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Anabanti, Chimere and Aroh, A.B. and Hart, Sarah and Oodo, A.R. (2021)  
A question of Zhou, Shi and Duan on nonpower subgroups of finite groups.  
Quaestiones Mathematicae , ISSN 1607-3606.

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To cite this article: C.S. Anabanti, A.B. Aroh, S.B. Hart & A.R. Oodo (2021): A question of zhou, shi and duan on nonpower subgroups of finite groups, Quaestiones Mathematicae, DOI: [10.2989/16073606.2021.1924891](https://doi.org/10.2989/16073606.2021.1924891)

To link to this article: <https://doi.org/10.2989/16073606.2021.1924891>



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Published online: 01 Jul 2021.



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# A QUESTION OF ZHOU, SHI AND DUAN ON NONPOWER SUBGROUPS OF FINITE GROUPS

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**ABSTRACT.** A subgroup  $H$  of a group  $G$  is called a *power subgroup* of  $G$  if there exists a non-negative integer  $m$  such that  $H = \langle g^m : g \in G \rangle$ . Any subgroup of  $G$  which is not a power subgroup is called a *nonpower subgroup* of  $G$ . Zhou, Shi and Duan, in a 2006 paper, asked whether for every integer  $k$  ( $k \geq 3$ ), there exist groups possessing exactly  $k$  nonpower subgroups. We answer this question in the affirmative by giving an explicit construction that leads to at least one group with exactly  $k$  nonpower subgroups, for all  $k \geq 3$ , and infinitely many such groups when  $k$  is composite and greater than 4. Moreover, we describe the number of nonpower subgroups for the cases of elementary abelian groups, dihedral groups, and 2-groups of maximal class.

*Mathematics Subject Classification (2020):* 20D25, 20D60, 20E34.

*Key words:* Counting subgroups, nonpower subgroups, finite groups.

**1. Introduction.** A subgroup  $H$  of a group  $G$  is called a *power subgroup* of  $G$  if there exists a non-negative integer  $m$  such that  $H = G^m$ , where  $G^m := \langle g^m : g \in G \rangle$ . The identity subgroup and the whole group are examples of power subgroups of any group  $G$ . If  $H$  is a power subgroup of  $G$ , then  $H$  is normal in  $G$ ; but the converse is not necessarily true. For instance, no subgroup of index 2 in the quaternion group  $Q_8$  of order 8 is a power subgroup of  $Q_8$ , even though they are

normal subgroups. A subgroup of  $G$  which is not a power subgroup is called a *nonpower subgroup* of  $G$ .

Let  $k$  be the number of nonpower subgroups of a group  $G$ . The authors (Zhou, Shi and Duan) of [4] proved the following:

- (a)  $k \in (0, \infty)$  if and only if  $G$  is a finite noncyclic group;
- (b)  $k = 0$  if and only if  $G$  is a cyclic group;
- (c)  $k = \infty$  if and only if  $G$  is an infinite noncyclic group.

They also remarked that neither  $k = 1$  nor  $k = 2$  is possible in any group. With respect to the case  $k \geq 3$ , they asked (see [4, Problem]):

QUESTION 1. (Zhou, Shi and Duan) *For any integer  $k$  ( $k \geq 3$ ), do there exist groups possessing exactly  $k$  nonpower subgroups?*

In this paper, we show that the answer to this question is yes. In fact, we prove that there is at least one group possessing exactly  $k$  nonpower subgroups for each  $k \geq 3$  (see Theorem 5). Our method of proof also shows that there are infinitely many such groups for each  $k > 4$  and  $k$  not prime. The constructions we used are given in Section 2; part of it involves the direct product of a dihedral group with a carefully chosen cyclic group.

There are further questions one could ask. For example, given a positive integer  $n$ , what is the maximum number of nonpower subgroups in a group of order  $n$ ? To supply further examples of the possible numbers of nonpower subgroups in a group of a given order, we also explore in Section 3 some special cases: elementary abelian  $p$ -groups, dihedral groups, and 2-groups of maximal class. For example, we observe (see Corollary 10) that the elementary abelian  $p$ -group  $C_p \times C_p$  ( $p$  prime) contains exactly  $p + 1$  nonpower subgroups, and the generalised quaternion group  $Q_{2^n}$  (where  $n \geq 3$ ) contains exactly  $2^{n-1} - 1$  nonpower subgroups (see Theorem 16). All the groups studied here are finite.

We end this introductory section by briefly establishing the notation we will use. For a positive integer  $n$ , we write  $C_n$  for the cyclic group of order  $n$ , with  $D_{2n}$  being the dihedral group of order  $2n$ .

NOTATION. Let  $G$  be a group. We write  $s(G)$  for the total number of subgroups in  $G$ . Also, we write  $ps(G)$  for the number of power subgroups, and  $nps(G)$  for the number of non-power subgroups. For example, in  $C_2 \times C_2$  we have  $s(G) = 5$ ,  $ps(G) = 2$  and  $nps(G) = 3$ .

**2. Groups with exactly  $k$  nonpower subgroups.** In this section, we give constructions that supply, for each  $k \geq 3$ , at least one finite group containing exactly  $k$  nonpower subgroups. Moreover, for  $k \neq 4$  and  $k$  not prime, our constructions give infinitely many finite groups containing exactly  $k$  nonpower subgroups.

REMARK 2. Let  $G$  be a finite group. If  $n$  is coprime to  $|G|$ , then  $G^n = G$  as the map  $g \mapsto g^n$ , while not a homomorphism, is certainly a bijection from  $G$  to itself in this case. More generally,  $G^{mn} = G^m$  for any positive integer  $m$ .

LEMMA 3. *Let  $A$  and  $B$  be finite groups such that  $|A|$  and  $|B|$  are coprime. Then every subgroup of  $A \times B$  is of the form  $U \times V$ , where  $U \leq A$  and  $V \leq B$ . Moreover, a subgroup of  $A \times B$  is a power subgroup if and only if it is of the form  $U \times V$ , where  $U$  is a power subgroup of  $A$  and  $V$  is a power subgroup of  $B$ . In particular,*

- (1)  $s(A \times B) = s(A) \times s(B);$
- (2)  $nps(A \times B) = s(A) \times s(B) - ps(A) \times ps(B).$

*Proof.* Let  $G = A \times B$ . The fact that the subgroups of  $G$  in this case are the direct products of subgroups of  $A$  and  $B$  is well-known, but we include the proof for completeness. Suppose  $H \leq G$  and let  $(a, b) \in H$ . Since  $|A|$  and  $|B|$  are coprime, the orders  $r$  and  $s$  of  $a$  and  $b$  respectively are also coprime. Therefore, there exist integers  $q$  and  $t$  such that  $rq + st = 1$ . Now  $(a, b)^{st} = (a, 1)$  and  $(a, b)^{rq} = (1, b)$ . Hence,  $(a, 1)$  and  $(1, b)$  are elements of  $H$ . It follows that  $H = U \times V$ , where  $U = \{a \in A : (a, 1) \in H\}$  and  $V = \{b \in B : (1, b) \in H\}$ . Therefore,  $s(G) = s(A) \times s(B)$ .

Consider the power subgroup  $G^m$  of  $G$ , for a positive integer  $m$ . We have that  $G^m = A^m \times B^m$ , because this group is generated by elements  $(x, y)^m = (x^m, y^m)$ , and we have observed that  $(x^m, y^m)$  is contained in a subgroup  $H$  if and only if  $(x^m, 1) \in H$  and  $(1, y^m) \in H$ . For the converse, suppose that  $U = A^\ell$  and  $V = B^m$ , for some positive integers  $m$  and  $\ell$ . We may assume that  $\ell$  divides  $|A|$  and  $m$  divides  $|B|$ , by Remark 2. Now, let  $n = \ell m$ . Since  $\ell$  and  $m$  are therefore coprime, we have that  $A^n = A^\ell$ , and  $B^n = B^m$ . Therefore,  $U \times V = G^n$ . Thus, a subgroup of  $G$  is a power subgroup if and only if it is of the form  $U \times V$ , where  $U$  is a power subgroup of  $A$  and  $V$  is a power subgroup of  $B$ . In particular,  $ps(G) = ps(A) \times ps(B)$ . Hence,  $nps(G) = s(G) - ps(G) = s(A) \times s(B) - ps(A) \times ps(B)$ .  $\square$

Let  $n$  be a positive integer. Zhou et al. showed that  $nps(C_n) = 0$ . We also note that  $s(C_n) = ps(C_n) = \tau(n)$ , where  $\tau(n)$  is the number of divisors of  $n$ .

COROLLARY 4. *Suppose  $G = A \times C_n$ , where  $n$  is a positive integer and  $A$  is a finite group whose order is coprime to  $n$ . Then  $nps(G) = \tau(n) \times nps(A)$ .*

*Proof.* We have that  $s(C_n) = ps(C_n) = \tau(n)$ . Therefore in Equation (2), we have  $nps(G) = (s(A) - ps(A))\tau(n) = \tau(n) \times nps(A)$ .  $\square$

Before the next result we note that if  $p$  is an odd prime, then  $nps(D_{2p}) = p$ . This is because  $D_{2p}$  has exactly  $p + 3$  subgroups; the  $p$  cyclic subgroups of order 2 are the nonpower subgroups. The remaining groups (the trivial subgroup, the cyclic subgroup of index 2, and the whole group) are the power subgroups  $D_{2p}^{2p}$ ,  $D_{2p}^2$  and  $D_{2p}^1$ , respectively. For a full description of nonpower subgroups in arbitrary dihedral groups, see Section 3.

THEOREM 5. *Let  $k$  be a positive integer, with  $k \geq 3$ . Then there exists a finite group  $G$  with exactly  $k$  nonpower subgroups. If  $k$  is composite and  $k > 4$ , then there are infinitely many such groups.*

*Proof.* Let  $k$  be a positive integer with  $k \geq 3$ . Then either  $k$  is divisible by 4, or  $k$  is divisible by an odd prime  $p$  (or both). Suppose first that  $k$  is divisible by an odd prime  $p$ . Let  $q$  be any odd prime other than  $p$ , and let  $r = \frac{k}{p} - 1$ . Then  $\tau(q^r) = \frac{k}{p}$ . We observe that  $nps(D_{2p}) = p$ . Therefore, by Corollary 4, we get  $nps(D_{2p} \times C_{q^r}) = k$ . On the other hand, if  $k$  is divisible by 4, then let  $r = \frac{k}{4} - 1$ , and let  $q$  be any prime greater than 3. A quick calculation shows that  $nps(C_3 \times C_3) = 4$ ; whence  $nps((C_3 \times C_3) \times C_{q^r}) = k$ . We note that, in each case, if  $k > 4$  and  $k$  is composite, then the exponent  $r$  is strictly positive. Therefore, since there are infinitely many choices for  $q$ , there are infinitely many finite groups  $G$  with exactly  $k$  nonpower subgroups.  $\square$

### 3. Special cases.

NOTATION. For a prime  $p$  and a positive integer  $n$ , we write  $C_p^n$  for the elementary abelian  $p$ -group of finite rank  $n$ , and denote the number of subgroups of rank  $r$  in  $C_p^n$  by  $N_p(n, r)$ .

THEOREM 6. ([3, Theorem 1]) *Let  $V$  be a vector space of dimension  $n$  over the finite field  $GF(q)$ , where  $q$  is a prime power. The number of subspaces of  $V$  of dimension  $r$  is*

$$\binom{q^n - 1}{q - 1} \binom{q^{n-1} - 1}{q^2 - 1} \cdots \binom{q^{n-r+1} - 1}{q^r - 1}.$$

REMARK. (a) The group  $G = C_p^n$  can be realised as an  $n$ -dimensional vector space (say  $V$ ) over  $GF(p)$ . Now, the number of subgroups of rank  $r$  in  $C_p^n$  is equal to the number of subspaces of dimension  $r$  in  $V$ . In the light of Theorem 6 therefore, given any prime  $p$  and positive integers  $n$  and  $r$ , with  $n > r \geq 2$ , we have that

$$(3) \quad N_p(n, r) = \binom{p^n - 1}{p - 1} \binom{p^{n-1} - 1}{p^2 - 1} \cdots \binom{p^{n-r+1} - 1}{p^r - 1} = \prod_{k=0}^{r-1} \binom{p^{n-k} - 1}{p^{k+1} - 1}.$$

(b)  $N_p(n, 0) = 1 = N_p(n, n)$  for any prime  $p$  and natural number  $n$ , and for  $n > 1$ ,

$$N_p(n, 1) = \frac{p^n - 1}{p - 1} = \sum_{k=0}^{n-1} p^k = N_p(n, n - 1).$$

PROPOSITION 7. *For prime  $p$  and positive integers  $n$  and  $r$  (with  $n > r \geq 2$ ), we have:*

- (a)  $N_p(n - 1, r) = \binom{p^{n-r} - 1}{p^{r-1} - 1} N_p(n - 1, r - 1)$ ;  
(b)  $N_p(n, r) = p^r N_p(n - 1, r) + N_p(n - 1, r - 1)$ .

*Proof.* Setting  $n = n - 1$  and  $r = r - 1$  in Equation (3), we have that

$$(4) \quad N_p(n - 1, r - 1) = \binom{p^{n-1} - 1}{p - 1} \cdots \binom{p^{n-r+1} - 1}{p^{r-1} - 1} = \prod_{k=0}^{r-2} \binom{p^{n-(k+1)} - 1}{p^{k+1} - 1}.$$

Setting  $n = n - 1$  in Equation (3), we have that

$$\begin{aligned}
 N_p(n-1, r) &= \left(\frac{p^{n-1} - 1}{p - 1}\right) \cdots \left(\frac{p^{n-r+1} - 1}{p^{r-1} - 1}\right) \left(\frac{p^{n-r} - 1}{p^r - 1}\right) = \prod_{k=0}^{r-1} \left(\frac{p^{n-(k+1)} - 1}{p^{k+1} - 1}\right) \\
 (5) \quad &= N_p(n-1, r-1) \left(\frac{p^{n-r} - 1}{p^r - 1}\right) \text{ (from Equation (4))},
 \end{aligned}$$

which settles the (a) part. For the (b) part, we multiply Equation (5) by  $p^r$ , add the result to Equation (4) and regroup the terms to get the desired result.  $\square$

The recurrence relations given in Proposition 7 would be a good source for OEIS <https://oeis.org/>. We now turn to the first main result of this study; see Theorem 8.

**THEOREM 8.** *For prime,  $p$  and a natural number  $n > 1$ ,*

$$nps(C_p^n) = s(C_p^n) - 2 = \sum_{r=1}^{n-1} N_p(n, r).$$

*Proof.* Let  $p$  be a prime and  $n > 1$  be an integer. We write  $G = C_p^n$ . For  $m \in \mathbb{N} \cup \{0\}$ ,

$$G^m = \begin{cases} \{1\}, & \text{if } m \equiv 0 \pmod{p} \\ G, & \text{if } m \not\equiv 0 \pmod{p}. \end{cases}$$

This tells us that the only power subgroups of  $G$  are the unique subgroups of ranks 0 and  $n$  (viz; the two trivial subgroups). That is,  $nps(G) = s(G) - 2$ . In particular, the nonpower subgroups of  $G$  are the subgroups of ranks  $1, 2, \dots, n - 1$ . Thus, the number of nonpower subgroups of  $G$  is  $\sum_{r=1}^{n-1} N_p(n, r)$ .  $\square$

The following result is an immediate consequence of Theorem 8.

**COROLLARY 9.** *Let  $n > 1$  and  $p$  be prime. Then the elementary abelian  $p$ -group  $C_p^n$  contains exactly  $\sum_{r=1}^{n-1} N_p(n, r)$  nonpower subgroups.*

In particular, when  $n = 2$ , we have the following.

**COROLLARY 10.** *Let  $p$  be prime. The elementary abelian  $p$ -group  $C_p^2$  contains exactly  $p + 1$  nonpower subgroups.*

**DEFINITION.** A 2-group of maximal class is a group of order  $2^n$  and nilpotency class  $n - 1$  for  $n \geq 3$ .

**REMARK.** It is known (for instance, see Theorem 1.2 and Corollary 1.7 of [1]) that any 2-group of maximal class belongs to one of the following three classes:

- (i)  $\langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{-1} \rangle$ ,  $n \geq 3$  (Dihedral);

(ii)  $\langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, xy = yx^{-1} \rangle$ ,  $n \geq 3$  (Generalised quaternion);

(iii)  $\langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{2^{n-2}-1} \rangle$ ,  $n \geq 4$  (Semidihedral).

DEFINITION. For  $n \geq 3$ , we write

$$D_{2n} := \langle x, y \mid x^n = 1 = y^2, xy = yx^{-1} \rangle$$

for the dihedral group of order  $2n$ .

REMARK.  $D_{2n} = \{1, x, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\}$ . In  $D_{2n}$ , each element of  $\{y, xy, \dots, x^{n-1}y\}$  is an involution. In particular, there are  $n + 1$  involutions in  $D_{2n}$  when  $n$  is even.

THEOREM 11. ([2]) For  $n > 2$ ,  $s(D_{2n}) = \tau + u$ , where  $\tau$  is the number of positive divisors of  $n$  and  $u$  is the sum of the positive divisors of  $n$ .

PROPOSITION 12. Let  $G = D_{2n}$ ,  $n > 2$ . Writing  $u$  for the sum of positive divisors of  $n$  and  $r$  for the number of even proper divisors of  $n$ , we have the following: (i) if  $n$  is odd, then  $nps(G) = u - 1$ ; (ii) if  $n$  is even, then  $nps(G) = s(G) - (r + 2)$ ; (iii) if  $n$  is a power of 2, then  $nps(G) = u$ ; (iv) if  $n = 2p$  for an odd prime  $p$ , then  $nps(G) = s(G) - 3 = 3p + 4$ .

*Proof.* Let  $\tau$  denote the number of positive divisors of  $n$  and  $u$  denote the sum of positive divisors of  $n$ . By Theorem 11,  $s(G) = \tau + u$ .

Let  $m \in \mathbb{N} \cup \{0\}$  be arbitrary. Then

$$G^{2m+1} = \langle 1, x^{2m+1}, \dots, x^{-(2m+1)}, y, xy, \dots, x^{n-1}y \rangle.$$

As  $\{1, y, xy, \dots, x^{n-1}y\} \subseteq G^{2m+1}$ , we see immediately that  $|G^{2m+1}| > \frac{1}{2}|G|$ . The fact that  $G^{2m+1}$  is a subgroup of  $G$  helps us to conclude that  $G^{2m+1} = G$ .

On the other hand,

$$G^{2m} = \langle 1, x^{2m}, x^{4m}, \dots, x^{-4m}, x^{-2m} \rangle = \langle x^{2m} \rangle.$$

(i) Let  $n$  be odd. Then  $\langle x^{2m} \rangle$  is of the form  $\langle x^v \rangle$ , where  $v$  is a positive divisor of  $n$ . Therefore the set of all power subgroups of  $G$  is given as

$$\{G\} \cup \{\langle x^v \rangle \mid v \text{ is a positive divisor of } n\}.$$

Thus  $ps(G) = \tau + 1$ , and we conclude that  $nps(G) = (\tau + u) - (\tau + 1) = u - 1$ .

(ii) Let  $n$  be even. Then  $\langle x^{2m} \rangle$  is of the form  $\langle x^\mu \rangle$ , where  $\mu$  is an even proper divisor of  $n$ . Therefore the set of all power subgroups of  $G$  is given as

$$(6) \quad \{\{1\}, G\} \cup \{\langle x^\mu \rangle \mid \mu \text{ is an even proper divisor of } n\}.$$

So  $ps(G) = r + 2$ , where  $r$  is the number of even proper divisors of  $n$ . Whence,  $nps(G) = s(G) - (r + 2)$ .



(iii) Let  $n = 2^\ell \geq 4$ . In the light of (6), the set of power subgroups of  $G$  is

$$\{\{1\}, G, \langle x^2 \rangle, \langle x^4 \rangle, \langle x^8 \rangle, \dots, \langle x^{n/2} \rangle\},$$

where  $\langle x^2 \rangle \cong C_{n/2}$ ,  $\langle x^4 \rangle \cong C_{n/4}$ ,  $\langle x^8 \rangle \cong C_{n/8}$ ,  $\dots$ ,  $\langle x^{n/2} \rangle \cong C_2$ . So  $ps(G) = \tau$ . Therefore,  $nps(G) = s(G) - ps(G) = (\tau + u) - \tau = u$ .

(iv) Let  $n = 2p$  for an odd prime  $p$ . In the light of (6), the set of power subgroups of  $G$  is

$$\{\{1\}, G\} \cup \{\langle x^\mu \rangle \mid \mu \text{ is an even proper divisor of } 2p\} = \{\{1\}, G, \langle x^2 \rangle\},$$

where  $\langle x^2 \rangle \cong C_p$ . Hence,  $ps(G) = 3$ , and we conclude that  $nps(G) = s(G) - 3 = \tau + u - 3 = 4 + (1 + 2 + p + 2p) - 3 = 3p + 4$ .  $\square$

**COROLLARY 13.** *Given an integer  $n \geq 3$ ,  $s(D_{2^n}) = 2^n + n - 1$  and  $nps(D_{2^n}) = 2^n - 1$ .*

*Proof.* The results follow from a direct application of Theorem 11 and Proposition 12(iii) since the number of positive divisors of  $2^{n-1}$ , which is the same as the number of subgroups of  $D_{2^n}$  in  $\langle x \rangle$ , is  $n$ , and the sum of positive divisors of  $2^{n-1}$ , which is the same as the number of subgroups of  $D_{2^n}$  not contained in  $\langle x \rangle$ , is  $2^n - 1$ .  $\square$

**DEFINITION.** For  $n \geq 3$ , we write

$$Q_{2^n} := \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, xy = yx^{-1} \rangle$$

for the generalised quaternion group of order  $2^n$ .

**REMARK.**  $Q_{2^n} = \{1, x, \dots, x^{2^{n-1}-1}, y, xy, \dots, x^{2^{n-1}-1}y\}$ . Each element of  $\{y, xy, \dots, x^{2^{n-1}-1}y\}$  has order 4 in  $Q_{2^n}$ , and the element  $x^{2^{n-2}}$  is the unique involution in  $Q_{2^n}$ .

**DEFINITION.** For  $n \geq 4$ , we write

$$SD_{2^n} := \langle x, y \mid x^{2^{n-1}} = y^2 = 1, xy = yx^{2^{n-2}-1} \rangle$$

for the semidihedral group of order  $2^n$ .

**REMARK.**  $SD_{2^n} = \{1, x, \dots, x^{2^{n-1}-1}, y, xy, \dots, x^{2^{n-1}-1}y\}$ . In  $SD_{2^n}$ , any element of  $\{xy, x^3y, \dots, x^{2^{n-1}-1}y\} \cup \{x^{2^{n-3}}, x^{-(2^{n-3})}\}$  has order 4 while elements of  $\{y, x^2y, \dots, x^{2^{n-1}-2}y\} \cup \{x^{2^{n-2}}\}$  are involutions.  $SD_{2^n}$  contains  $2^{n-2} + 1$  involutions and  $2^{n-2} + 2$  elements of order 4.

**LEMMA 14.** *Let  $G$  be any of the three 2-groups of maximal class. If  $A$  is a noncyclic proper normal subgroup of  $G$ , then  $[G : A] = 2$ .*

*Proof.* Let  $G$  be any of the three 2-groups of maximal class and of order  $2^n$ , and let  $A$  be a noncyclic proper normal subgroup of  $G$ . Clearly,  $A \not\subseteq \langle x \rangle$ . Let  $a \in A$  be such that  $a \in \{y, xy, \dots, x^{2^{n-1}-1}y\}$ . Now, suppose  $G$  is either dihedral or generalised quaternion. We have that  $a = x^i y$  for some  $i \in \{0, 1, \dots, 2^{n-1} - 1\}$ . Using the relation  $xy = yx^{-1}$ , we obtain that  $axa^{-1} = x^2(x^i y) = x^2 a$ . As  $A$  is normal in  $G$  and  $a \in A$ , we deduce that  $(axa^{-1})a^{-1} = x^2 \in A$ . So  $\langle x^2 \rangle \subseteq A$ . Let  $G$  be a semidihedral group. If  $a = x^{2i+1}y$  for some  $i \in \{0, 1, \dots, 2^{n-2} - 1\}$ , then using the relation  $xy = yx^{2^{n-2}-1}$ , we obtain that  $axa^{-1} = yx^{-2i-3}$ . Therefore  $a(axa^{-1}) = x^{2i+1}yyx^{-2i-3} = x^{-2}$ . As  $A$  is normal in  $G$  and  $a \in A$ , we conclude that  $x^{-2} \in A$ ; whence  $\langle x^{-2} \rangle = \langle x^2 \rangle \subseteq A$ . If  $a = x^{2i}y$  for some  $i \in \{0, 1, \dots, 2^{n-2} - 1\}$ , then using the relation  $xy = yx^{2^{n-2}-1}$ , we obtain that  $axa^{-1} = yx^{2^{n-2}-2i-2}$ . So  $a(axa^{-1}) = x^{2i}yyx^{2^{n-2}-2i-2} = x^{2^{n-2}-2} \in A$ . But the order of  $x^{2^{n-2}-2}$  is the same as the order of  $x^2$ ; whence  $\langle x^{2^{n-2}-2} \rangle = \langle x^2 \rangle \subseteq A$ . In all the cases, we have these three in common:  $[G : \langle x^2 \rangle] = 4$ ,  $\langle x^2 \rangle \subseteq A \subseteq G$  and  $\langle x^2 \rangle \neq A \neq G$ . Therefore  $[G : A] = 2$ .  $\square$

**PROPOSITION 15.** *Let  $G$  be any of the three 2-groups of maximal class, and of order  $2^n$  for some  $n \geq 4$ . Given  $k \in \{1, 2, \dots, n-2\}$ , the number of subgroups of order  $2^{n-k}$  is  $2^k + 1$ .*

*Proof.* Let  $G = G_{2^n}$  be any of the three 2-groups of maximal class, and of order  $2^n$  for some  $n \geq 4$ , and let  $k \in \{1, 2, \dots, n-2\}$  be arbitrary. We show that there are  $2^k + 1$  subgroups of size  $2^{n-k}$ . The first case ( $k = 1$ ) follows from the well-known fact that there are 3 subgroups of index 2 in  $G$ ; the subgroups of index 2 in  $G$  are

$$\langle x \rangle, \langle x^2, y \rangle \text{ and } \langle x^2, xy \rangle,$$

where

$$\langle x \rangle \cong C_{2^{n-1}} \text{ and } \langle x^2, y \rangle \cong G_{2^{n-1}} \cong \langle x^2, xy \rangle.$$

Let  $H$  be a non-trivial subgroup of  $G$ . Recall that every non-trivial subgroup of a 2-group is contained in an index 2-subgroup of the group. Let  $k \in \{1, 2, \dots, n-2\}$ , and suppose  $H$  is a subgroup of size  $2^{n-k}$  in  $G$ . In the light of Lemma 14,  $H$  is contained in either  $\langle x \rangle$  or one of the noncyclic subgroups of index 2 in any (non-cyclic) subgroup of  $G$  which is isomorphic to  $G_{2^{n-k+1}}$ . But there are  $2^k$  noncyclic subgroups of index  $2^k$  in  $G_{2^n}$  for any  $k \in \{1, 2, \dots, n-2\}$ , where  $n \geq 4$ . Thus, the subgroups of size  $2^{n-k}$  (i.e., subgroups of index  $2^k$ ) in  $G_{2^n}$  are the unique cyclic subgroup of size  $2^{n-k}$  and the  $2^k$  non-cyclic subgroups of index  $2^k$ . Therefore there are  $1 + 2^k$  subgroups of size  $2^{n-k}$  in  $G_{2^n}$ .  $\square$

**THEOREM 16.** *Given an integer  $n \geq 3$ ,  $s(Q_{2^n}) = 2^{n-1} + n - 1$  and  $nps(Q_{2^n}) = 2^{n-1} - 1$ .*

*Proof.* In the light of Proposition 15, the number of subgroups of size  $2^k$  in  $Q_{2^n}$  and  $D_{2^n}$  are equal for each  $k \in \{2, 3, \dots, n-1\}$ . As the the number of subgroups of index 2 in both  $D_8$  and  $Q_8$  is 3, one sees immediately that the assertion is also true

for both  $D_8$  and  $Q_8$ . The distinction between the number of subgroups of various sizes in  $Q_{2^n}$  and  $D_{2^n}$  (where  $n \geq 3$ ) is in the subgroups of size 2. In particular, we have only one subgroup of size 2 in  $Q_{2^n}$  as opposed in  $D_{2^n}$ , where there are  $2^{n-1} + 1$  subgroups of size 2. Thus,

$$\begin{aligned} s(Q_{2^n}) &= s(D_{2^n}) - (2^{n-1} + 1) + 1 \\ &= 2^{n-1} + n - 1 \text{ (by Corollary 13)}. \end{aligned}$$

For the second part, let  $m \in \mathbb{N} \cup \{0\}$  be arbitrary, and  $G = Q_{2^n}$  for  $n \geq 3$ . Firstly,  $G^{4m+1} = \langle 1, x^{4m+1}, \dots, x^{-(4m+1)}, y, xy, \dots, x^{2^{n-1}-1}y \rangle$ . But  $\{1, y, xy, \dots, x^{2^{n-1}-1}y\} \subseteq G^{4m+1}$ ; whence  $|G^{4m+1}| > \frac{1}{2}|G|$ . As  $G^{4m+1}$  is a subgroup of  $G$ , we conclude that  $G^{4m+1} = G$ . Secondly,  $G^{4m+3} = \langle 1, x^{4m+3}, \dots, x^{-(4m+3)}, y^{-1}, (xy)^{-1}, \dots, (x^{2^{n-1}-1}y)^{-1} \rangle$ . As  $|\{1, y^{-1}, (xy)^{-1}, \dots, (x^{2^{n-1}-1}y)^{-1}\}| > \frac{1}{2}|G|$ , we deduce that  $G^{4m+3} = G$ . Thirdly,  $G^{4m+2} = \langle 1, x^{4m+2}, \dots, x^{-(4m+2)}, x^{2^{n-2}} \rangle = \langle x^2 \rangle \cong C_{2^{n-2}}$ . Finally,  $G^{4m} = \langle 1, x^{4m}, x^{8m}, \dots, x^{-8m}, x^{-4m} \rangle = \langle x^{4m} \rangle$ . If  $G = Q_8$ , then  $\langle x^{4m} \rangle \cong \{1\}$ . If  $G = Q_{16}$ , then  $\langle x^{4m} \rangle \cong \{1\}$  or  $\langle x^4 \rangle$ , where  $\langle x^4 \rangle \cong C_2$ . Now, let  $n \geq 5$ , and suppose  $\langle x^{4m} \rangle \neq \{1\}$ . Then  $\langle x^{4m} \rangle$  is exactly one of the following occurring subgroups of  $Q_{2^n}$ :

$$\langle x^{2^{n-2}} \rangle, \langle x^{2^{n-3}} \rangle, \dots, \langle x^4 \rangle,$$

where

$$\langle x^{2^{n-2}} \rangle \cong C_2, \langle x^{2^{n-3}} \rangle \cong C_4, \dots, \langle x^4 \rangle \cong C_{2^{n-3}}.$$

Therefore,  $ps(Q_{2^n}) = n$ ; whence  $nps(Q_{2^n}) = 2^{n-1} + (n-1) - n = 2^{n-1} - 1$ .  $\square$

**THEOREM 17.** *Given an integer  $n \geq 4$ ,*

$$s(SD_{2^n}) = 3(2^{n-2}) + n - 1 \text{ and } nps(SD_{2^n}) = 3(2^{n-2}) - 1.$$

*Proof.* In the light of Proposition 15, the number of subgroups of size  $2^k$  in  $SD_{2^n}$  and  $D_{2^n}$  are equal for each  $k \in \{2, 3, \dots, n-1\}$ . The distinction between the number of subgroups of various sizes in  $SD_{2^n}$  and  $D_{2^n}$  is in the subgroups of size 2. In particular, we have only  $2^{n-2} + 1$  subgroups of size 2 in  $SD_{2^n}$  whilst there are  $2^{n-1} + 1$  subgroups of size 2 in  $D_{2^n}$ . Thus,

$$\begin{aligned} s(SD_{2^n}) &= s(D_{2^n}) - (2^{n-1} + 1) + (2^{n-2} + 1) \\ &= 3(2^{n-2}) + n - 1 \text{ (by Corollary 13)}. \end{aligned}$$

For the second part, let  $m \in \mathbb{N} \cup \{0\}$  be arbitrary, and  $G = SD_{2^n}$  for  $n \geq 4$ . Then

$$G^{4m+1} = G = G^{4m+3}$$

follows from similar arguments as in the proof of Theorem 16. On the other hand, the results for  $G^{4m}$  and  $G^{4m+2}$  are also the same with the results for the generalised quaternion cases. Thus,  $ps(SD_{2^n}) = n$ ; whence  $nps(SD_{2^n}) = 3(2^{n-2}) + (n-1) - n = 3(2^{n-2}) - 1$ .  $\square$

*Acknowledgement.* The first author was supported by the Austrian Science Fund (FWF): P30934-N35, F05503 and F05510.

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*Received 22 December, 2020.*