Risk contributions of lambda quantiles

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Risk contributions of portfolios form an indispensable part of risk adjusted performance measurement. The risk contribution of a portfolio, e.g., in the Euler or Aumann-Shapley framework, is given by the partial derivatives of a risk measure applied to the portfolio return in direction of the asset weights. For risk measures that are not positively homogeneous of degree 1, however, known capital allocation principles do not apply. We study the class of lambda quantile risk measures, that includes the well-known Value-at-Risk as a special case, but for which no known allocation rule is applicable. We prove differentiability and derive explicit formulae of the derivatives of lambda quantiles with respect to their portfolio composition, that is their risk contribution. For this purpose, we define lambda quantiles on the space of portfolio compositions and consider generic (also non-linear) portfolio operators.

We further derive the Euler decomposition of lambda quantiles for generic portfolios and show that lambda quantiles are homogeneous in the space of portfolio compositions, with a homogeneity degree that depends on the portfolio composition and the lambda function. This result is in stark contrast to the positive homogeneity properties of risk measures defined on the space of random variables which admit a constant homogeneity degree. We introduce a generalised version of Euler contributions and Euler allocation rule, which are compatible with risk measures of any homogeneity degree and non-linear portfolios. We further provide financial interpretations of the homogeneity degree of lambda quantiles and introduce the notion of event-specific homogeneity of portfolio operators.

Keywords: Lambda Quantiles; Capital Allocation; Risk Contribution; Lambda Value-at-Risk; Euler Allocation

1. Introduction

Calculating firm-wide or portfolio-level risk is at the heart of modern financial risk management. Financial institutions use risk measures to determine economic capital, that is a capital buffer to absorb unexpected losses during adverse market scenarios and to preserve solvency. However, understanding how firm-wide or portfolio-level risk are formed and affected by their respective constituents is of equal importance to risk management processes. Determining contributions of assets or sub-portfolios to the overall portfolio risk, or contributions of business lines to the firm-wide risk enables market practitioners to make informed decisions on capital allocations to protect each business line's profitability and secure its solvency.

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In this paper we focus on lambda quantile risk measures, a class of law-invariant risk measures that generalises the well-known risk measure Value-at-Risk (VaR). Lambda quantiles were first proposed by Frittelli *et al.* (2014) to overcome two of the most criticised aspects of VaR. First VaR's inability to distinguish between different tail behaviours and second its failure to capture extreme losses. Indeed, lambda quantiles have the ability to (a) penalise heavy-tailed (return) distributions and (b) identify extreme losses dynamically, e.g., by recalibrating the lambda function of lambda quantiles, see Hitaj *et al.* (2018). The key difference between VaR and a lambda quantile is, that while VaR_λ is the negative of a quantile function at fixed level λ, a lambda quantile is the negative of a generalised quantile at a level determined by a function – the so-called lambda function.

Throughout, we consider generic, not necessarily linear, portfolio operators, that are collections of linear and/or non-linear assets where both long and short positions are permitted. A portfolio thus consists of a random vector of asset returns and its portfolio composition, a vector containing the number of units of each asset, or their weights. To calculate risk contributions, we define lambda quantiles on the space of portfolio compositions, a subset of $\mathbb{R}^n$, instead of the portfolio loss, the space of random variables. Using this novel domain for lambda quantiles, we study how the portfolio's risk – the lambda quantile of a portfolio – is affected by changes in its composition. Specifically, we address the question of what each asset's contribution to the overall portfolio risk is. These risk contributions quantify the extent of change in portfolio risk due to changes in an assets' exposure; an important metric in portfolio rebalancing.

Lambda quantiles are the subject of extensive study in Burzoni *et al.* (2017), Hitaj *et al.* (2018), and Corbetta and Peri (2018), where lambda quantiles are referred to as Lambda Value-at-Risk. When defined on the set of probability measures, lambda quantiles possess the properties of monotonicity and quasi-convexity (Frittelli *et al.* 2014). Burzoni *et al.* (2017) study robustness, elicitability, and consistency properties of lambda quantiles. A theoretical framework for backtesting lambda quantiles is provided in Corbetta and Peri (2018), who propose three backtesting methodologies. Moreover, Hitaj *et al.* (2018) argues to estimate the lambda function of lambda quantiles using major equity market indices, such as S&P500, FTSE100, and EURO STOXX 50, which provides a dynamic macro approach to measuring market risk. The axiomatisation and further properties of lambda quantiles are studied in Bellini and Peri (2019). These previous studies on lambda quantiles have either defined lambda quantiles on the space of probability measures or on the space of almost surely finite random variables. For the purpose of risk contributions, however, we define lambda quantiles on subsets of $\mathbb{R}^n$; the domain of asset compositions of a portfolio. Defining lambda quantiles on the space of portfolio compositions provides a natural way of comparing rates of change in portfolio risk with respect to asset weights. Understanding changes in portfolio risk that may arise from portfolio rebalancing is highly important from a performance measurement perspective and relevant for risk capital allocation.

There exist a plethora of risk capital allocation methods that firms use for risk management and performance measurement, see Balog *et al.* (2017) for a review and comparison of risk capital allocation methods and their properties. It should be noted that not all capital allocation methods are compatible with a specific risk measure, and applicability is determined by the properties of the risk measure in question. For example, the axiomatic approach taken in Denault (2001) to define a coherent risk capital allocation principle derived from the Aumann-Shapley value (Aumann and Shapley 1974) only applies to coherent risk measures (Artzner *et al.* 1999); a property lambda quantiles do not possess. Furthermore, the Aumann-Shapley capital allocation rule introduced in Tsanakas (2009), which was also inspired from the Aumann-Shapley value, is defined for Gateaux differentiable risk measures on linear portfolios. Explicit formulae of the Aumann-Shapley allocation rule is provided, for the class of convex risk measures, in Tsanakas (2009). For positive homogeneous (but not necessarily coherent) risk measures, the Euler capital allocation (Patrik *et al.* (1999), Tasche (1999), Denault (2001), Tasche (2007)) can be used, which, on the space of coherent risk measures, coincides with the Aumann-Shapley allocation. It is worth noting that both Denault (2001) and Tasche (1999) arrive (for coherent risk measures) at the same capital allocation rule using different theoretical approaches: the former uses a game-theoretic approach whilst the latter the notion of risk-adjusted performance measurement, a common practice for internal economic capital calculations. The general class of non-Gateaux differentiable but convex or quasi-convex risk measures is treated in Centrone and Gianin (2018). These capital allocation rules are not applicable to this study since they apply to linear portfolio operators, whereas we treat generic (not necessarily linear) portfolio operators. While Pesenti *et al.* (2018) define Euler allocation rules for non-linear portfolios, they only apply to positive homogeneous risk measures with homogeneity degree equal to 1. The homogeneity degree of lambda quantiles, as we show in this paper, is however a function of the portfolio composition and the lambda function.

In this paper, we define risk contributions of lambda quantiles defined on the space of portfolio compositions as the partial derivatives of lambda quantiles with respect to asset weights. We derive risk contributions of individual assets to the overall portfolio risk, measured via the lambda quantile of the portfolio composition. In doing so, we prove that lambda quantiles are continuously partially differentiable in the space of portfolio compositions using two independent methods which assume different properties. Furthermore, we prove that lambda quantiles are continuously differentiable in smaller subsets of $\mathbb{R}^n$, for a lambda function that may contain discontinuities, as long as it is continuously differentiable within a specific interval of $\mathbb{R}$.

Risk contributions calculated as directional derivatives of positive homogeneous risk measures of degree 1 are known as Euler contributions, where the assignment of capital using Euler contributions is known as Euler allocation. We show in this paper that lambda quantiles can be written as a weighted sum of their partial derivatives scaled by a factor. This property is then used to show that lambda quantiles are homogeneous in the space of portfolio compositions, with a homogeneity degree that depends on both the portfolio composition and the lambda function. Only for the special case of a constant lambda function, the lambda quantile reduces to the VaR and has a homogeneity degree of 1. Therefore, the Euler allocation rule may not always be applicable to lambda quantiles, since their homogeneity degree is not universally equal to 1. Due to the variable nature of lambda quantiles' homogeneity degrees, we introduce a generalised Euler capital allocation rule, that is compatible with risk measures of any homogeneity degree and non-linear portfolios. We prove that the generalised Euler allocations of lambda quantiles have the full allocation property. We further provide financial interpretations to explain how the lambda quantile homogeneity degree is a function of both the portfolio composition and the lambda function.

This paper builds upon methods and results relating to risk contributions and differentiability of VaR, whose literature is extensive and well established; indicatively see Tasche (1999), Hallerbach (2003), Hong (2009), Tsanakas and Millossovich (2016), Saporito and Targino (2020), and Pesenti *et al.* (2021). Specifically, the papers of Tasche (1999), Hong (2009), and Tsanakas and Millossovich (2016) provide a stepping stone for proving differentiability and calculating risk contributions of lambda quantiles from a portfolio performance measurement perspective, which are all novel pursuits in risk measure theory. Indeed, for lambda quantiles to be partially differentiable, we require additional smoothness assumptions. A first set of assumptions relates to the invertibility property of portfolio returns, similar to Tasche (1999). As this may not always be the satisfied for generic portfolio returns, we further prove our results using the condition that the portfolio return possesses a (locally) continuous density, akin to the assumptions in Hong (2009).

The paper is organised as follows: Section 2 introduces the necessary notation and definitions. In Section 3, we prove continuous partial differentiability of lambda quantiles in subsets of $\mathbb{R}^n$ and derive explicit formulae of risk contributions of lambda quantiles with respect to portfolio compositions. We introduce the novel definition of event-specific $\mathbb{P}$-a.s. homogeneity of portfolio operators in Section 4 and use this to prove the Euler decomposition of lambda quantiles. Furthermore, we also define the generalised Euler contributions and generalised Euler allocation rule in Section 4 and prove that generalised Euler contributions of lambda quantiles fulfil the full allocation property. We provide financial interpretations of the homogeneity degree of lambda quantiles in Section 4.1 and study the homogeneity properties of generic portfolio operators in Section 4.2.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote by $\mathcal{X}$ the set of random variables and by $\mathcal{X}^n$ for $n \geq 2$ the set of random vectors on that space, taking values in $\mathbb{R}$ and $\mathbb{R}^n$ respectively. The joint probability distribution function of $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ is represented by $F_{\boldsymbol{X}}(\boldsymbol{x}) := \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$, where each $X_1, \ldots, X_n \in \mathcal{X}$. We will use $\boldsymbol{X}_{-1} := (X_2, \ldots, X_n) \in \mathcal{X}^{n-1}$ and $\boldsymbol{x}_{-1} := (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ to indicate, respectively, random and ordinary vectors with first components removed. Define ϕ to be the density of the conditional probability distribution of X_1 given $X_2 = x_2 \ldots, X_n = x_n$. Also, $U \subset \mathbb{R}\backslash\{0\} \times \mathbb{R}^{n-1}$ is a bounded set of n-dimensional real vectors with at least one non-zero component, which we set w.l.o.g. to the first component. Note that the choice of the first component is arbitrary and ϕ could represent the density of the conditional distribution of X_i given $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_{i+1} = x_{i+1}, \ldots, X_n = x_n$ for any $i = 1, \ldots, n$, provided that the i^{th} component of U does not contain zero.

In this paper, we treat a portfolio of n assets. Random returns of assets are represented by $\boldsymbol{X}$ and the portfolio composition, or asset weights, is given by $\boldsymbol{u} \in U$.

Definition 1 The mapping $g : U \times \mathcal{X}^n \to \mathcal{X}$ is called a *portfolio operator* and $g[\boldsymbol{u}, \boldsymbol{X}]$ represents the overall random portfolio return for a composition $\boldsymbol{u}$ and return vector $\boldsymbol{X}$.

For fixed $\boldsymbol{X} \in \mathcal{X}^n$, the mapping $g_{\boldsymbol{X}} : U \to \mathcal{X}$ such that $g_{\boldsymbol{X}}(\boldsymbol{u}) = g[\boldsymbol{u}, \boldsymbol{X}]$ is the *portfolio return as a function of the composition* $\boldsymbol{u}$ or *portfolio return* for short.

Finally, if the random vector $\boldsymbol{X}$ is realised, i.e. $\boldsymbol{X}(\omega) = \boldsymbol{x} \in \mathbb{R}^n$ for some outcome $\omega \in \Omega$, then we denote the portfolio return using the mapping $g_{\boldsymbol{x}} : U \to \mathbb{R}$ such that $g_{\boldsymbol{x}}(\boldsymbol{u}) = g[\boldsymbol{u}, \boldsymbol{X}(\omega)]$, and call it the *realised portfolio return.*

The portfolio operator g is subject to stochastic variability because the value taken by $\boldsymbol{X}$ at each outcome $\omega \in \Omega$ is random and we assume that g is independent of the probability distribution $F_{\boldsymbol{X}}$. The portfolio operator is also subject to distributional variability because we consider all random vectors in $\mathcal{X}^n$ – we are not restricted to a class of random vectors of a specific distribution. If the random vector $\boldsymbol{X}$, and hence its joint probability distribution $F_{\boldsymbol{X}}$, is fixed, then the portfolio operator is only subject to stochastic variability and we consider the portfolio return $g_{\boldsymbol{X}}(\cdot)$, i.e. the asset returns of our portfolio vary randomly at each observation determined by their joint probability distribution. Note that $g_{\boldsymbol{X}}(\boldsymbol{u})$, for any $\boldsymbol{u} \in U$, is a random variable, because $\boldsymbol{X}$ has not been realised. Moreover, $g_{\boldsymbol{X}}(\boldsymbol{u})$ varies (deterministically) with the dynamics of portfolio composition $\boldsymbol{u}$. As we must distinguish between the joint probability distribution $F_{\boldsymbol{X}}$ and the probability distribution function of the portfolio return $Y := g_{\boldsymbol{X}}(\boldsymbol{u})$, we denote the probability distribution and density functions of the portfolio return Y by $F_Y(y) = \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y)$ and $f_Y(y) = dF_Y/dy(y)$, respectively, for all $y \in \mathbb{R}$.

The nature of assets in a portfolio ultimately determine the operator g, or in other words, the structure of the portfolio is determined by the payoff functions of the assets. A portfolio operator is said to be *linear in* U if $g[c_1\boldsymbol{u} + c_2\boldsymbol{v}, \boldsymbol{X}] = c_1 g[\boldsymbol{u}, \boldsymbol{X}] + c_2 g[\boldsymbol{v}, \boldsymbol{X}]$ for any $\boldsymbol{u}, \boldsymbol{v} \in U$ and $c_1, c_2 \in \mathbb{R}$. For example, the return of a portfolio of only equity positions can be represented using a linear operator because the payoff of an equity position is linearly dependent on the stock price. However, one must consider a non-linear operator for derivatives positions with non-linear payoff functions, such as equity options, or for reinsurance contracts. The following example shows the conceptual difference between the portfolio operator g and the portfolio return $g_{\boldsymbol{X}}$.

Example 1 *Let* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{u} = (u_1, u_2)$. *Consider the portfolio operator:*

$$g[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2 - \mathbb{E}[u_1 X_1 + u_2 X_2].$$

This operator represents the difference between actual and expected returns of a portfolio, or in other words, the unexpected return. Even though the portfolio return $g_{\boldsymbol{X}}$ *has the same form as the* *operator g, they are fundamentally different objects and we may choose to write $g_{\boldsymbol{X}}$ as:*

$$g_{\boldsymbol{X}}(\boldsymbol{u}) = u_1 X_1 + u_2 X_2 - \mu_Y,$$

where μ_Y is the mean of the random variable $Y := u_1 X_1 + u_2 X_2$ with fixed $\boldsymbol{X}$. This is because for a fixed composition $\hat{\boldsymbol{u}} := (\hat{u}_1, \hat{u}_2)$ and fixed return $\boldsymbol{X}$, the distribution of $\hat{Y} := \hat{u}_1 X_1 + \hat{u}_2 X_2$ is also fixed and therefore the mean $\mu_{\hat{Y}}$ is a constant. On the other hand, if we do not fix $\boldsymbol{X}$, then the expectation $\mathbb{E}[\hat{u}_1 X_1 + \hat{u}_2 X_2]$, that appears in the operator $g[\hat{\boldsymbol{u}}, \boldsymbol{X}]$, is a function of $\boldsymbol{X}$.

In practice, if the distribution of asset returns is known, practitioners are interested in changing the portfolio composition $\boldsymbol{u}$ to achieve a higher risk-adjusted return for their portfolio. The process of selecting assets by comparing their expected returns and contribution to overall portfolio risk is known as risk-adjusted performance measurement. In order to do this, one must know the per-unit contribution of each asset to the overall portfolio risk, and, in particular, risk contributions that are suitable for performance measurement.

In order to measure the risk, we use lambda quantiles that are traditionally defined on distributions. However, the purpose of this paper is to calculate the per-unit risk contribution of each asset to the overall portfolio risk. Hence, as one of the novelties of this paper, we define lambda quantiles on the set of the portfolio compositions U and calculate partial derivatives of lambda quantiles with respect to asset weights. The partial derivative with respect to asset weights is the only definition of risk contribution that is suitable for performance measurement (Tasche 1999).

Definition 2 The *lambda quantile* $\rho_\Lambda : U \to \mathbb{R} \cup \{+\infty\}$ with respect to $g_{\boldsymbol{X}}(\boldsymbol{u}) \in \mathcal{X}$ is defined as follows:

$$\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) := -\inf\{y \in \mathbb{R} \,|\, \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y) > \Lambda(y)\},$$

where $\Lambda : \mathbb{R} \to (0,1)$ is bounded and referred to as the *lambda function.*

The lambda quantile at $\boldsymbol{u}$ is the negative of the smallest intersection point of the distribution F_Y and the lambda function $\Lambda(\cdot)$, provided they are both continuous. Otherwise, it is the negative of the smallest point from which the distribution F_Y dominates the lambda function. The negative of the right quantile is used, as we work with asset returns. Moreover, we use right quantiles of a distribution function as we are working with returns; left quantiles are typically used when asset losses are considered, e.g., in an insurance context. The lambda quantile $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$ represents a positive amount to be allocated to absorb losses. A positive amount is allocated only if the return is negative (or loss is positive), i.e. $-\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) < 0$, otherwise the risk measure suggests there is a surplus of money which can be removed from the portfolio and still ensures its solvency. For the time being, we assume that the lambda function is bounded and we denote the derivative of the lambda function by $\Lambda'(x) := d\Lambda/dx(x)$. We will, however, assume additional properties in subsequent sections.

From Definition 2, we observe that the lambda quantile is a generalisation of the quantile function, in that the lambda quantile is the negative of the quantile function at a level determined by the lambda function $\Lambda(\cdot)$. If the lambda function is a constant, i.e. $\Lambda(y) = \lambda \in (0,1)$ for all $y \in \mathbb{R}$, then the lambda quantile simplifies to the well-known *Value-at-Risk* (VaR). In particular, the $VaR_\lambda : U \to \mathbb{R} \cup \{+\infty\}$ is given by:

$$VaR_\lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) := -\inf\{y \in \mathbb{R} | \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y) > \lambda\}.$$

Here, we view the VaR at (fixed) level λ as a function of $\boldsymbol{u}$, whereas typically the VaR is considered as a function of the random variable $g_{\boldsymbol{X}}(\boldsymbol{u})$. For simplicity, we write $\rho_\Lambda(\boldsymbol{u})$ and $VaR_\lambda(\boldsymbol{u})$ when there is no ambiguity on the portfolio return $g_{\boldsymbol{X}}$.

We now provide examples of portfolio operators which we will use as running examples. Operators (ii)-(v) are studied in Major (2018), which we have adapted to our notation.

EXAMPLE 2 *Let $n = 2$, $\boldsymbol{X} = (X_1, X_2)$, $\boldsymbol{u} = (u_1, u_2)$ and $Y := u_1 X_1 + u_2 X_2$. The following are examples of portfolio operators defined on the Cartesian product $U \times \mathcal{X}^2$:*

(i) $g[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2$
(ii) $g[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2 - \mathbb{E}[u_1 X_1 + u_2 X_2]$
(iii) $g[\boldsymbol{u}, \boldsymbol{X}] = \max\{0, u_1 X_1 + u_2 X_2 - \mathbb{E}[u_1 X_1 + u_2 X_2]\}$
(iv) $g[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2 - VaR_\lambda(\boldsymbol{u}; Y)$
(v) $g[\boldsymbol{u}, \boldsymbol{X}] = \max\{0, \min\{u_1 X_1 + u_2 X_2 - VaR_{\lambda_1}(\boldsymbol{u}; Y), VaR_{\lambda_2}(\boldsymbol{u}; Y) - VaR_{\lambda_1}(\boldsymbol{u}; Y)\}\}$
(vi) $g[\boldsymbol{u}, \boldsymbol{X}] = u_1^\tau X_1 + u_2^\tau X_2 - \rho_\Lambda(\boldsymbol{u}; Y)$*, where* $\tau \in \mathbb{R}$.

For the exposition, we require additional notation. Define $(x_1, \boldsymbol{x}_{-1}) := \boldsymbol{x}$, so that the realised portfolio returns, for all $\omega \in \Omega$ with $\boldsymbol{X}(\omega) = \boldsymbol{x}$, becomes:

$$g_{\boldsymbol{X}(\omega)}(\boldsymbol{u}) = g_{\boldsymbol{x}}(\boldsymbol{u}) = g_{(x_1, \boldsymbol{x}_{-1})}(\boldsymbol{u})\,.$$

Further, the partial derivatives of $g_{\boldsymbol{x}}$ with respect to u_i and x_1 are denoted by $\partial_{u_i} g_{\boldsymbol{x}}(\boldsymbol{u}) := \partial g_{\boldsymbol{x}} / \partial u_i(\boldsymbol{u})$ and $\partial_{x_1} g_{\boldsymbol{x}}(\boldsymbol{u}) := \partial g_{\boldsymbol{x}} / \partial x_1(\boldsymbol{u})$, respectively. Similarly, we denote the $\mathbb{P}$-a.s. partial derivatives of $g_{\boldsymbol{X}}$ with respect to u_i by $\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) := \partial g_{\boldsymbol{X}} / \partial u_i(\boldsymbol{u})$.

We write $A_1 \subset \mathbb{R}$ for the support of the random variable X_1, i.e. $A_1 \subset \mathbb{R}$ is the smallest closed set such that $\mathbb{P}(X_1 \in A_1) = \mathbb{P}(\{\omega \in \Omega \mid X_1(\omega) \in A_1\}) = 1$. We say that $g_{\boldsymbol{X}}$ is *invertible with respect to $X_1 = x_1$, for all $X_2 = x_2, \ldots, X_n = x_n$*, if for all $x_1 \in A_1$, the function $g_{(x_1, \boldsymbol{x}_{-1})}$ is invertible with respect to x_1, for all $x_2, \ldots, x_n$. We denote the inverse of $g_{(x_1, \boldsymbol{x}_{-1})}$ by $l_{(y, \boldsymbol{x}_{-1})} : U \to \mathbb{R}$ such that

$$l_{(y, \boldsymbol{x}_{-1})}(\boldsymbol{u}) = x_1 \iff g_{(x_1, \boldsymbol{x}_{-1})}(\boldsymbol{u}) = y,$$

for all $x_1 \in A_1$, and its partial derivatives with respect to u_i and y are respectively $\partial_{u_i} l_{(y, \boldsymbol{x}_{-1})}(\boldsymbol{u}) := \partial l_{(y, \boldsymbol{x}_{-1})} / \partial u_i(\boldsymbol{u})$ and $\partial_y l_{(y, \boldsymbol{x}_{-1})}(\boldsymbol{u}) := \partial l_{(y, \boldsymbol{x}_{-1})} / \partial y(\boldsymbol{u})$.

3. Differentiability and risk contributions of lambda quantiles

In this section, we study differentiability of lambda quantiles in the set U. Derivatives of the lambda quantile risk measure have not been studied in previous literature. Derivatives of the special case, namely the VaR measure, however, have an extensive literature, see for example, Tasche (1999), Gourieroux *et al.* (2000), Hallerbach (2003), Hong (2009), Tsanakas and Millossovich (2016). Although these studies calculate derivatives of VaR, they differ in both methods used and assumptions made in their respective settings.

Partial derivatives of risk measures with respect to asset weights are crucial in portfolio risk management as they represent the risk contribution of each asset to the overall portfolio risk. This definition of risk contributions is consistent with risk-adjusted performance measurement of portfolios (see Definition 6 and Lemma 4 in the Appendix A). This section extends the literature to include partial derivatives of lambda quantiles and recover previous results on VaR as special cases. In light of this motivation, this section has two objectives. The first objective is to provide conditions under which lambda quantiles are continuously partially differentiable in the set U. The second objective is to calculate these partial derivatives explicitly. There are several approaches one may follow to achieve the latter, which ultimately depend on assumptions made regarding the portfolio return $g_{\boldsymbol{X}}$, the random vector $\boldsymbol{X}$, and the lambda function $\Lambda(\cdot)$. Partial derivatives of lambda quantiles will be calculated using two different approaches, each having its own set of assumptions.

In the first approach we generalise the treatment in Tasche (1999), who calculated partial derivatives of VaR_λ for linear portfolios, to lambda quantiles for generic portfolios. We extend this method to take into account the lambda function (instead of a fixed level λ) and to cover non-linear portfolios (by defining lambda quantiles on generic portfolios). These generalisations require additional assumptions for lambda quantiles to be continuously differentiable in the set U.

In the second approach we utilise the closed-form representation of probability sensitivities, proposed by Hong (2009). The probability sensitivity corresponds, in our context, to the partial derivative of the portfolio return's probability distribution function with respect to asset weights. Note that the two approaches mentioned above allow us to prove the same property (continuously partially differentiable in U) of lambda quantiles. Furthermore, partial derivatives of lambda quantiles are the same under both approaches.

Assumption 1 We say that Assumption 1 is satisfied if:

(i) $g_{(x_1,\boldsymbol{x}_{-1})}$ is monotonically increasing in x_1.
(ii) $g_{\boldsymbol{X}}$ is $\mathbb{P}$-a.s. differentiable in all of its arguments for all $\boldsymbol{u} \in U$.
(iii) For fixed $\boldsymbol{x}_{-1}$, the density $y \mapsto \phi(y|\boldsymbol{x}_{-1})$ is continuous in y.
(iv) For fixed $\boldsymbol{x}_{-1}$, $l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})$ is continuously differentiable with respect to y and u_i, $i = 1, \ldots, n$, and for all $\boldsymbol{u} \in U$.
(v) For fixed $\boldsymbol{u}$ and all $i = 1, \ldots, n$, the following maps are uniformly bounded:

$$
\begin{aligned}
&y \mapsto \mathbb{E}[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})], \\
&y \mapsto \mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})].
\end{aligned}
$$

(vi) For $g_{(x_1,\boldsymbol{x}_{-1})}$ monotonically increasing in x_1 and for each $\boldsymbol{u} \in U$, assume:

$$
\mathbb{E}[\partial_y l_{(-\rho_\Lambda(\boldsymbol{u}),\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(-\rho_\Lambda(\boldsymbol{u}),\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})] > \Lambda'(-\rho_\Lambda(\boldsymbol{u})).
$$

Note that Assumption 1 (i) implies that $g_{\boldsymbol{X}}$ is invertible with respect to $X_1 = x_1$, for all $X_2 = x_2, \ldots, X_n = x_n$. Assumption 1 (i) means that the realised portfolio return $g_{\boldsymbol{x}}$ increases with the realised return x_1 of the first asset. Although we assume $g_{(x_1,\boldsymbol{x}_{-1})}$ is monotonically increasing in x_1 in this paper, the results also hold for the monotonic decreasing case, with some sign changes. The proofs of the decreasing case are similar to those of the increasing case and thus omitted.

Remark 1 Assumption 1 (iii) and (iv) imply the following mappings are continuous for fixed $\boldsymbol{x}_{-1}$ and $i = 1, \ldots, n$

$$
\begin{aligned}
&(y, \boldsymbol{u}) \mapsto \partial_y l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1}), \\
&(y, \boldsymbol{u}) \mapsto \partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1}),
\end{aligned}
$$

and the following mappings are continuous, for all $i = 1, \ldots, n$,

$$
\begin{aligned}
&(y, \boldsymbol{u}) \mapsto \mathbb{E}[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})], \\
&(y, \boldsymbol{u}) \mapsto \mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})].
\end{aligned}
$$

The second set of assumptions relates to the probability sensitivity of Hong (2009), which have been adapted to our setting. In contrast to the approach taken in Hong (2009), we require Assumption 2 (vi) to account for the lambda function in our treatment.

Assumption 2 We say that Assumption 2 is satisfied if:

(i) $g_{\boldsymbol{X}}$ is $\mathbb{P}$-a.s. differentiable in all of its arguments for all $\boldsymbol{u} \in U$.

(ii) There exists a random variable $m(\boldsymbol{X})$ with $\mathbb{E}[m(\boldsymbol{X})] < \infty$ such that for all $\boldsymbol{u}, \boldsymbol{v} \in U$:

$$|g_{\boldsymbol{X}}(\boldsymbol{u}) - g_{\boldsymbol{X}}(\boldsymbol{v})| \;\leq\; m(\boldsymbol{X})\|\boldsymbol{u} - \boldsymbol{v}\| \quad \mathbb{P}\text{-a.s.}\,,$$

where $\|\cdot\|$ denotes the Euclidean norm in U.

(iii) For all $\boldsymbol{u} \in U$, the random variable $g_{\boldsymbol{X}}(\boldsymbol{u})$ has a continuous density $f_Y(y)$ in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$.

(iv) For the function $F : \mathbb{R}\times U \to [0,1]$ defined as $F(y,\boldsymbol{u}) := \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y)$, the partial derivatives $\partial_{u_i} F(y,\boldsymbol{u})$ exist, for all $i = 1,\ldots,n$, and are continuous in $\boldsymbol{u}$ and y in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$.

(v) For all $\boldsymbol{u} \in U$ and for $i = 1,\ldots,n$, the following mappings are continuous at $y = -\rho_\Lambda(\boldsymbol{u})$:

$$y \mapsto \mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = y].$$

(vi) $f_Y(-\rho_\Lambda(\boldsymbol{u})) > \Lambda'(-\rho_\Lambda(\boldsymbol{u}))$ for all $\boldsymbol{u} \in U$.

Assumption 2 (iii) implies that each distribution of the random field $(g_{\boldsymbol{X}}(\boldsymbol{u}))_{\boldsymbol{u}\in U}$ is continuous in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$ for its respective $\boldsymbol{u} \in U$.

Using Assumption 1, we demonstrate Lemmas 1 and 2, which we require to prove Theorem 1 with condition (i) and Proposition 2. Lemmas 1 and 2 are generalisations of Lemmas 3.2 and 2.2 in Tasche (2001), respectively, as they apply to both linear and non-linear portfolio returns.

Lemma 1 *Suppose Assumption 1 (i)-(v) are satisfied. Then the function $F : \mathbb{R}\times U \to [0,1]$ defined as $F(y,\boldsymbol{u}) := \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y)$ is partially differentiable in y and u_i for $i = 1,\ldots,n$. The continuous derivatives are given by:*

$$\frac{\partial F}{\partial y}(y,\boldsymbol{u}) = \mathbb{E}\left[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})\right], \tag{1}$$

$$\frac{\partial F}{\partial u_i}(y,\boldsymbol{u}) = \mathbb{E}\left[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})\right]. \tag{2}$$

Proof. We generalise the approach taken in the proof of Lemma 5.3 in Tasche (1999) to prove that F is continuously differentiable. Our method applies to a generic random variable $g_{\boldsymbol{X}}(\boldsymbol{u})$ whilst the proof provided in Tasche (1999) applies only to linear portfolios, that is to $\sum_{i=1}^n u_i X_i$.

We first introduce the following integral using the density ϕ of the conditional distribution of X_1 given $\boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}$

$$G(y,\boldsymbol{u},\boldsymbol{x}_{-1}) := \int_{-\infty}^{l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})} \phi(t|\boldsymbol{x}_{-1})dt. \tag{3}$$

Note that G can be written in the following form

$$\begin{aligned}
G(y,\boldsymbol{u},\boldsymbol{x}_{-1}) &= \mathbb{P}(X_1 \leq l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}) \\
&= \mathbb{P}(\{\omega \in \Omega | X_1(\omega) \leq l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})\}) \\
&= \mathbb{P}(\{\omega \in \Omega | g_{(X_1(\omega),\boldsymbol{x}_{-1})}(\boldsymbol{u}) \leq g_{(l_{(y,\boldsymbol{x}_{-1})},\boldsymbol{x}_{-1})}(\boldsymbol{u})\}) \\
&= \mathbb{P}(\{\omega \in \Omega | g_{(X_1(\omega),\boldsymbol{x}_{-1})}(\boldsymbol{u}) \leq y\}) \\
&= \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y|\boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}),
\end{aligned}$$

and F can be written in terms of G:

$$F(y,\boldsymbol{u}) = \mathbb{E}[\mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y|\boldsymbol{X}_{-1})] = \mathbb{E}[G(y,\boldsymbol{u},\boldsymbol{X}_{-1})]. \tag{4}$$

We show that F is continuously differentiable in y and u_i for $i = 1,\dots,n$, and that its derivatives can be computed by changing the order of integration and differentiation on the right-hand side of (4). In order to do this, we apply Lemma 3 (see Appendix A) to the function $G : \mathbb{R}\times U\times\mathbb{R}^{n-1} \to \mathbb{R}$ to the components y, and $u_1,\dots,u_n$. For this, we define $S_y := U\times\mathbb{R}^{n-1}$, $S_{u_1} := U\setminus(0,1]\times\mathbb{R}\times\mathbb{R}^{n-1}$, and $S_{u_j} := U \setminus [0,1] \times \mathbb{R} \times \mathbb{R}^{n-1}$ for $j = 2,\dots,n$. Note that we distinguish u_1 from $u_2,\dots,u_n$ as u_1 cannot be zero.

For differentiability in the first component y, condition (i) of Lemma 3 is satisfied as:

$$\int_{S_y} |G(y,\boldsymbol{u},\boldsymbol{x}_{-1})|\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, d\boldsymbol{u} = \int_U \mathbb{E}[|G(y,\boldsymbol{u},\boldsymbol{X}_{-1})|]\, d\boldsymbol{u} = \int_U F(y,\boldsymbol{u})\, d\boldsymbol{u} < \infty.$$

The finiteness follows from the observation that for any fixed $\boldsymbol{u} \in U$, $F(y,\boldsymbol{u}) = F_Y(y)$ is the distribution function of $Y = g_{\boldsymbol{X}}(\boldsymbol{u})$ and that U is bounded. The same condition is satisfied for differentiability of u_1 as:

$$\begin{aligned}\int_{S_{u_1}} |G(y,\boldsymbol{u},\boldsymbol{x}_{-1})|\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1},y)\, d\boldsymbol{u}_{-1} &= \int_{U\setminus(0,1]} \mathbb{E}[G(Y,\boldsymbol{u},\boldsymbol{X}_{-1})]\, d\boldsymbol{u}_{-1} \\ &= \int_{U\setminus(0,1]} \mathbb{E}[F(Y,\boldsymbol{u})]\, d\boldsymbol{u}_{-1} < \infty,\end{aligned}$$

where $\boldsymbol{u}_{-1} := (u_2,\dots,u_n) \in U \setminus (0,1]$ and $F_{\boldsymbol{X}_{-1},Y} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is the cumulative distribution function of the joint probability distribution of $\boldsymbol{X}_{-1}$ and Y. Similarly, for $u_2,\dots,u_n$ we have:

$$\begin{aligned}\int_{S_{u_j}} |G(y,\boldsymbol{u},\boldsymbol{x}_{-1})|\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1},y)\, d\boldsymbol{u}_{-j} &= \int_{U\setminus[0,1]} \mathbb{E}[G(Y,\boldsymbol{u},\boldsymbol{X}_{-1})]\, d\boldsymbol{u}_{-j} \\ &= \int_{U\setminus[0,1]} \mathbb{E}[F(Y,\boldsymbol{u})]\, d\boldsymbol{u}_{-j} < \infty,\end{aligned}$$

where for $j > 1$, $\boldsymbol{u}_{-j} := (u_1,\dots,u_{j-1},u_{j+1},\dots,u_n) \in U \setminus [0,1]$. For condition (ii) of Lemma 3, we differentiate G partially with respect to y and u_i for $i = 1,\dots,n$ using (3):

$$\begin{aligned}\frac{\partial G}{\partial y}(y,\boldsymbol{u},\boldsymbol{x}_{-1}) &= \partial_y l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1})\,, \\ \frac{\partial G}{\partial u_i}(y,\boldsymbol{u},\boldsymbol{x}_{-1}) &= \partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1}),\end{aligned}$$

which are all continuous for fixed $\boldsymbol{x}_{-1}$ by Remark 1. For condition (iii) of Lemma 3, observe that the integral of $\partial G/\partial y$ in the domain S_y is continuous in y by Remark 1:

$$\begin{aligned}\int_{S_y} \frac{\partial G}{\partial y}(y,\boldsymbol{u},\boldsymbol{x}_{-1})\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, d\boldsymbol{u} &= \int_U \mathbb{E}\left[\frac{\partial G}{\partial y}(y,\boldsymbol{u},\boldsymbol{X}_{-1})\right] d\boldsymbol{u} \\ &= \int_U \mathbb{E}[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]\, d\boldsymbol{u}.\end{aligned}$$

Similarly, by Remark 1 integral of $\partial G/\partial u_1$ in S_{u_1} is continuous in u_1:

$$\begin{aligned}\int_{S_{u_1}} \frac{\partial G}{\partial u_1}(y, \boldsymbol{u}, \boldsymbol{x}_{-1})\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-1} &= \int_{U\setminus(0,1]} \mathbb{E}\left[\frac{\partial G}{\partial u_1}(Y, \boldsymbol{u}, \boldsymbol{X}_{-1})\right] d\boldsymbol{u}_{-1} \\ &= \int_{U\setminus(0,1]} \mathbb{E}[\partial_{u_1} l_{(Y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(Y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]\, d\boldsymbol{u}_{-1},\end{aligned}$$

and the integral of $\partial G/\partial u_j$ in S_{u_j} is continuous in u_j for $j = 2, \ldots, n$:

$$\begin{aligned}\int_{S_{u_j}} \frac{\partial G}{\partial u_j}(y, \boldsymbol{u}, \boldsymbol{x}_{-1})\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-j} &= \int_{U\setminus[0,1]} \mathbb{E}\left[\frac{\partial G}{\partial u_j}(Y, \boldsymbol{u}, \boldsymbol{X}_{-1})\right] d\boldsymbol{u}_{-j} \\ &= \int_{U\setminus[0,1]} \mathbb{E}[\partial_{u_j} l_{(Y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(Y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]\, d\boldsymbol{u}_{-j}.\end{aligned}$$

For condition (iv) of Lemma 3, we show that for $\delta > 0$, the following integrals are finite:

$$\begin{aligned}I_y &:= \int_{S_y} \int_{-\delta}^{\delta} \left|\frac{\partial G}{\partial y}(y + \theta, \boldsymbol{u}, \boldsymbol{x}_{-1})\right| d\theta\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, d\boldsymbol{u}, \\ I_{u_1} &:= \int_{S_{u_1}} \int_{-\delta}^{\delta} \left|\frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{x}_{-1})\right| d\theta\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-1}, \\ I_{u_j} &:= \int_{S_{u_j}} \int_{-\delta}^{\delta} \left|\frac{\partial G}{\partial u_j}(y, \boldsymbol{u} + \theta \boldsymbol{e}_j, \boldsymbol{x}_{-1})\right| d\theta\, dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-j},\end{aligned}$$

for $j = 2, \ldots, n$ where $(\boldsymbol{e}_k)_l = 1$ for $k = l$ and 0 otherwise. Recall that ϕ is a continuous density by Assumption 1 (iii) and $\partial_y l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \geq 0$ because $l_{(y,\boldsymbol{x}_{-1})}$ is monotonically increasing in y. Therefore,

$$\frac{\partial G}{\partial y}(y + \theta, \boldsymbol{u}, \boldsymbol{x}_{-1}) = \partial_y l_{(y+\theta,\boldsymbol{x}_{-1})}(\boldsymbol{u})\phi(l_{(y+\theta,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1}) \geq 0,$$

for all $y \in \mathbb{R}$, $\theta \in (-\delta, \delta)$, $\boldsymbol{u} \in U$ and $\boldsymbol{x}_{-1} \in \mathbb{R}^{n-1}$. We can also see the above inequality, by noting that G is the conditional probability distribution of Y given $\boldsymbol{X}_{-1}$, and therefore its partial derivative w.r.t y is a conditional probability density. Now, observe that:

$$\begin{aligned}I_y &= \int_{S_y} \int_{-\delta}^{\delta} \frac{\partial G}{\partial y}(y + \theta, \boldsymbol{u}, \boldsymbol{x}_{-1})\, d\theta\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, d\boldsymbol{u} \\ &= \int_{S_y} (G(y + \delta, \boldsymbol{u}, \boldsymbol{x}_{-1}) - G(y - \delta, \boldsymbol{u}, \boldsymbol{x}_{-1}))\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, d\boldsymbol{u} \\ &= \int_U \mathbb{E}[G(y + \delta, \boldsymbol{u}, \boldsymbol{X}_{-1}) - G(y - \delta, \boldsymbol{u}, \boldsymbol{X}_{-1})]\, d\boldsymbol{u} \\ &= \int_U \Big(F(y + \delta, \boldsymbol{u}) - F(y - \delta, \boldsymbol{u})\Big)\, d\boldsymbol{u} < \infty.\end{aligned}$$

To prove I_{u_1} is finite, observe that since the integrand is positive, we can change the order of integration by Tonelli's theorem:

$$\begin{aligned}
I_{u_1} &= \int_{-\delta}^{\delta} \int_{S_{u_1}} \left| \frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{x}_{-1}) \right| dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-1}\, d\theta \\
&= \int_{-\delta}^{\delta} \int_{S_{u_1}} \max\left\{ \frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{x}_{-1}), -\frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{x}_{-1}) \right\} dF_{\boldsymbol{X}_{-1},Y}(\boldsymbol{x}_{-1}, y)\, d\boldsymbol{u}_{-1}\, d\theta \\
&= \int_{-\delta}^{\delta} \int_{U\setminus(0,1]} \mathbb{E}\left[\mathbb{E}\left[\max\left\{ \frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{X}_{-1}), -\frac{\partial G}{\partial u_1}(y, \boldsymbol{u} + \theta \boldsymbol{e}_1, \boldsymbol{X}_{-1}) \right\} \;\middle|\; Y = y \right]\right] d\boldsymbol{u}_{-1}\, d\theta < \infty,
\end{aligned}$$

where we note that by Assumption 1 (v), the conditional expectation is bounded. The proof of $I_{u_j} < \infty$ for $j = 2, \ldots, n$ follows the same approach. Therefore, by Lemma 3 we conclude that F is continuously partially differentiable in y and u_i for $i = 1, \ldots, n$ with derivatives:

$$\frac{\partial F}{\partial y}(y, \boldsymbol{u}) = \mathbb{E}\left[\frac{\partial G}{\partial y}(y, \boldsymbol{u}, \boldsymbol{X}_{-1}) \right] \quad \text{and} \quad \frac{\partial F}{\partial u_i}(y, \boldsymbol{u}) = \mathbb{E}\left[\frac{\partial G}{\partial u_i}(y, \boldsymbol{u}, \boldsymbol{X}_{-1}) \right].$$

□

Remark 2 Lemma 1 also holds if $g_{(x_1, \boldsymbol{x}_{-1})}$ is monotonically decreasing in x_1 with a change of sign of the partial derivatives in Equations (1) and (2). The proof of the decreasing case follows using similar steps as the increasing case, and noting that for the decreasing case:

$$G(y, \boldsymbol{u}, \boldsymbol{x}_{-1}) = \mathbb{P}(X_1 \leq l_{(y, \boldsymbol{x}_{-1})}(\boldsymbol{u}) | \boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}) = 1 - \mathbb{P}(g_{\boldsymbol{X}}(\boldsymbol{u}) \leq y | \boldsymbol{X}_{-1} = \boldsymbol{x}_{-1}),$$

and again F can be written in terms of G:

$$F(y, \boldsymbol{u}) = 1 - \mathbb{E}[G(y, \boldsymbol{u}, \boldsymbol{X}_{-1})].$$

Finally, to prove that $I_y < \infty$, we note that:

$$\left| \frac{\partial G}{\partial y}(y + \theta, \boldsymbol{u}, \boldsymbol{x}_{-1}) \right| = -\frac{\partial G}{\partial y}(y + \theta, \boldsymbol{u}, \boldsymbol{x}_{-1}),$$

since $\partial_y l_{(y+\theta, \boldsymbol{x}_{-1})}(\boldsymbol{u}) \leq 0$ for all $y \in \mathbb{R}$, $\theta \in (-\delta, \delta)$, $\boldsymbol{u} \in U$, and $\boldsymbol{x} \in \mathbb{R}^{n-1}$.

Remark 3 If Assumption 1 (i)-(v) are satisfied then, for any $\boldsymbol{u} \in U$, the random variable $Y = g_{\boldsymbol{X}}(\boldsymbol{u})$ has a continuous probability density function given by:

$$f_Y(y) = \mathbb{E}[\partial_y l_{(y, \boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y, \boldsymbol{X}_{-1})}(\boldsymbol{u}) | \boldsymbol{X}_{-1})].$$

To see this, note that for fixed $\boldsymbol{u} \in U$, F and F_Y are equivalent, i.e. $F(y, \boldsymbol{u}) = F_Y(y)$ for all $y \in \mathbb{R}$. The continuous partial derivative of F with respect to y is then given in Lemma 1 as:

$$\frac{\partial F}{\partial y}(y, \boldsymbol{u}) = \frac{\partial F_Y}{\partial y}(y) = f_Y(y) = \mathbb{E}[\partial_y l_{(y, \boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y, \boldsymbol{X}_{-1})}(\boldsymbol{u}) | \boldsymbol{X}_{-1})].$$

Therefore, we see that Assumption 1 (vi) corresponds to the gradient of the distribution function F_Y being greater than that of the lambda function at the point $y = -\rho_\Lambda(\boldsymbol{u})$.

The monotonically decreasing case follows from the same argument with a change of sign. Note that the density $f_Y(y)$ is indeed positive for the decreasing case, because $l_{(y, \boldsymbol{x}_{-1})}$ is decreasing in y and therefore $\partial_y l_{(y, \boldsymbol{x}_{-1})}(\boldsymbol{u}) \leq 0$ for all $y \in \mathbb{R}$, $\boldsymbol{u} \in U$, and $\boldsymbol{x}_{-1} \in \mathbb{R}^{n-1}$.

Lemma 2 *If Assumption 1 (i)-(v) are satisfied then, for any $\boldsymbol{u} \in U$ and $i = 1, \ldots, n$, we have:*

$$\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = y] = -\frac{\mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}{\mathbb{E}[\partial_{y} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) \phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}. \tag{5}$$

Proof. The proof method is inspired by the proof of Lemma 1 in Tsanakas and Millossovich (2016). Our proof, however, considers a portfolio return $g_{\boldsymbol{X}}$ on the set U, whereas Tsanakas and Millossovich (2016) do not use asset weights.

Consider the following expectation for an absolutely integrable function k, i.e. $\int_{\mathbb{R}} |k(y)| \mathrm{d}y < \infty$, and fixed $\boldsymbol{u}$:

$$\begin{aligned}
\mathbb{E}[k(Y)\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u})] &= \mathbb{E}[\mathbb{E}[k(Y)\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid \boldsymbol{X}_{-1}]] \\
&= \mathbb{E}\left[\int_{-\infty}^{+\infty} \partial_{u_i} g_{(x_1,\boldsymbol{X}_{-1})}(\boldsymbol{u}) k(g_{(x_1,\boldsymbol{X}_{-1})}(\boldsymbol{u}))\phi(x_1|\boldsymbol{X}_{-1}) dx_1\right]. \qquad (6)
\end{aligned}$$

We now apply a change of variable:

$$x_1 = l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \iff y = g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u}). \tag{7}$$

For any $\boldsymbol{u} \in U$ and $\boldsymbol{x}_{-1} \in \mathbb{R}^{n-1}$, we can write (7) as:

$$x_1 = l_{(g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u}),\boldsymbol{x}_{-1})}(\boldsymbol{u}) \iff y = g_{(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}),\boldsymbol{x}_{-1})}(\boldsymbol{u}). \tag{8}$$

so that:

$$\frac{\mathrm{d}x_1}{\mathrm{d}y} = \partial_y l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{y=g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})} = \left(\partial_{x_1} g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{x_1=l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})}\right)^{-1}, \tag{9}$$

where we used the representation of derivatives of inverse functions. Next, we compute the partial derivative of the equation $y = g_{(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}),\boldsymbol{x}_{-1})}(\boldsymbol{u})$ in (8) with respect to u_i, $i = 1, \ldots, n$, and note that the derivatives of the LHS are zero, i.e. $\partial y/\partial u_i = 0$ for $i = 1, \ldots, n$. For the RHS, we have:

$$\partial_{u_i} g_{(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}),\boldsymbol{x}_{-1})}(\boldsymbol{u}) = \partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \partial_{x_1} g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{x_1=l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})} + \partial_{u_i} g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{x_1=l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})}.$$

From this we deduce that:

$$\partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u}) \partial_{x_1} g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{x_1=l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})} = -\partial_{u_i} g_{(x_1,\boldsymbol{x}_{-1})}(\boldsymbol{u})|_{x_1=l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})}. \tag{10}$$

Using (9) and (10), our expectation in (6) now becomes:

$$\begin{aligned}
\mathbb{E}[k(Y)\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u})] &= \mathbb{E}\left[-\int_{-\infty}^{+\infty} k(y)\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u}) dy\right] \\
&= \mathbb{E}\left[-\int_{-\infty}^{+\infty} k(y)\frac{\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})}{f_Y(y)} f_Y(y) dy\right] \qquad (11) \\
&= -\int_{-\infty}^{+\infty} k(y)\frac{\mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}{f_Y(y)} f_Y(y) dy \qquad (12) \\
&= \mathbb{E}[k(Y)q(Y)],
\end{aligned}$$

where:

$$q(y) = -\frac{\mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}{f_Y(y)}.$$

Notice that we have switched the order of integration and expectation to move from (11) to (12). This can be justified by considering the following integral on the product space $\mathbb{R} \times \mathbb{R}^{n-1}$:

$$I_k := \int_{\mathbb{R}\times\mathbb{R}^{n-1}} |k(y)\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1})\partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\, dy\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1}). \tag{13}$$

If the integral (13) is finite then changing the order of integrals in (12) is justified by Fubini's theorem. Observe that since the integrand is non-negative, we can apply Tonelli's theorem to (13):

$$\begin{aligned}
I_k &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |k(y)\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1})\partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |k(y)||\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1})\partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, dy \\
&= \int_{\mathbb{R}} |k(y)| \int_{\mathbb{R}^{n-1}} |\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1})\partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\, dF_{\boldsymbol{X}_{-1}}(\boldsymbol{x}_{-1})\, dy \\
&= \int_{\mathbb{R}} |k(y)| \left(\mathbb{E}\left[\max\left(\frac{\partial G}{\partial u_i}(y, \boldsymbol{u}, \boldsymbol{X}_{-1}), -\frac{\partial G}{\partial u_i}(y, \boldsymbol{u}, \boldsymbol{X}_{-1}) \right) \right] \right) dy, \qquad (14)
\end{aligned}$$

where, as in the proof of Lemma 1, we used that:

$$\frac{\partial G}{\partial u_i}(y, \boldsymbol{u}, \boldsymbol{x}_{-1}) = \partial_{u_i} l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{x}_{-1})}(\boldsymbol{u})|\boldsymbol{x}_{-1}).$$

By Assumption 1 (v), the expectation in the integrand of (14) is finite and since k is absolutely integrable, we conclude that $I_k < \infty$. Using the explicit form of f_Y from Remark 3, we conclude that:

$$\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = y] = -\frac{\mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}{\mathbb{E}[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]}.$$

□

Remark 4 Using Lemmas 1 and 2 and Remark 3, one can write the derivative of the portfolio return with respect to its composition as:

$$\begin{aligned}
\frac{\partial F}{\partial u_i}(y, \boldsymbol{u}) &= \mathbb{E}[\partial_{u_i} l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})] \\
&= -\mathbb{E}[\partial_y l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})\phi(l_{(y,\boldsymbol{X}_{-1})}(\boldsymbol{u})|\boldsymbol{X}_{-1})]\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = y] \\
&= -f_Y(y)\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = y]. \qquad (15)
\end{aligned}$$

Using Assumption 2, we demonstrate Proposition 1 which proves partial differentiability of lambda quantiles without assuming that $g_{\boldsymbol{X}}$ is neither monotone nor invertible. Instead, we assume that the portfolio return has a continuous density in a neighbourhood of the lambda quantile. Assumption 2 and Proposition 1 are then used to prove Theorem 1 with condition (ii).

Proposition 1 *Suppose Assumption 2* (iii), (iv) *and* (vi) *are satisfied and* Λ *is continuously differentiable in a neighbourhood of* $-\rho_\Lambda(\boldsymbol{u})$. *Then,* ρ_Λ *is continuously partially differentiable in* U *with derivatives:*

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = \left(\frac{\partial H}{\partial y}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u})\right)^{-1} \frac{\partial H}{\partial u_i}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}),$$

where $H(y, \boldsymbol{u}) := F(y, \boldsymbol{u}) - \Lambda(y)$.

Proof. Fix $\boldsymbol{u} \in U$. Then, $g_{\boldsymbol{X}}(u)$ is a continuous random variable in a neighbourhood of $-\rho_\Lambda(\boldsymbol{u})$. Therefore, it holds that:

$$F(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}) = \Lambda(-\rho_\Lambda(\boldsymbol{u})).$$

Then, $y = -\rho_\Lambda(\boldsymbol{u})$ is a solution of $H(y, \boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in U$, i.e. $H(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in U$. Note that H is continuously partially differentiable in y and u_i for $i = 1, \ldots, n$ since by assumption, both f_Y and Λ' are continuous in the same neighbourhood of $-\rho_\Lambda(\boldsymbol{u})$. Also, observe that:

$$\left.\frac{\partial H}{\partial y}(y, \boldsymbol{u})\right|_{y=-\rho_\Lambda(\boldsymbol{u})} = f_Y(-\rho_\Lambda(\boldsymbol{u})) - \Lambda'(-\rho_\Lambda(\boldsymbol{u})) > 0$$

by Assumption 2 (vi). Applying the implicit function theorem to H and using Assumption 2 (iv), we conclude that $-\rho_\Lambda$ is continuously partially differentiable in U with derivatives:

$$\frac{\partial(-\rho_\Lambda)}{\partial u_i}(\boldsymbol{u}) = -\left(\frac{\partial H}{\partial y}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u})\right)^{-1} \frac{\partial H}{\partial u_i}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}).$$

□

We now define the return density adjustment which is important for both the risk contributions and Euler decomposition of lambda quantiles. Also, we will show that the return density adjustment evaluated at the point $y = -\rho_\Lambda(\boldsymbol{u})$ corresponds to the homogeneity degree of lambda quantiles.

Definition 3 For a continuous random variable $Y \in \mathcal{X}$ and continuously differentiable lambda function, define the *return density adjustment of* Y *with respect to* Λ as the function $\eta_{\Lambda,Y} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\eta_{\Lambda,Y}(y) := \frac{f_Y(y)}{f_Y(y) - \Lambda'(y)}, \tag{16}$$

where we use the convention that $\frac{1}{0} = +\infty$.

Remark 5 Observe that $\eta_{\Lambda,Y}(y) = 1$ at a given y if, and only if, $\Lambda'(y) = 0$. Also, for a fixed lambda function, the return density adjustment $\eta_{\Lambda,Y}$ is law invariant with respect to the random variable Y, that is, for random variables $Y_1, Y_2 \in \mathcal{X}$ that are equal in distribution, i.e. $Y_1 \overset{d}{=} Y_2$, it holds $\eta_{\Lambda,Y_1}(y) = \eta_{\Lambda,Y_2}(y)$ for all $y \in \mathbb{R}$.

The following theorem states the conditions under which lambda quantiles are continuously partially differentiable in the space of portfolio compositions and provides closed form formulae of lambda quantile risk contributions. We prove Theorem 1 using two different approaches, which correspond to the use of Assumption 1 and Assumption 2. Also, note that the assumptions for the lambda function are different for each approach.

THEOREM 1 *Suppose either:*

(i) Λ *is continuously differentiable on* $\mathbb{R}$ *and Assumption 1 is satisfied,* *or*

(ii) Λ *is continuously differentiable in a neighbourhood of* $y = -\rho_\Lambda(\boldsymbol{u})$ *and Assumption 2 is satisfied.*

Then, ρ_Λ *is continuously partially differentiable in* U *with partial derivatives:*

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -\rho_\Lambda(\boldsymbol{u})], \tag{17}$$

for $i = 1, \ldots, n$.

Proof of Theorem 1. Proof using condition (i). By Lemma 1 and Remark 3, $Y = g_{\boldsymbol{X}}(\boldsymbol{u})$ is a continuous random variable with a continuous probability density function f_Y and the partial derivatives $\partial_{u_i} F(y, \boldsymbol{u})$ are continuous in y and $\boldsymbol{u}$ for $i = 1, \ldots, n$. Furthermore, with Assumption 1 (vi), we invoke Proposition 1 to deduce that ρ_Λ is continuously partially differentiable in U with derivatives:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = \frac{\partial_{u_i} F(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u})}{f_Y(-\rho_\Lambda(\boldsymbol{u})) - \Lambda'(-\rho_\Lambda(\boldsymbol{u}))},$$

for $i = 1, \ldots, n$. By Remark 4, we note that:

$$\partial_{u_i} F(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}) = -f_Y(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = -\rho_\Lambda(\boldsymbol{u})],$$

which concludes the proof using condition (i), since:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = -\frac{f_Y(-\rho_\Lambda(\boldsymbol{u}))}{f_Y(-\rho_\Lambda(\boldsymbol{u})) - \Lambda'(-\rho_\Lambda(\boldsymbol{u}))}\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = -\rho_\Lambda(\boldsymbol{u})].$$

Proof using condition (ii). By Proposition 1, ρ_Λ is continuously partially differentiable in U with partial derivatives:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = \left(\frac{\partial H}{\partial y}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u})\right)^{-1} \frac{\partial H}{\partial u_i}(-\rho_\Lambda(\boldsymbol{u}), \boldsymbol{u}),$$

where $H(y, \boldsymbol{u}) := F(y, \boldsymbol{u}) - \Lambda(y)$. By Theorem 1 of Hong (2009), we have:

$$\partial_{u_i} F(y, \boldsymbol{u}) = -f_Y(y)\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}}(\boldsymbol{u}) = y],$$

for $i = 1, \ldots, n$, which are continuous in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$ by Assumption 2 (iv). Furthermore, observe that:

$$\frac{\partial H}{\partial y}(y, \boldsymbol{u}) = f_Y(y) - \Lambda'(y),$$

which, again, is continuous in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$ by Assumption 2 (iii) and by the assumption that Λ is continuously differentiable in a neighbourhood of $y = -\rho_\Lambda(\boldsymbol{u})$. We conclude that the continuous partial derivatives of ρ_Λ are given by:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = -\frac{f_Y(-\rho_\Lambda(\boldsymbol{u}))}{f_Y(-\rho_\Lambda(\boldsymbol{u})) - \Lambda'(-\rho_\Lambda(\boldsymbol{u}))}\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid g_{\boldsymbol{X}} = -\rho_\Lambda(\boldsymbol{u})],$$

for $i = 1, \dots, n$. $\square$

Theorem 1 with condition (ii) is a generalisation of the quantile sensitivity of VaR derived in Theorem 2 of Hong (2009) to the class of lambda quantiles.

For the special case of VaR_λ, we observe that the return density adjustment is equal to one, i.e. $\eta_{\lambda,Y}(y) = 1$ for all $y \in \mathbb{R}$, which leads to the following result.

Corollary 1 *Suppose $\Lambda(x) = \lambda \in (0,1)$ for all $x \in \mathbb{R}$. If Assumption 1 or Assumption 2 is satisfied, then $\rho_\lambda \equiv VaR_\lambda$ is continuously partially differentiable in U with partial derivatives:*

$$\frac{\partial VaR_\lambda}{\partial u_i}(\boldsymbol{u}) = -\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -VaR_\lambda(\boldsymbol{u})], \tag{18}$$

for $i = 1, \dots, n$.

The Proof of Corollary 1 follows straightforwardly from Theorem 1, and is thus omitted. Corollary 1 with Assumption 1 generalises Lemma 5.3 in Tasche (1999) to generic portfolio returns $g_{\boldsymbol{X}}$. Furthermore, even though (18) with Assumption 1 is of the same form as the partial derivative given in Theorem 2 of Hong (2009) (except that here $\boldsymbol{u}$ is multivariate as opposed to one-dimensional), the assumptions used to obtain these results differ. In Hong (2009), the simulation output is assumed to be a continuous random variable. Corollary 1 with Assumption 1, does not require this assumption, we do, however, assume that at least one of the returns X_i has a continuous density.

The following example shows that the risk contributions of VaR_λ in Tasche (1999), who considers linear portfolios, are a special case of those of the lambda quantiles.

Example 3 *For the linear portfolio operator given in Example 2 (i), we fix the random vector $\boldsymbol{X}$ to obtain the portfolio return:*

$$g_{\boldsymbol{X}}(\boldsymbol{u}) := u_1 X_1 + u_2 X_2.$$

Then the lambda quantile's risk contribution of asset i to the portfolio is given by:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}[X_i | u_1 X_1 + u_2 X_2 = -\rho_\Lambda(\boldsymbol{u})].$$

If $\Lambda(x) = \lambda \in (0,1)$ is a constant, then we retrieve partial derivatives of VaR_λ as obtained in Tasche (1999), Gourieroux et al. (2000), and Hallerbach (2003):

$$\frac{\partial VaR_\lambda}{\partial u_i}(\boldsymbol{u}) = -\mathbb{E}[X_i | u_1 X_1 + u_2 X_2 = -VaR_\lambda(\boldsymbol{u})].$$

So far we proved in Theorem 1 that, under smoothness assumptions, lambda quantiles are continuously partially differentiable in U. Next, we consider differentiability in subsets of U. This is important for situations when portfolio selection is restricted to specific classes of compositions, or in other words, to subsets of U. In the following proposition, we use Assumption 1 to prove that lambda quantiles are continuously partially differentiable in subsets of U. This result allows for flexibility in the choice of lambda function of lambda quantiles. Recall that in Theorem 1 with condition (i), $\Lambda(\cdot)$ was assumed to be continuously differentiable in $\mathbb{R}$. In Proposition 2, we only require continuous differentiability of the lambda function within an interval, thus generalising to lambda functions that may be discontinuous on $\mathbb{R}$.

Consider a subset $V \subset U$ such that for all $\boldsymbol{v} \in V$, the smallest intersection point of F and $\Lambda(\cdot)$ lies in the interval $(\alpha, \beta) \subset \mathbb{R}$, i.e. $-\rho_\Lambda(\boldsymbol{v}) \in (\alpha, \beta)$ for all $\boldsymbol{v} \in V$. The following result provides the necessary conditions to ensure lambda quantiles are continuously partially differentiable in V, and hence allow us to calculate risk contributions of lambda quantiles in V.

Proposition 2 *Assume that:*

(i) Λ *is continuously differentiable in the interval* $(\alpha, \beta) \subset \mathbb{R}$*;*
(ii) Assumption 1 is satisfied;
(iii) $-\rho_\Lambda(\boldsymbol{v}) \in (\alpha, \beta)$ *for all* $\boldsymbol{v} \in V \subset U$*.*

Then, ρ_Λ *is continuously partially differentiable in* V*, where the partial derivatives are given by:*

$$\frac{\partial \rho_\Lambda}{\partial v_i}(\boldsymbol{v}) = -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{v}))\mathbb{E}[\partial_{v_i} g_{\boldsymbol{X}}(\boldsymbol{v}) \mid Y = -\rho_\Lambda(\boldsymbol{v})], \tag{19}$$

for $i = 1, \ldots, n$*.*

Proof. The proof follows the same approach as that of Theorem 1 with condition (i). The major difference is that we calculate partial derivatives with respect to y in the interval (α, β) to ensure $\Lambda'(y)$ exists and is well defined. We further point out that for fixed $\boldsymbol{v} \in V$, $-\rho_\Lambda(\boldsymbol{v})$ is the smallest intersection point of F and Λ on (α, β), since they're both continuous on this interval, i.e. we have:

$$F(-\rho_\Lambda(\boldsymbol{v}), \boldsymbol{v}) = \Lambda(-\rho_\Lambda(\boldsymbol{v})),$$

for all $\boldsymbol{v} \in V$. □

4. Euler decomposition and the generalised Euler allocation rule

In this section, we aggregate the risk contributions of lambda quantiles to prove a relationship known as the *Euler decomposition* for lambda quantiles. The Euler decomposition is, for homogeneous risk measures in U, the property that the risk measure, scaled by its homogeneity degree, can be written as a weighted sum of its partial derivatives. We show that the homogeneity degree of lambda quantiles is determined by the portfolio composition, the density function of the portfolio return, and the gradient of the lambda function, both evaluated at the lambda quantile. This implies that lambda quantile homogeneity degree is not a constant over choices of portfolio compositions or lambda functions. Furthermore, the homogeneity degree varies across different distributions of the portfolio return. This is in contrast to other risk measures, such as VaR, where the homogeneity degree is constant.

In risk measure theory, the property of homogeneity is typically studied for risk measures defined on the space of random variables. A risk measure, defined on random variables – representing asset returns – is positive homogeneous, if the risk of an asset scales linearly with its return, e.g. doubling the return doubles the position's risk. In this case, the risk measure has a homogeneity degree of 1. The positive homogeneity (of degree 1) property of risk measures forms part of the definition of coherent risk measures, introduced in the seminal paper by Artzner *et al.* (1999). However, the property of a risk measure having a homogeneity degree of 1 has been questioned in Föllmer and Schied (2002). They argue that large multiples of a position may introduce additional liquidity risk and, therefore, the position risk and size may not increase linearly.

In this paper, we study the homogeneity property of lambda quantiles on the set U and explore the relationship between asset weights and portfolio risk for lambda quantiles. Therefore, our treatment of the homogeneity property should not be confused with homogeneity of risk measures defined on set of random variables $\mathcal{X}$.

The Euler decomposition of a positive homogeneous (of degree 1) risk measure, defined on the set of random variables, is known as the *Euler allocation rule* (Patrik *et al.* 1999, Denault 2001, Tasche 2007), which is one of the most well established allocation methods in risk measure theory. This allocation rule assigns economic capital to assets using directional derivatives (in the direction of asset returns) of positive homogeneous risk measures. Furthermore, Euler allocation rule is used for portfolios with linear risk aggregation or linear portfolio operators in Tasche (2007) and Tsanakas (2009).

Our treatment considers a more general setup, where we consider a generic portfolio operator and lambda quantiles, that are risk measures with homogeneity degree not necessarily equal to 1.

DEFINITION 4 Let $\tau \in \mathbb{R}$ be fixed. For an event $E \in \mathcal{F}$, an operator $g : U \times \mathcal{X}^n \to \mathcal{X}$ is said to be $\mathbb{P}$-*almost surely homogeneous of degree* τ *in* U *and in the event* E if for all $\boldsymbol{X} \in \mathcal{X}^n$, and all $\boldsymbol{u} \in U$ and $t > 0$ with $t\boldsymbol{u} \in U$, we have:

$$\mathbb{P}\left(\{\omega \in E : g[t\boldsymbol{u}, \boldsymbol{X}](\omega) = t^\tau g[\boldsymbol{u}, \boldsymbol{X}](\omega)\}\right) = 1\,. \tag{20}$$

If $E = \Omega$, we say g is $\mathbb{P}$-*a.s.* τ-*homogeneous in* U.

Observe that if g is $\mathbb{P}$-a.s. τ-homogeneous in U and in the event $E \in \mathcal{F}$, then $g_{\boldsymbol{X}}$ is also $\mathbb{P}$-a.s. τ-homogeneous in U and in the event E for all $\boldsymbol{X} \in \mathcal{X}^n$. Moreover, any portfolio operator that is linear in U is $\mathbb{P}$-a.s. 1-homogeneous in U. For a non-linear operator g, however, the homogeneity property may not hold for all $\omega \in \Omega$. Positive homogeneity is traditionally defined for real-valued functions $r : U \to \mathbb{R}$ such that for all $t > 0$ and $\boldsymbol{u} \in U$ with $t\boldsymbol{u} \in U$, one has $r(t\boldsymbol{u}) = t^\tau r(\boldsymbol{u})$ for some $\tau \in \mathbb{R}$ (see Definition 7 in Appendix A). In contrast, Definition 4 applies to functions which map onto random variables, where there may exist outcomes ω for which the operator g is not homogeneous in U. Therefore, $\mathbb{P}$-a.s. homogeneity is especially appealing to non-linear portfolio operators.

THEOREM 2 *Suppose* Λ *is continuously differentiable on* $\mathbb{R}$*,* $g_{\boldsymbol{X}}$ *is* $\mathbb{P}$*-a.s.* τ*-homogeneous in* U *and* $g_{\boldsymbol{x}}$ *is totally differentiable in* U *for any* $\boldsymbol{x} \in \mathbb{R}^n$*. If either Assumption 1 or Assumption 2 is satisfied, then for all* $\boldsymbol{u} \in U$*,* ρ_Λ *satisfies:*

$$\tau\, \eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\, \rho_\Lambda(\boldsymbol{u}) = \sum_{i=1}^n u_i \frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}). \tag{21}$$

Furthermore, ρ_Λ *is homogeneous in* U *of degree* $\tau\, \eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))$ *in the sense of Definition 7 in Appendix A.*

Proof. If $g_{\boldsymbol{X}}$ is $\mathbb{P}$-a.s. τ-homogeneous in U and $g_{\boldsymbol{x}}$ is totally differentiable in U for any $\boldsymbol{x} \in \mathbb{R}^n$, then by Euler's homogeneous function theorem, the Euler decomposition for almost all $\omega \in \Omega$ of the real functions $g_{\boldsymbol{X}(\omega)} : U \to \mathbb{R}$ is

$$\tau\; g_{\boldsymbol{X}(\omega)}(\boldsymbol{u}) = \sum_{i=1}^n u_i \partial_{u_i} g_{\boldsymbol{X}(\omega)}(\boldsymbol{u}), \tag{22}$$

for all $\boldsymbol{u} \in U$. We note that (22) is equivalent to

$$\tau g_{\boldsymbol{X}}(\boldsymbol{u}) = \sum_{i=1}^n u_i \partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \quad \mathbb{P}\text{-a.s.}, \tag{23}$$

for all $\boldsymbol{u} \in U$. In (23), we have equivalence of two random variables in a $\mathbb{P}$-a.s. sense, thus they have $\mathbb{P}$-a.s. equal conditional expectations, i.e.

$$\mathbb{E}[\tau g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y] = \mathbb{E}\left[\sum_{i=1}^n u_i \partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y\right] \quad \mathbb{P}\text{-a.s.}, \tag{24}$$

where $Y \in \mathcal{X}$. Note that under Assumption 1, we apply Theorem 1 with condition (i), and under Assumption 2, we apply Theorem 1 with condition (ii), to obtain partial derivatives of ρ_Λ. Recall expression (17) from Theorem 1:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) = -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -\rho_\Lambda(\boldsymbol{u})],$$

for $i = 1, \ldots, n$. Note that the conditioning event $Y = -\rho_\Lambda(\boldsymbol{u})$ in the expectation is the same for all i. Therefore, summing the weighted risk contributions over i and using (24), we obtain

$$\begin{aligned}
\sum_{i=1}^n u_i \frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}) &= -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u})) \sum_{i=1}^n u_i \mathbb{E}[\partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -\rho_\Lambda(\boldsymbol{u})] \\
&= -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}\Bigg[\sum_{i=1}^n u_i \partial_{u_i} g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -\rho_\Lambda(\boldsymbol{u})\Bigg] \\
&= -\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\mathbb{E}[\tau g_{\boldsymbol{X}}(\boldsymbol{u}) \mid Y = -\rho_\Lambda(\boldsymbol{u})] \\
&= \tau\, \eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))\, \rho_\Lambda(\boldsymbol{u}),
\end{aligned}$$

where the last equation holds by continuity of F_Y. □

Theorem 2 has several interesting implications. To begin with, the homogeneity degree of a lambda quantile is $\tau\, \eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))$, a composition of the homogeneity degree of the portfolio return τ and the return density adjustment $\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))$. Thus, the homogeneity degree of ρ_Λ of a linear portfolio operator (i.e. $\tau = 1$) is $\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))$. It is straightforward that the homogeneity degree of ρ_Λ with a constant lambda function, $\Lambda(x) = \lambda \in (0,1)$, and for a linear portfolio operator is precisely the homogeneity degree of the VaR_λ measure, that is 1. Indeed, for $\mathbb{P}$-a.s. 1-homogeneous portfolio operators, ρ_Λ is 1-homogeneous if, and only if, $\Lambda'(y) = 0$ for all $y \in \mathbb{R}$. Note that for VaR_λ the homogeneity degree is independent of the portfolio composition $\boldsymbol{u}$. This is in contrast to a non-constant Λ function, in which case the lambda quantile homogeneity degree may differ for each portfolio composition $\boldsymbol{u} \in U$. Moreover, for fixed portfolio composition $\boldsymbol{u}$, the homogeneity degree of lambda quantiles may change for different choices of the Λ function.

We now use Theorem 2 to define a new capital allocation rule, which generalises the well-known Euler allocation. For a linear portfolio operator, risk contributions calculated as directional derivatives of positive homogeneous risk measures of degree 1 are known as Euler contributions (Tasche 2007). Furthermore, the assignment of capital using Euler contributions is known as Euler allocation. Defining for Euler contributions is that they possess the full allocation property, i.e. their sum over all assets equals the risk measure. We propose a generalisation of Euler contributions that is compatible with η-homogeneous risk measures, for any $\eta \in \mathbb{R}$, and generic portfolio operators, thus applicable to lambda quantiles, and satisfies the full allocation property.

Definition 5 Suppose the portfolio return $g_{\boldsymbol{X}}$ is $\mathbb{P}$-a.s. τ-homogeneous in U, $\Phi : \mathcal{X} \to \mathbb{R}$ is a positive homogeneous risk measure[1] of degree $\eta \in \mathbb{R}$ and $\mathbf{1} := (1, \ldots, 1)$. Then, the functionals $\psi_j : \mathcal{X} \to \mathbb{R}$ defined by:

$$\psi_j(g_{\boldsymbol{X}}(\mathbf{1})) := \frac{1}{\tau\eta} \frac{\partial \Phi}{\partial u_j}(g_{\boldsymbol{X}}(\boldsymbol{u}))\bigg|_{\boldsymbol{u}=\mathbf{1}} \quad \text{for} \quad j = 1, \ldots, n, \tag{25}$$

[1]Here, we use the definition of η-positive homogeneity for risk measures defined on the set of random variables, i.e. $\Phi(tX) = t^\eta \Phi(X)$ for all $t > 0$ and all $X \in \mathcal{X}$ (Artzner *et al.* 1999).

are called *generalised Euler contributions.* Furthermore, we call the process of allocating capital to sub-portfolios using generalised Euler contributions, the *generalised Euler allocation rule.*

Note that Euler contributions (Tasche 2007) and the Euler allocation rule (Patrik *et al.* 1999, Denault 2001, Tasche 2007) are special cases of Definition 5 for $\eta = \tau = 1$.

PROPOSITION 3 *Suppose Λ is continuously differentiable on $\mathbb{R}$, $g_{\boldsymbol{X}}$ is $\mathbb{P}$-a.s. τ-homogeneous in U and $g_{\boldsymbol{x}}$ is totally differentiable in U for any $\boldsymbol{x} \in \mathbb{R}^n$. Also, suppose that either Assumption 1 or Assumption 2 is satisfied. Then, the generalised Euler contributions of the lambda quantile are given by:*

$$\psi_j(g_{\boldsymbol{X}}(\mathbf{1})) = -\frac{1}{\tau}\mathbb{E}[\partial_{u_j} g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})] \quad \textit{for} \quad j = 1, \ldots, n. \tag{26}$$

Furthermore, allocations $\psi_j(\cdot)$ define a generalised Euler allocation rule for lambda quantiles with the full allocation property:

$$\sum_{j=1}^{n} \psi_j(g_{\boldsymbol{X}}(\mathbf{1})) = \rho_\Lambda(\mathbf{1}). \tag{27}$$

Proof. For fixed $\boldsymbol{X} \in \mathcal{X}^n$, define $\Phi_\Lambda : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ by $\Phi_\Lambda(g_{\boldsymbol{X}}(\boldsymbol{u})) = \rho_\Lambda(\boldsymbol{u})$, for all $\boldsymbol{u} \in U$. Then, we obtain using Theorem 1, in the third equality:

$$\begin{aligned}
\psi_j(g_{\boldsymbol{X}}(\mathbf{1})) &= \frac{1}{\tau \eta_{\Lambda, g_{\boldsymbol{X}}(\mathbf{1})}(-\rho_\Lambda(\mathbf{1}))} \frac{\partial \Phi_\Lambda}{\partial u_j}(g_{\boldsymbol{X}}(\boldsymbol{u}))\Big|_{\boldsymbol{u}=\mathbf{1}} \\
&= \frac{1}{\tau \eta_{\Lambda, g_{\boldsymbol{X}}(\mathbf{1})}(-\Phi_\Lambda(g_{\boldsymbol{X}}(\mathbf{1})))} \frac{\partial \rho_\Lambda}{\partial u_j}(\boldsymbol{u})\Big|_{\boldsymbol{u}=\mathbf{1}} \\
&= -\frac{\eta_{\Lambda, g_{\boldsymbol{X}}(\mathbf{1})}(-\Phi_\Lambda(g_{\boldsymbol{X}}(\mathbf{1})))}{\tau \eta_{\Lambda, g_{\boldsymbol{X}}(\mathbf{1})}(-\Phi_\Lambda(g_{\boldsymbol{X}}(\mathbf{1})))} \mathbb{E}[\partial_{u_j} g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})] \\
&= -\frac{1}{\tau}\mathbb{E}[\partial_{u_j} g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})].
\end{aligned}$$

Observe that by the $\mathbb{P}$-a.s. τ-homogeneity property of $g_{\boldsymbol{X}}$, we can write

$$\begin{aligned}
\sum_{j=1}^{n} \psi_j(g_{\boldsymbol{X}}(\mathbf{1})) &= -\frac{1}{\tau}\sum_{j=1}^{n} \mathbb{E}[\partial_{u_j} g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})] \\
&= -\frac{1}{\tau}\mathbb{E}\Big[\sum_{j=1}^{n} \partial_{u_j} g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})\Big] \\
&= -\frac{1}{\tau}\mathbb{E}[\tau g_{\boldsymbol{X}}(\mathbf{1}) \mid g_{\boldsymbol{X}}(\mathbf{1}) = -\rho_\Lambda(\mathbf{1})] \\
&= \rho_\Lambda(\mathbf{1}) = \Phi_\Lambda(g_{\boldsymbol{X}}(\mathbf{1})).
\end{aligned}$$

□

4.1. *Interpretation of homogeneity degrees*

The homogeneity degree of lambda quantiles in U is dictated by the slope of the lambda function $\Lambda(y)$ at the point $y = -\rho_\Lambda(\boldsymbol{u})$, for the portfolio composition $\boldsymbol{u}$. If the lambda function represents a benchmark return distribution, then the lambda quantile homogeneity degree changes day-over-day depending on the benchmark's performance. Furthermore, the homogeneity degree also changes as the portfolio is rebalanced. Therefore, a position's risk and size may not scale linearly under all market conditions (measured by the benchmark) and portfolio selections. We formulate intuition behind this observation.

We begin by assuming that the lambda function represents the equity market's return distribution, here denoted by $\Lambda_m : \mathbb{R} \to (0, 1)$. Thus, $\Lambda_m(y)$ is, for example, the probability that the equity market observes a return less than or equal to $y \in \mathbb{R}$, where the equity market return may be tracked, for example, using a market index or aggregated stock returns. Suppose that we hold an equally-weighted portfolio of $n \geq 2$ long equity positions, so that the portfolio composition is given by $\boldsymbol{u}_n = (\frac{1}{n}, \dots, \frac{1}{n}) \in U$. Also, let $X_i \in \mathcal{X}$ represent the return of position i. Then, the portfolio return is given by:

$$R := g_{\boldsymbol{X}}(\boldsymbol{u}_n) = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

and the lambda quantile of our portfolio with respect to the benchmark Λ_m is $\rho_{\Lambda_m}(\boldsymbol{u}_n)$. Next, we increase the position size of each equity in our portfolio by the same number of units $(t-1)/n$, where $t \geq 2$ is an integer, resulting in a portfolio composition of $(\frac{t}{n}, \dots, \frac{t}{n}) = t\boldsymbol{u_n}$. According to Theorem 2, the lambda quantile of our new portfolio is $\rho_{\Lambda_m}(t\boldsymbol{u}_n) = t^{\eta} \rho_{\Lambda_m}(\boldsymbol{u}_n)$, with homogeneity degree η given by:

$$\eta = \frac{f_R(-\rho_{\Lambda_m}(\boldsymbol{u}_n))}{f_R(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) - \Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n))},$$

where f_R is the probability density function of our portfolio return R. For simplicity of illustration, we assume the following:

(i) R has a continuous density in a neighbourhood of $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$. This is equivalent to Assumption 2 (iii).
(ii) $f_R(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) > \Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n)))$. This is equivalent to Assumption 2 (vi).
(iii) There exists $y_1 < y_2$, such that $\Lambda_m(\cdot)$ is increasing on (y_1, y_2) and constant otherwise.

Under these assumptions, the homogeneity degree satisfies $\eta \geq 1$. We now discuss and interpret the cases $\eta = 1$ and $\eta > 1$ separately.

Notice that $\eta = 1$ if, and only if, $\Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) = 0$. Therefore, our lambda quantile ρ_{Λ_m} is a 1-homogeneous risk measure if, and only if, $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$ lies in a region where $\Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) = 0$. Since Λ_m is a bounded return distribution, its gradient is 0 on its right and left tails. Here, we focus on the left tail, denoted by $S_{tail} := \{y \in \mathbb{R}^- : y \leq y_1\}$, since the lambda quantile is the smallest intersection point of F_R and Λ_m, which lies on the negative part of the return axis. If this intersection point is a negative return, then the lambda quantile is a positive loss amount. Note that $\Lambda'(y) = 0$ for all $y \in S_{tail}$. Figure 1 illustrates this scenario, where the smallest intersection point of F_R and Λ_m is indeed a negative return and the portfolio risk $\rho_{\Lambda_m}(\boldsymbol{u}_n)$ is a positive loss amount. Thus, in the case $\eta = 1$, our portfolio risk scales linearly with asset weights, i.e. $\rho_{\Lambda_m}(t\boldsymbol{u}) = t\rho_{\Lambda_m}(\boldsymbol{u})$ for all $t > 0$ and $\boldsymbol{u} \in U$ (since our portfolio choice $\boldsymbol{u}_n$ is arbitrary). We use the notation $\lambda_{tail} := \Lambda_m(-\rho_{\Lambda_m}(\boldsymbol{u}_n))$, when $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$ lies on the left tail of Λ_m. Then, the probability of the portfolio return exceeding $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$, or equivalently, the probability of the portfolio loss not exceeding $\rho_{\Lambda_m}(\boldsymbol{u}_n)$ is $\mathbb{P}(R > -\rho_{\Lambda_m}(\boldsymbol{u}_n)) = 1 - \lambda_{tail}$. Note that λ_{tail} is fixed since the left tail is assumed to be flat. Also, λ_{tail} is independent of the choice of portfolio composition $\boldsymbol{u}_n$, as long as $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$ lies on the left tail of Λ_m.

On the other hand, $\eta > 1$ if, and only if, $\Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) > 0$. Therefore, lambda quantile ρ_{Λ_m} is not 1-homogeneous if, and only if, $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$ lies in a region where $\Lambda_m'(-\rho_{\Lambda_m}(\boldsymbol{u}_n)) > 0$. Since, by assumption, Λ_m is increasing in (y_1, y_2), its gradient is strictly positive on the body of the

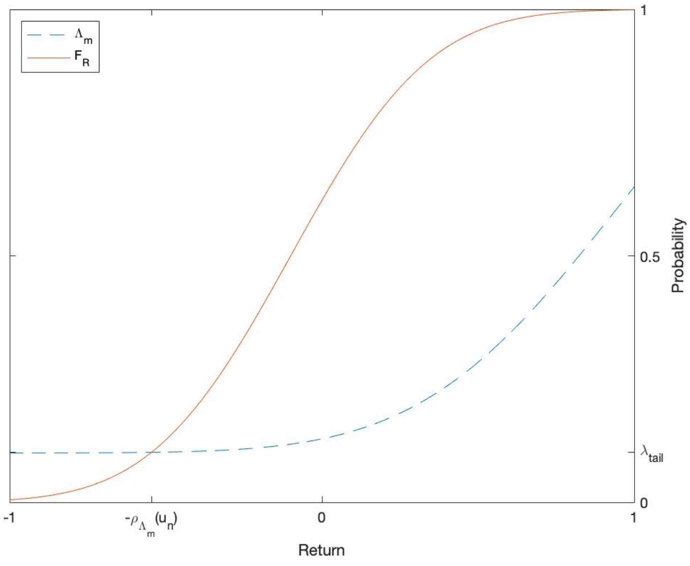

Figure 1. Homogeneity degree $\eta = 1$

distribution, i.e. everywhere except for its right and left tails. We define the body of Λ_m as the set $S_{body} := \{y \in \mathbb{R}^- : y \in (y_1, y_2)\}$. Again, we note that we are only interested in the negative part of the return axis, which corresponds to positive loss amounts. Therefore, our portfolio risk scales of order η with asset weights, i.e. $\rho_{\Lambda_m}(t\boldsymbol{u}) = t^\eta \rho_{\Lambda_m}(\boldsymbol{u})$ for all $t > 0$ and $\boldsymbol{u} \in U$. We use the notation $\lambda_{body} := \Lambda_m(-\rho_{\Lambda_m}(\boldsymbol{u}_n))$, when $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$ lies in the body of Λ_m. Then, by definition the probability of the portfolio loss not exceeding $\rho_{\Lambda_m}(\boldsymbol{u}_n)$ is $\mathbb{P}(R > -\rho_{\Lambda_m}(\boldsymbol{u}_n)) = 1 - \lambda_{body}$. This case, i.e. $\eta > 1$, is illustrated in Figure 2. Note that, contrary to λ_{tail}, λ_{body} is not a constant and depends on the portfolio composition.

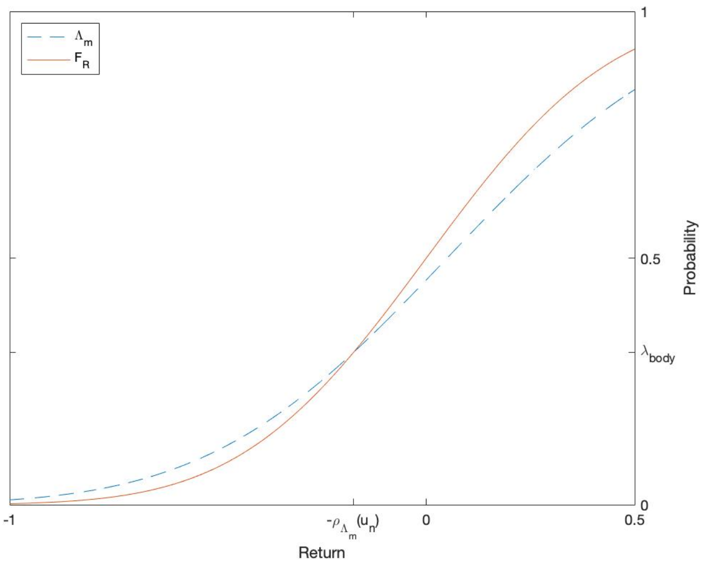

Figure 2. Homogeneity degree $\eta > 1$

We have seen that $\eta = 1$ if, and only if, $\mathbb{P}(R \leq -\rho_{\Lambda_m}(\boldsymbol{u}_n)) = \lambda_{tail}$, and $\eta > 1$ if, and only if, $\mathbb{P}(R \leq -\rho_{\Lambda_m}(\boldsymbol{u}_n)) = \lambda_{body}$. Observe that $0 < \lambda_{tail} < \lambda_{body} < 1$, since we have kept the lambda function Λ_m fixed in our arguments above. Therefore, the increase in the homogeneity degree from

λ_{tail} to λ_{body}, as we move along the negative return axis towards the origin, can be explained by an increase in the probability of exceeding the loss amount $\rho_{\Lambda_m}(\boldsymbol{u}_n)$. Recall that the probabilities λ_{tail} and λ_{body} are determined by the evaluation of the lambda function at the point $-\rho_{\Lambda_m}(\boldsymbol{u}_n)$, i.e. the portfolio return. In other words, they are determined by the likeliness of the equity market observing the loss amount $\rho_{\Lambda_m}(\boldsymbol{u}_n)$. With this understanding, we conclude that the probabilities λ_{tail} and λ_{body} are determined by the "market" as opposed to being kept constant, which is the current practice for both VaR and Expected Shortfall. If the market suggests a small probability of exceeding a portfolio loss, i.e. λ_{tail}, then that portfolio composition is 1-homogeneous in U and the portfolio risk scales linearly with asset weights. Otherwise, the portfolio risk scales with asset weights of order greater than 1, where the order, that is the homogeneity degree, depends on the market conditions through the lambda function. This implies that, for portfolio compositions that are riskier than the market, doubling its weights increases the risk by a factor of more than two, indeed by 2^η. In contrast, note that $VaR_{\lambda_{tail}}$ and $VaR_{\lambda_{body}}$ have the same homogeneity degree of 1, even though they represent different possible losses at different probabilities.

Next, we consider two portfolio compositions $\boldsymbol{u} = (u_1, \ldots, u_n) \in U$ and $\boldsymbol{v} = (v_1, \ldots, v_n) \in U$, with $\boldsymbol{u} \neq \boldsymbol{v}$. Both $\boldsymbol{u}$ and $\boldsymbol{v}$ represent asset weights for the same group of assets indexed by $1, \ldots, n$. We assume that $\rho_{\Lambda_m}(\boldsymbol{u}) > \rho_{\Lambda_m}(\boldsymbol{v})$, i.e. portfolio composition $\boldsymbol{u}$ is "riskier" than $\boldsymbol{v}$. Further, let $\Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{u})) = 0$ and $\Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{v})) > 0$. Then, using the above arguments, we have that $\eta_{\boldsymbol{u}} = 1$ and $\eta_{\boldsymbol{v}} > 1$, where $\eta_{\boldsymbol{u}}$ and $\eta_{\boldsymbol{v}}$ are the homogeneity degrees of $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively. Analogous to above, we define the portfolio returns $R_{\boldsymbol{u}}$ and $R_{\boldsymbol{v}}$ by:

$$R_{\boldsymbol{u}} := g_{\boldsymbol{X}}(\boldsymbol{u}) = \sum_{i=1}^{n} u_i X_i \quad \text{and} \quad R_{\boldsymbol{v}} := g_{\boldsymbol{X}}(\boldsymbol{v}) = \sum_{i=1}^{n} v_i X_i$$

and further assume that:

(i) $R_{\boldsymbol{u}}$ and $R_{\boldsymbol{v}}$ have continuous densities in neighbourhoods of $-\rho_{\Lambda_m}(\boldsymbol{u})$ and $-\rho_{\Lambda_m}(\boldsymbol{v})$ respectively.
(ii) $f_{R_{\boldsymbol{u}}}(-\rho_{\Lambda_m}(\boldsymbol{u})) > \Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{u})))$ and $f_{R_{\boldsymbol{v}}}(-\rho_{\Lambda_m}(\boldsymbol{v})) > \Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{v})))$.
(iii) There exists $y_1 < y_2$, such that $\Lambda_m(\cdot)$ is increasing on (y_1, y_2) and otherwise constant.

Using the same arguments as above, we conclude that $\rho_{\Lambda_m}(\boldsymbol{u})$ lies in the tail and $\rho_{\Lambda_m}(\boldsymbol{v})$ in the body of Λ_m. Thus, we denote $\lambda_{tail} := \Lambda_m(-\rho_{\Lambda_m}(\boldsymbol{u}))$ and $\lambda_{body} := \Lambda_m(-\rho_{\Lambda_m}(\boldsymbol{v}))$, where again $0 < \lambda_{tail} < \lambda_{body} < 1$. This scenario is illustrated in Figure 3, which shows that $-\rho_{\Lambda_m}(\boldsymbol{u}) < -\rho_{\Lambda_m}(\boldsymbol{v})$, i.e. the return of portfolio composition $\boldsymbol{u}$ is more negative than that of $\boldsymbol{v}$ (and hence the loss of $\boldsymbol{u}$ is greater than the loss of $\boldsymbol{v}$). Figure 3 also shows that $\Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{u})) = 0$ and $\Lambda'_m(-\rho_{\Lambda_m}(\boldsymbol{v})) > 0$, which corresponds to $F_{R_{\boldsymbol{u}}}$ intersecting the left tail of Λ_m and $F_{R_{\boldsymbol{v}}}$ intersecting the increasing part (body) of Λ_m respectively. We conclude that the portfolio composition $\boldsymbol{u}$ has a larger potential loss amount $\rho_{\Lambda_m}(\boldsymbol{u})$, but a smaller probability λ_{tail} of exceeding this loss and homogeneity degree of 1. On the other hand, the portfolio composition $\boldsymbol{v}$ has a smaller potential loss amount $\rho_{\Lambda_m}(\boldsymbol{v})$, but a larger probability λ_{body} of exceeding this loss and a homogeneity degree greater than 1. Therefore, the two portfolio compositions $\boldsymbol{u}$ and $\boldsymbol{v}$ may appeal to different risk appetites, where the former composition enjoys a lower probability of exceeding the portfolio loss and linear position size and risk scaling, but poses a larger potential loss, whilst the latter composition may realise a smaller loss with a higher probability and a position size versus risk scaling of order greater than 1.

This notion of local homogeneity, i.e. the variability of homogeneity degree by portfolio selection preferences and market conditions, is in favour of the criticism that positive homogeneous risk measures defined on random variables have received. Föllmer and Schied (2002), for example, argue that large position multiples may induce additional liquidity risk, causing the portfolio risk to increase non-linearly compared to position size. This criticism is related to position sizes and the inadvertent effects of exiting large positions on asset prices. It is not, however, a criticism on the failure or non-existence of positive homogeneity in certain market conditions. Any positive

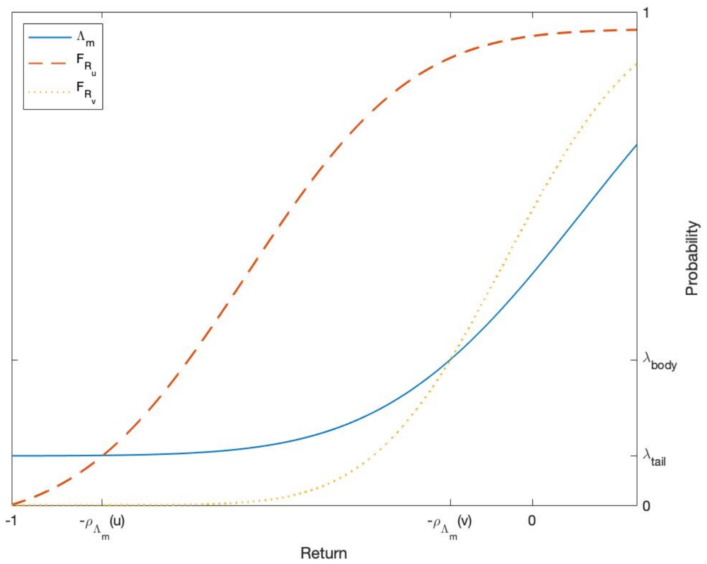

Figure 3. Homogeneity degrees of distinct portfolio compositions $\boldsymbol{u}$ and $\boldsymbol{v}$

homogeneous risk measure defined on random variables is, by definition, universally positive homogeneous, i.e. under all market conditions and across all portfolio choices. Increased liquidity risk of large positions merely addresses one of the pillars of universal positive homogeneity – the position size. It does not address the potential impact of market conditions on the homogeneity degree, or position size-risk relationship.

An example, which illustrates how market conditions may affect homogeneity degree, is times of economic stress, where credit contagion risk is high. In a market with high credit contagion risk, a bank is exposed to and imposes more counterparty risk from and to its counterparties respectively. This increased counterparty risk prevalent in the market may result in losses should a bank's counterparty default and fail to meets its contractual obligations. Increasing position sizes, under such market conditions, may exacerbate counterparty risk exposures and increase the overall portfolio risk non-linearly.

4.2. *Homogeneity of portfolio operators*

The central assumption for the Euler decomposition of lambda quantiles is the $\mathbb{P}$-a.s. τ-homogeneity of $g_{\boldsymbol{X}}$. Thus, in this section, we study properties that ensure $\mathbb{P}$-a.s. homogeneity in U of generic portfolio operators. For this, we first consider operators g of the following additive form to motivate some preliminary results:

$$g[\boldsymbol{u}, \boldsymbol{X}] = \mathfrak{a}[\boldsymbol{u}, \boldsymbol{X}] + \mathfrak{b}(\boldsymbol{u}, \boldsymbol{X}), \tag{28}$$

where $\mathfrak{a} : U \times \mathcal{X}^n \to \mathcal{X}$ and $\mathfrak{b} : U \times \mathcal{X}^n \to \mathbb{R}$. We refer to $\mathfrak{a}$ as the *stochastic* part of g because it depends on a given $\omega \in \Omega$ and $\mathfrak{b}$ as the *deterministic* part of g because it is a constant over all choices of $\omega \in \Omega$ (in Major (2018), $\mathfrak{a}$ and $\mathfrak{b}$ are referred to as the pointwise and constant functions respectively). In what follows and unless otherwise stated, homogeneity of g and $\mathfrak{a}$ is understood in the $\mathbb{P}$-a.s. sense (see Definition 4), whereas homogeneity of $\mathfrak{b}$ and ρ_Λ is understood in the sense of Definition 7 in Appendix A.

Example 4 *Consider the portfolio operator given in Example 2 (ii):*

$$g[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2 - \mathbb{E}[u_1 X_1 + u_2 X_2].$$

Then, we have:

$$\mathfrak{a}[\boldsymbol{u}, \boldsymbol{X}] = u_1 X_1 + u_2 X_2,$$

and

$$\mathfrak{b}(\boldsymbol{u}, \boldsymbol{X}) = -\mathbb{E}[u_1 X_1 + u_2 X_2].$$

In this example, $\mathfrak{a}$ *is* $\mathbb{P}$*-a.s. 1-homogeneous in* U *and* $\mathfrak{b}$ *1-homogeneous in* U *in the sense of Definition 7 in Appendix A. As a result,* g *is* $\mathbb{P}$*-a.s. 1-homogeneous in* U*.*

PROPOSITION 4 *Suppose the portfolio operator* g *can be written in the form* (28)*. Then,* g *is* $\mathbb{P}$*-a.s.* τ*-homogeneous in* U *if* $\mathfrak{a}$ *is* $\mathbb{P}$*-a.s.* τ*-homogeneous in* U *and* $\mathfrak{b}$ *is* τ*-homogeneous in* U*.*

Proof. If $\mathfrak{a}$ is $\mathbb{P}$-a.s. τ-homogeneous in U, then for almost all $\omega \in \Omega$ and for any $t > 0$ and $\boldsymbol{u} \in U$ with $t\boldsymbol{u} \in U$, we have:

$$\begin{aligned} g[t\boldsymbol{u}, \boldsymbol{X}](\omega) =& \mathfrak{a}[t\boldsymbol{u}, \boldsymbol{X}](\omega) + \mathfrak{b}(t\boldsymbol{u}, \boldsymbol{X}) \\ =& t^\tau \mathfrak{a}[\mathbf{u}, \boldsymbol{X}](\omega) + t^\tau \mathfrak{b}(\boldsymbol{u}, \boldsymbol{X}) \\ =& t^\tau g[\mathbf{u}, \boldsymbol{X}](\omega), \end{aligned}$$

and hence g is $\mathbb{P}$-a.s. τ-homogeneous. □

PROPOSITION 5 *Suppose the portfolio operator* g *can be written in the form* (28) *and* $\mathfrak{a}$ *is given by:*

$$\mathfrak{a}[\boldsymbol{u}, \boldsymbol{X}] = \sum_{i=1}^{n} u_i^\tau h_i(X_i),$$

where $h_i : \mathcal{X} \to \mathcal{X}$ *for* $i = 1, \ldots, n$ *and* $\tau \in \mathbb{R}$*. Then,* g *is* $\mathbb{P}$*-a.s.* τ*-homogeneous in* U *if, and only if,* $\mathfrak{b}$ *is* τ*-homogeneous in* U*.*

Proof. We note that $\mathfrak{a}$ is $\mathbb{P}$-a.s. τ-homogeneous in U because of the powers of the u_i's. If $\mathfrak{b}$ is also τ-homogeneous, then g is $\mathbb{P}$-a.s. τ-homogeneous by an argument similar to the proof of Proposition 4. For the opposite case, assume g is $\mathbb{P}$-a.s. τ-homogeneous. Then for any $t > 0$ and $\boldsymbol{u}$ with $t\boldsymbol{u} \in U$ and almost all $\omega \in \Omega$, we have:

$$\begin{aligned} g[t\boldsymbol{u}, \boldsymbol{X}](\omega) &= \sum_{i=1}^{n} (u_i t)^\tau h_i(X_i(\omega)) + \mathfrak{b}(t\boldsymbol{u}, \boldsymbol{X}) \\ &= t^\tau \mathfrak{a}[\mathbf{u}, \boldsymbol{X}](\omega) + \mathfrak{b}(t\boldsymbol{u}, \boldsymbol{X}). \end{aligned}$$

By our assumption, we can write:

$$g[t\boldsymbol{u}, \boldsymbol{X}](\omega) = t^\tau g[\mathbf{u}, \boldsymbol{X}](\omega) = t^\tau (\mathfrak{a}[\mathbf{u}, \boldsymbol{X}](\omega) + \mathfrak{b}(\boldsymbol{u}, \boldsymbol{X})).$$

Hence, $\mathfrak{b}(t\boldsymbol{u}, \boldsymbol{X}) = t^\tau \mathfrak{b}(\boldsymbol{u}, \boldsymbol{X})$ and $\mathfrak{b}$ is τ-homogeneous in U. □

EXAMPLE 5 *Consider the operators (i)-(v) from Example 2. These operators are* $\mathbb{P}$*-a.s. 1-homogeneous in* U *since the* min *and* max *functions are 1-homogeneous. Further, the function*

$\hat{\mathfrak{a}}$ *given by:*

$$\hat{\mathfrak{a}}(\boldsymbol{u}, \boldsymbol{X}) := u_1 X_1 + u_2 X_2,$$

is $\mathbb{P}$*-a.s. 1-homogeneous in* U*, and*

$$\hat{\mathfrak{b}}_1(\boldsymbol{u}, \boldsymbol{X}) := \mathbb{E}[u_1 X_1 + u_2 X_2],$$
$$\hat{\mathfrak{b}}_2(\boldsymbol{u}, \boldsymbol{X}) := VaR_\lambda(\boldsymbol{u}),$$

are both 1-homogeneous in U*. Functions* $\hat{\mathfrak{a}}$*,* $\hat{\mathfrak{b}}_1$ *and* $\hat{\mathfrak{b}}_2$ *may be regarded as "building blocks" for applications in finance and insurance.*

Example 6 *Consider the operator (vi) from Example 2, with* $\tau = 1$ *and suppose* $\Lambda(\cdot)$ *is monotonically increasing. Observe that* $\hat{\mathfrak{a}}(\boldsymbol{u}, \boldsymbol{X}) = u_1 X_1 + u_2 X_2$ *is* $\mathbb{P}$*-a.s. 1-homogeneous in* U*. However, the lambda quantile* ρ_Λ *has homogeneity degree:*

$$\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u})) = \frac{f_Y(-\rho_\Lambda(\boldsymbol{u}))}{f_Y(-\rho_\Lambda(\boldsymbol{u})) - \Lambda'(-\rho_\Lambda(\boldsymbol{u}))} \neq 1,$$

for any $\boldsymbol{u} \in U$ *since* $\Lambda'(y) \neq 0$ *for all* $y \in \mathbb{R}$*. Therefore, the operator* g *from Example 2 with* $\tau = 1$ *and monotonically increasing* $\Lambda(\cdot)$ *is not homogeneous in* U *by Proposition 4.*

Corollary 2 *An operator of the form* $g = \mathfrak{a}[\boldsymbol{u}, \boldsymbol{X}] + \rho_\Lambda(\boldsymbol{u})$*, where* $\mathfrak{a}$ *is linear in* $\boldsymbol{u}$ *and* $\Lambda(\cdot)$ *is monotonically increasing, is not homogeneous in* U*.*

The assumption that Λ is monotonically increasing in Corollary 2 implies that the homogeneity degree of the lambda quantile is strictly greater than 1. Therefore, by Proposition 4, operator g is not homogeneous.

Example 7 *Consider again the operator (vi) from Example 2 and suppose* $\Lambda(\cdot)$ *is monotonically increasing. Then,* g *is* $\mathbb{P}$*-a.s.* τ*-homogeneous in* U *if* $\tau = \eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}))$*.*

The next result shows that if a deterministic variable $\mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u})$ (of portfolio compositions), i.e. a positive cash amount determined by asset weights, is added to, or subtracted from, the portfolio return, then the lambda quantile is reduced or increased, respectively, by the same amount $\mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u})$. Furthermore, the lambda function is shifted along the return axis by this deterministic variable. The following result is related to the Λ-translation invariance property of lambda quantiles in Frittelli *et al.* (2014), that is the cash additivity property of lambda quantiles defined on the set of probability measures. However, it should not be confused with the translation invariance property defined on U, since we are not adding or subtracting from the portfolio composition $\boldsymbol{u}$ but from $\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u})$ instead.

Proposition 6 *For fixed* $\boldsymbol{X} \in \mathcal{X}^n$*, consider the portfolio return:*

$$g_{\boldsymbol{X}}(\boldsymbol{u}) = \mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u}) + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}), \tag{29}$$

where $\mathfrak{a}_{\boldsymbol{X}}(u) := \mathfrak{a}[u, \boldsymbol{X}]$ *and* $\mathfrak{b}_{\boldsymbol{X}}(u) := \mathfrak{b}(u, \boldsymbol{X})$*. Also, define* $\Gamma(z) := \Lambda(z + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}))$ *for all* $z \in \mathbb{R}$*. Then, for all* $\boldsymbol{u} \in U$*, it holds that:*

$$\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) = \rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}) - \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}).$$

Proof. Observe that for all $\boldsymbol{u} \in U$, we can write:

$$\begin{aligned}\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) &:= -\inf\{y \mid \mathbb{P}[\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u}) + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}) \le y] > \Lambda(y)\} \\ &= -\inf\{y \mid \mathbb{P}[\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u}) \le y - \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u})] > \Lambda(y)\} \\ &= -\inf\{z + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}) \mid \mathbb{P}[\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u}) \le z] > \Lambda(z + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}))\} \\ &= -\inf\{z \mid \mathbb{P}[\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u}) \le z] \ge \Gamma(z)\} - \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}) \\ &= \rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}) - \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}).\end{aligned}$$

□

In Proposition 6, $\mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u})$ corresponds to the cash amount determined by the asset weights $\boldsymbol{u}$, which is added to the existing portfolio return $\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u})$ to obtain the return $g_{\boldsymbol{X}}(\boldsymbol{u})$ of the newly formed portfolio. As a result, the risk of the new portfolio, i.e. $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$, is obtained by subtracting $\mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u})$ from the risk of the existing portfolio, i.e. $\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})$. Note that the cash injection causes the lambda quantile of existing and new portfolios to be calculated using different, but related lambda functions, with the relationship given by $\Gamma(z) := \Lambda(z + \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}))$.

PROPOSITION 7 *Suppose $g_{\boldsymbol{X}}$ can be written in the form* (29). *If $\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})$ and $\mathfrak{b}_{\boldsymbol{X}}$ are homogeneous in U with the same degree, then $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$ is homogeneous in U with degree:*

$$\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})) = \eta_{\Gamma,Z}(-\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})), \tag{30}$$

where $Z := \mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u})$.

Proof. Suppose $\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})$ and $\mathfrak{b}_{\boldsymbol{X}}$ have homogeneity degree $\tau \in \mathbb{R}$. Then, by Proposition 6 we can write the following for any $t > 0$:

$$\begin{aligned}\rho_\Lambda(t\boldsymbol{u}; g_{\boldsymbol{X}}) &= \rho_\Gamma(t\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}) - \mathfrak{b}_{\boldsymbol{X}}(t\boldsymbol{u}) \\ &= t^\tau \rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}) - t^\tau \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}) \\ &= t^\tau \rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}),\end{aligned}$$

which implies $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$ is τ-homogeneous in U. Then by Lemma 5 in Appendix A, both $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$ and $\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})$ can be written in the form (21) from Theorem 2. Therefore, the homogeneity degrees of $\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}})$ and $\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})$ are given by $\eta_{\Lambda,Y}(-\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}))$ and $\eta_{\Gamma,Z}(-\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}))$ respectively, which proves our result. □

PROPOSITION 8 *Let $g_{\boldsymbol{X}}$ be of the form* (29) *and assume:*

(i) Assumption 1 is satisfied for $\mathfrak{a}_{\boldsymbol{X}}$ and Γ;
(ii) Γ is continuously differentiable on $\mathbb{R}$;
(iii) $\mathfrak{b}_{\boldsymbol{X}}$ is continuously differentiable by u_i for $i = 1, \ldots, n$ for all $\boldsymbol{u} \in U$.

Then, $\rho_\Lambda(\cdot; g_{\boldsymbol{X}})$ is partially differentiable in U with continuous derivatives given by:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u}; g_{\boldsymbol{X}}) = -\eta_{\Gamma,Z}(-\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}))\mathbb{E}[\partial_{u_i}\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u})|Z = -\rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}})] - \partial_{u_i}\mathfrak{b}(\boldsymbol{u}), \tag{31}$$

for $i = 1, \ldots, n$.

Proof. By Proposition 6, we can write

$$\rho_\Lambda(\boldsymbol{u}; g_{\boldsymbol{X}}) = \rho_\Gamma(\boldsymbol{u}; \mathfrak{a}_{\boldsymbol{X}}) - \mathfrak{b}_{\boldsymbol{X}}(\boldsymbol{u}).$$

By Theorem 1 with condition (i), $\rho_\Gamma(\cdot;\mathfrak{a}_{\boldsymbol{X}})$ is partially differentiable in U with continuous derivatives given by:

$$\frac{\partial \rho_\Gamma}{\partial u_i}(\boldsymbol{u};\mathfrak{a}_{\boldsymbol{X}}) = -\eta_{\Gamma,Z}(-\rho_\Gamma(\boldsymbol{u};\mathfrak{a}_{\boldsymbol{X}}))\mathbb{E}[\partial_{u_i}\mathfrak{a}_{\boldsymbol{X}}(\boldsymbol{u})|Z=-\rho_\Gamma(\boldsymbol{u};\mathfrak{a}_{\boldsymbol{X}})],$$

for $i=1,\dots,n$. Since both $\rho_\Gamma(\cdot;\mathfrak{a}_{\boldsymbol{X}})$ and $\mathfrak{b}_{\boldsymbol{X}}$ are continuously differentiable in U, we conclude that $\rho_\Lambda(\cdot;g_{\boldsymbol{X}})$ is continuously partially differentiable in U. □

Example 8 *Consider the operator in Example 2 (vi) with $\tau=1$ such that $g_{\boldsymbol{X}}$ is given by:*

$$g_{\boldsymbol{X}}(\boldsymbol{u}) = u_1X_1 + u_2X_2 - \rho_\Lambda(\boldsymbol{u};Y),$$

where $Y = u_1X_1+u_2X_2$. Then,

$$\rho_\Lambda(\boldsymbol{u};g_{\boldsymbol{X}}) = \rho_\Gamma(\boldsymbol{u};Y) - \rho_\Lambda(\boldsymbol{u};Y),$$

where $\Gamma(y) = \Lambda(y+\rho_\Lambda(\boldsymbol{u};Y))$ and its risk contributions, for $i=1,\dots,n$, are:

$$\frac{\partial \rho_\Lambda}{\partial u_i}(\boldsymbol{u};g_{\boldsymbol{X}}) = \eta_{Y,\Gamma}(\rho_\Gamma(\boldsymbol{u};Y))\mathbb{E}[X_i \mid Y=\rho_\Gamma(\boldsymbol{u};Y)] - \eta_{\Lambda,Y}(\rho_\Lambda(\boldsymbol{u};Y))\mathbb{E}[X_i \mid Y=\rho_\Lambda(\boldsymbol{u};Y)].$$

As illustrated in the examples above, the function $\mathfrak{a}$ is typically a polynomial in $\boldsymbol{X}$ whilst $\mathfrak{b}$ a moment or quantile. A portfolio operator may also be constructed via a function of these "building blocks", as in Example 2. Thus, we consider g to be a composition of a function $f:\mathcal{X}\times\mathbb{R}\to\mathcal{X}$ with $\mathfrak{a}$ and $\mathfrak{b}$, i.e. we consider operators of the form

$$g[\boldsymbol{u},\boldsymbol{X}] = (f\circ(\mathfrak{a},\mathfrak{b}))(\boldsymbol{u},\boldsymbol{X}) = f(\mathfrak{a}[\boldsymbol{u},\boldsymbol{X}],\mathfrak{b}(\boldsymbol{u},\boldsymbol{X})). \tag{32}$$

Since f acts on $\mathfrak{a}[\boldsymbol{u},\boldsymbol{X}]\in\mathcal{X}$ and $\mathfrak{b}(\boldsymbol{u},\boldsymbol{X})\in\mathbb{R}$, homogeneity of the function f is discussed in $\mathcal{X}$ and $\mathbb{R}$, but not in U. The function f in this case may be implicitly homogeneous in U.

Proposition 9 *Suppose the function $\mathfrak{a}:U\times\mathcal{X}^n\to\mathcal{X}$ is $\mathbb{P}$-a.s. τ-homogeneous in U and $\mathfrak{b}:U\times\mathcal{X}^n\to\mathbb{R}$ is τ-homogeneous in U. Also, suppose the function $f:\mathcal{X}\times\mathbb{R}\to\mathcal{X}$ is $\mathbb{P}$-a.s. ν-homogeneous in both $\mathcal{X}$ and $\mathbb{R}$. Then, the operator in Equation (32) is $\mathbb{P}$-a.s. homogeneous of degree $\tau\nu$ in U.*

Proof. Noting that $\mathfrak{b}(\boldsymbol{u},\boldsymbol{X})$ is constant across outcomes $\omega\in\Omega$, we can write the following for almost all $\omega\in\Omega$ and for any $t>0$ and $\boldsymbol{u}\in U$:

$$\begin{aligned}
g[t\boldsymbol{u},\boldsymbol{X}](\omega) &= (f\circ(\mathfrak{a},\mathfrak{b}))(t\boldsymbol{u},\boldsymbol{X})(\omega)\\
&= f(\mathfrak{a}[t\boldsymbol{u},\boldsymbol{X}](\omega),\mathfrak{b}(t\boldsymbol{u},\boldsymbol{X}))\\
&= f(t^\tau\mathfrak{a}[\boldsymbol{u},\boldsymbol{X}](\omega),t^\tau\mathfrak{b}(\boldsymbol{u},\boldsymbol{X}))\\
&= (t^\tau)^\nu f(\mathfrak{a}[\boldsymbol{u},\boldsymbol{X}](\omega),\mathfrak{b}(\boldsymbol{u},\boldsymbol{X}))\\
&= t^{\tau\nu} f(\mathfrak{a}[\boldsymbol{u},\boldsymbol{X}](\omega),\mathfrak{b}(\boldsymbol{u},\boldsymbol{X}))\\
&= t^{\tau\nu}(f\circ(\mathfrak{a},\mathfrak{b}))(\boldsymbol{u},\boldsymbol{X})(\omega)\\
&= t^{\tau\nu} g[\boldsymbol{u},\boldsymbol{X}](\omega).
\end{aligned}$$

Hence, g is $\mathbb{P}$-a.s. homogeneous in U of degree $\tau\nu$. □

5. Conclusion and future research

This paper presents a novel treatment of lambda quantile risk measures on subsets of $\mathbb{R}^n$. We prove that lambda quantiles are differentiable with respect to the portfolio composition, subject to smoothness conditions, and derive explicit formulae for the derivatives. These partial derivatives correspond to risk contributions of assets to the overall portfolio risk. We further provide the Euler decomposition of lambda quantiles, i.e. the property that lambda quantiles, scaled by a factor, can be written as weighted sums of their partial derivatives. This decomposition demonstrates that lambda quantiles are homogeneous risk measures in the space of portfolio compositions. Our results further show that the homogeneity degree of a lambda quantile is determined by the portfolio composition and the lambda function. This allows us to treat homogeneity as a dynamic property rather than constant and universal. Indeed, the lambda quantile homogeneity degree varies across portfolio risk profiles, rather than remains constant. This contrasts the case of risk measures that have a constant homogeneity degree, such as VaR. Due to the variable nature of lambda quantiles' homogeneity degrees, we introduce a generalised Euler capital allocation rule, that is compatible with risk measures of any homogeneity degree and non-linear portfolios. We prove that the generalised Euler allocations of lambda quantiles have the full allocation property.

Our interpretation that the lambda quantile homogeneity degree depends on market conditions and portfolio compositions may spark, in future research, to view homogeneity as a property of portfolio compositions rather than risk measures. Our approach to homogeneity poses interesting questions with practical implications: Is it the possible to achieve a targeted homogeneity degree? If ρ_Λ is τ-homogeneous for portfolio composition $\boldsymbol{u}$, does there exist non-trivial $\boldsymbol{v} \in U$ such that ρ_Λ is also τ-homogeneous for portfolio composition $\boldsymbol{u} + \boldsymbol{v} \in U$? Put differently, which changes in portfolio composition preserve a homogeneity degree? Another interesting question is how changes in the lambda function affect the homogeneity degree, i.e. under which market shocks does the homogeneity degree change? This would allow practitioners to understand how periods of economic stress or boost would impact the homogeneity degree of their portfolio. These questions are not unique to lambda quantiles; they apply to any risk measure with homogeneity degree that is impacted by portfolio composition and/or market conditions.

Appendix A: Auxiliary definitions and results

This appendix is a collection of results and definition relevant for the exposition of the paper.

Lemma 3 (Theorem A.5.1 of Durrett (2019)) *Let $(S, \mathcal{S}, \mu)$ be a measure space. Let f be a complex valued function defined on $\mathbb{R} \times S$. Let $\delta > 0$, and suppose that for $x \in (y-\delta, y+\delta)$ we have:*

(i) $u(x) = \int_S f(x,s)\mu(ds)$ *with* $\int_S |f(x,s)|\mu(ds) < \infty$,

(ii) for fixed s, $\partial f/\partial x(x,s)$ exists and is a continuous function of x,

(iii) $v(x) = \int_S \frac{\partial f}{\partial x}(x,s)\mu(ds)$ *is continuous at $x = y$,*

(iv) $\int_S \int_{-\delta}^{\delta} |\frac{\partial f}{\partial x}(y+\theta, s)| d\theta \mu(ds) < \infty$,

then $u'(y) = v(y)$.

Definition 6 (Definition 4.2 of Tasche (1999)) Let $U \neq \emptyset$ be a set in $\mathbb{R}^n$ and $r : U \to \mathbb{R}$ be a function defined on U. A vector field $\boldsymbol{a} := (a_1, \dots, a_n) : U \to \mathbb{R}^n$ is called *suitable for performance measurement* with the function r if the following conditions are satisfied:

(a) for all $\boldsymbol{m} \in \mathbb{R}^n$ and $\boldsymbol{u} \in U$ with $r(\boldsymbol{u}) \neq \boldsymbol{m'u}$ and $i \in \{1, \dots, n\}$ the inequality

$$m_i r(\boldsymbol{u}) > a_i(\boldsymbol{u})\boldsymbol{m'u} \tag{A1}$$

implies that there exists an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have

$$g_{r,\boldsymbol{m}}(\boldsymbol{u} - t\boldsymbol{e}_i) < g_{r,\boldsymbol{m}}(\boldsymbol{u}) < g_{r,\boldsymbol{m}}(\boldsymbol{u} + t\boldsymbol{e}_i).$$

(b) for all $\boldsymbol{m} \in \mathbb{R}^n$ and $\boldsymbol{u} \in U$ with $r(\boldsymbol{u}) \neq \boldsymbol{m'u}$ and $i \in \{1, \dots, n\}$ the inequality

$$m_i r(\boldsymbol{u}) < a_i(\boldsymbol{u})\boldsymbol{m'u} \tag{A2}$$

implies that there exists an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have

$$g_{r,\boldsymbol{m}}(\boldsymbol{u} - t\boldsymbol{e}_i) > g_{r,\boldsymbol{m}}(\boldsymbol{u}) > g_{r,\boldsymbol{m}}(\boldsymbol{u} + t\boldsymbol{e}_i),$$

where $g = g_{r,\boldsymbol{m}} : \{\boldsymbol{u} \in U | r(\boldsymbol{u}) \neq \boldsymbol{m'u}\} \to \mathbb{R}$ is the *return function* for r for a fixed $\boldsymbol{m} \in \mathbb{R}^n$ defined by

$$g_{r,\boldsymbol{m}}(\boldsymbol{u}) := \frac{\boldsymbol{m'u}}{r(\boldsymbol{u}) - \boldsymbol{m'u}}.$$

Lemma 4 (Theorem 4.4 of Tasche (1999)) *Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set and $r : U \to \mathbb{R}$ a function partially differentiable in U with continuous derivatives. Also, let $\boldsymbol{a} = (a_i, \dots, a_n) : U \to \mathbb{R}^n$ be a continuous vector field. Then $\boldsymbol{a}$ is suitable for performance measurement with function r if and only if:*

$$a_i(\boldsymbol{u}) = \frac{\partial r}{\partial u_i}(\boldsymbol{u})$$

with $i = 1, \dots, n$ and $\boldsymbol{u} \in U$.

Definition 7 Let τ be any fixed real number. A function $r : U \to \mathbb{R}$ is *τ-homogeneous* in U if for each $\boldsymbol{u} \in U$ and $t > 0$ with $t\boldsymbol{u} \in U$, we have $t^\tau r(\boldsymbol{u}) = r(t\boldsymbol{u})$.

Note that our definition of a homogeneous function on the set U doesn't assume U is a homogeneous subset of $\mathbb{R}^n$. The homogeneity assumption of U is used in previous literature (see for example Tasche (1999)) to ensure that the function r is well-defined for all arguments of the form $t\boldsymbol{u}$, where $\boldsymbol{u} \in U$ and $t > 0$. Instead, we assume U is a bounded set and impose the condition $t\boldsymbol{u} \in U$ for all $\boldsymbol{u} \in U$ and $t > 0$ in the definition of a homogeneous function to ensure r is well-defined for all arguments.

LEMMA 5 (Proposition 3.5(b) of Tasche (1999)) *Denote by $U \neq \emptyset$ a homogeneous open set in $\mathbb{R}^n$, let $r : U \to \mathbb{R}$ be a real-valued function and fix $\tau \in \mathbb{R}$. If r is totally differentiable then it is τ-homogeneous if and only if for all $\boldsymbol{u} \in U$:*

$$\tau r(\boldsymbol{u}) = \sum_{i=1}^{n} u_i \frac{\partial r}{\partial u_i}(\boldsymbol{u})\,.$$

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