mechanism. Changes in government debt issuance and maturity structure could have at least as much influence on the monetary policy stance as asset purchases or sales, as Figure 1.1 suggests may have been the case in the United Kingdom.

Moreover if the 'flow effects' of QE are as powerful as estimated in Chapter 1, it is important to unwind QE in good time for the next encounter with the zero bound, in order to restore firepower. In particular, the model suggests that permanently holding a sizable portion of long term debt on the central bank balance sheet will hamper the ability to respond to future zero bound episodes and may therefore reduce welfare.

The results of Chapter 2 suggest that macroeconomic policymakers were probably right not to use monetary financing as a stimulus measure in the aftermath of the financial crisis. According to the model I use, generating a moderate amount of stimulus would have required unprecedentedly large transfers to households. Moreover, the results are sensitive to alternative specifications of the frictions that give rise to a demand for money. Overall, the results do not suggest that there is any particular benefit to monetary financing relative to debt financing (which is better understood).

The results of Chapter 3 show that monetary policymakers should pay attention to government debt dynamics and debt sustainability. Even a relatively small risk of a move to active fiscal policy could have an economically meaningful effect on longer-term inflation expectations. Such movements in inflation expectations will, in general, make it more difficult for a monetary policymaker to achieve a flexible inflation targeting mandate.

More broadly, the finding that active fiscal policy may help monetary policymakers to stabilize the economy at the zero bound has implications for the design of macroeconomic policy frameworks. In particular, policymakers may wish to consider the extent to which policy frameworks can be designed to incorporate explicit shifts in the roles of fiscal and monetary policy behaviors in the vicinity of the zero lower bound.

Avenues for future work

The investigations in Chapters 1–3 adopt a common approach. In each case I start from a simple, textbook, model and introduce the minimal additional features required to study the question of interest. This approach is motivated by the fact that many key results in monetary economics are based on the textbook model, presented by Clarida, Galí, and Gertler (1999) as the foundation of the 'science' of monetary policy. It helps to isolate the effects of additional features and compare the results with a well-known benchmark. In some cases, it permits the derivation of analytical results.

It would, however, be interesting to explore the robustness of my findings in richer settings. Some of the most promising extensions are discussed below.

Even a decade after the introduction of quantitative easing, Bernanke's remark that QE works in practice but not in theory has some force. The results of Chapter 1 depend on the particular transmission mechanism through which QE is assumed to operate. It would be useful to compare the optimal policy prescriptions with models containing alternative transmission mechanisms through which QE may operate. Some results based on alternative theories of QE are indeed starting to emerge (for example Bhattarai et al., 2015; Carlstrom et al., 2017; Cui and Sterk, 2018).

More ambitiously, it would be instructive to analyze optimal policy in the presence of uncertainty over the transmission channel of QE. This uncertainty was pervasive at the inception of the policy (Benford et al., 2009). A particular aspect of the uncertainty relates to the observation that the effects of QE may be state contingent, in particular that the effects on asset prices (and hence the macroeconomy) may depend on the extent to which financial markets are disrupted (Joyce et al., 2011, Figure 8). Incorporating such effects would be a valuable addition to the literature.

As noted in Section 2.6.1, the analysis of Chapter 2 abstracts from banks and the banking system more generally. Interpreting the results in terms of the implications for a financial system underpinned by interest-bearing reserves would be a useful extension, requiring a more fully articulated treatment of the financial system.

The results of Chapter 2 also imply that there is no particular reason to favor monetary financing over debt financing. A richer assessment of this issue might usefully incorporate portfolio frictions along the lines of Chapter 1. Such a model would provide a richer framework for a cost-benefit assessment of alternative monetary policy tools.

As observed in Section 3.3.7, the duration of UK government debt is much longer than other advanced economies. One may be tempted to infer that the implications of active fiscal policy in the United Kingdom could be approximated by the 'long duration' variant of the model studied in Chapter 3. However, that model abstracts from two other important features of the UK economy: a flexible exchange rate and the substantial fraction of government debt that is index linked. The first feature provides a mechanism through which price level movements can be achieved relatively rapidly through large movements in the exchange rate. The second implies that a portion of the government debt stock cannot be revalued by changes in the path of the price level. Assessing the net effects of these competing channels in a richer model would be an interesting exercise.

Combining elements of the analysis of fiscal risks and the presence of the zero lower bound may produce some important insights. This would permit an analysis of whether it would be beneficial for fiscal policy to become *temporarily* active when the economy encounters the lower bound. If the expected duration of the active fiscal policy regime was long enough, it may raise inflation expectations enough to mitigate the deflationary effects of the lower bound. Moreover, if fiscal policy was ultimately expected to switch back to a passive regime, output and inflation may also be better stabilized when away from the zero bound.

In general, it would be interesting to explore how the conclusions of my research change when a richer menu of fiscal policy options is available. Studying jointly optimal monetary and fiscal policies could be productive in several ways. The portfolio balance mechanism embedded in the model of Chapter 1 implies a key role for the government debt stock (and maturity structure) in determining the policy stance. But in that model, government debt and the maturity structure of issuance are held fixed for simplicity. A simplification in Chapter 3 is the assumption that taxes are lump sum and fiscal policy operates according to a simple rule. Relaxing these assumptions and assuming that fiscal policy also acts optimally could be an fruitful avenue for future research.

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