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Groups with a given number of nonpower subgroups

C. S. Anabanti¹ and S. B. Hart²

¹chimere.anabanti@up.ac.za; Dept. of Mathematics and Applied Mathematics, University of Pretoria
²s.hart@bbk.ac.uk; Dept. of Economics, Mathematics and Statistics, Birkbeck, University of London

Abstract

It is well-known that no group has either exactly 1 or exactly 2 nonpower subgroups. In this paper, we obtain a classification of groups containing exactly 3 nonpower subgroups. Moreover, we show that there is a unique finite group with exactly 4 nonpower subgroups. Finally, we show that given any integer $k$ greater than 4, there are infinitely many groups with exactly $k$ nonpower subgroups.

1 Introduction

A subgroup $H$ of a group $G$ is called a power subgroup of $G$ if there exists a non-negative integer $m$ such that $H = \langle g^m : g \in G \rangle$. Any subgroup of $G$ which is not a power subgroup is called a nonpower subgroup of $G$. Zhou et al.⁴ proved that cyclic groups have no nonpower subgroups, and infinite noncyclic groups have an infinite number of nonpower subgroups. They showed further that no group has either exactly 1 or exactly 2 nonpower subgroups, and then asked: for each integer $k$ greater than 2, does there exist at least one group possessing exactly $k$ nonpower subgroups? This question was recently answered positively in [1], where it was also proved that for any integer $k$ greater than 4 and composite, there are infinitely many groups with exactly $k$ nonpower subgroups.

Let $p$ be an odd prime. For each positive integer $n$, we define the group $G_{n,p}$ as follows:

$$G_{n,p} := \langle x, y : x^{2^n} = 1 = y^p, yx = xy^{-1} \rangle.$$ 

We note that $G_{1,p}$ is the dihedral group of order $2p$, and $G_{2,p}$ is the generalized quaternion group of order $4p$ (we obtain its usual presentation $\langle a, b : a^{2p} = 1, b^2 = a^p, ba = a^{-1}b \rangle$ by setting $a = x^2 y$ and $b = x$). More generally, for any positive integer $n$, $G_{n,p}$ is the semidirect product $C_p \rtimes C_{2^n}$, and has order $2^{n+1}p$. We may now state our first result.

**Theorem 1.** There are infinitely many groups with an odd prime number of nonpower subgroups. In particular, for any odd prime $p$ and each positive integer $n$, the group $G_{n,p}$ has exactly $p$ nonpower subgroups.

**Key words and phrases:** Counting subgroups, nonpower subgroups, finite groups

2010 Mathematics Subject Classification: 20D25, 20D60, 20E34.
Theorem 1 combined with the fact that for composite $k$ greater than 4 there are infinitely many groups with $k$ nonpower subgroups [1, Theorem 5], gives the following immediate corollary.

**Corollary 2.** Let $k$ be an integer greater than 4. Then there are infinitely many groups with exactly $k$ nonpower subgroups.

The only unresolved cases are therefore $k = 3$ and $k = 4$. Our second main result deals with these cases.

**Theorem 3.**

(a) A group $G$ contains exactly three nonpower subgroups if and only if $G$ is isomorphic to one of $C_2 \times C_2$, $Q_8$ or $G_{n,3}$ for $n \in \mathbb{Z}^+$.

(b) Up to isomorphism, $C_3 \times C_3$ is the only group containing exactly four nonpower subgroups.

For the rest of this section, we recall some preliminaries. We note that each power subgroup is characteristic and hence normal in $G$. Following [1], we write $s(G)$ for the number of subgroups in a group $G$, $ps(G)$ for the number of power subgroups of $G$ and $nps(G)$ for the number of nonpower subgroups of $G$.

**Lemma 4.** [1, Lemma 3] If $A$ and $B$ are finite groups such that $|A|$ and $|B|$ are coprime, then

$$nps(A \times B) = nps(A)s(B) + ps(A)nps(B).$$

We denote by $\Phi(G)$ the Frattini subgroup of $G$, that is, the intersection of the maximal subgroups of $G$. It is a characteristic subgroup of $G$.

**Theorem 5** (Burnside’s Basis Theorem). Let $G$ be a $p$-group and suppose $[G : \Phi(G)] = p^d$.

(a) $G/\Phi(G)$ is elementary abelian of order $p^d$. Moreover, if $N \trianglelefteq G$ and $G/N$ is elementary abelian, then $\Phi(G) \leq N$.

(b) Every minimal system of generators of $G$ contains exactly $d$ elements.

(c) $\Phi(G) = G^{p}G'$. In particular, if $p = 2$, then $\Phi(G) = G^2$.

**Lemma 6** ([2] Theorem 1.10(a)). Let $G$ be a non-cyclic $p$-group, where $p > 2$. Then the number of subgroups of order $p$ in $G$ is congruent to $1 + p$ modulo $p^2$.

**Remark.** It is well-known that the only 2-groups with a unique involution are cyclic or generalised quaternion.

## 2 Proof of main results

We begin with a proof of Theorem 1.

**Proof of Theorem 1.** Let $p$ be an odd prime. Our goal is to show that for any positive integer $n$, and any odd prime $p$, the group $G_{n,p} = \langle x, y : x^{2^n} = 1 = y^p, yx = xy^{-1} \rangle$ contains exactly $p$ nonpower subgroups. We have that $|G_{n,p}| = 2^n p$. We first obtain a count on the number of subgroups in $G_{n,p}$. Since the Sylow 2-subgroup $\langle x \rangle$ is not a normal subgroup, the number of
Sylow 2-subgroups of $G_{n,p}$ must be $p$. On the other hand, since $y^x = y^{-1}$, there is a unique normal Sylow $p$-subgroup, namely the cyclic subgroup $\langle y \rangle$ of order $p$. Since $x^2$ is central in $G_{n,p}$ and each Sylow 2-subgroup of $G_{n,p}$ is cyclic, there is a unique subgroup of order $2^k$ (for each $k \in \{0, \ldots, n-1\}$) and a unique subgroup of order $2^k p$ (for each $k \in \{1, \ldots, n\}$). Along with the $p$ subgroups of order $2^n$, we see that $s(G_{n,p}) = 2n + p + 1$. As the subgroups of order $2^n$ are not normal, we know immediately that they are nonpower subgroups. Hence $n_{ps}(G_{n,p}) \geq p$. We now show that any subgroup of $G_{n,p}$ that is not a Sylow 2-subgroup of $G_{n,p}$ is a power subgroup of $G_{n,p}$. First, the unique subgroup of order $p$ is $G_{n,p}^p$. Secondly, for each $k \in \{0, \ldots, n-1\}$, the subgroup of order $2^k$ is $G_{n,p}^{2^k}$. Finally, for each $k \in \{1, \ldots, n\}$, the subgroup of order $2^k p$ is $G_{n,p}^{2^k p}$. Therefore, $n_{ps}(G_{n,p}) = 2n + 1$; whence $n_{ps}(G_{n,p}) = p$. 

We now move onto the proof of Theorem 7. Let $G$ be a finite noncyclic group. Then $G$ falls into one of the following three categories: (i) a noncyclic $p$-group; (ii) a noncyclic nilpotent group that is not a $p$-group; (iii) a non-nilpotent group. For each of these cases above, we classify all the finite groups with exactly 3 or 4 nonpower subgroups.

**Proposition 7.** Let $G$ be a finite noncyclic $p$-group. Then $n_{ps}(G) = 3$ if and only if $G$ is $C_2 \times C_2$ or $Q_8$, and $n_{ps}(G) = 4$ if and only if $G$ is $C_3 \times C_3$.

**Proof.** Let $G$ be noncyclic of order $p^n$. It was shown in [3] that if $N \trianglelefteq G$ and $A/N$ is a nonpower subgroup of $G/N$, then $A$ is a nonpower subgroup of $G$. Suppose $G$ has exactly $k$ nonpower subgroups, where $k \in \{3, 4\}$. Now, $G/\Phi(G) \cong C_p \times \cdots \times C_p$ $(d$-times), and $d \geq 2$ as $G$ is not cyclic. The $p^{d-1} \times p-1$ cyclic subgroups of order $p$ in $C_p^d$ are nonpower subgroups. Thus $G/\Phi(G)$, and hence $G$, has at least $1 + p + \cdots + p^{d-1}$ nonpower subgroups. Hence, $d = 2$, either $p = 2$ or $p = 3$, and $G$ has $p + 1$ maximal subgroups that are nonpower subgroups.

The power subgroups of $G$ are $G^1 = G$, $G^p$, $G^{p^2}, \ldots, G^{p^m}$, where $p^m$ is the exponent of $G$. There are thus at most $m + 1$ distinct power subgroups. Since $G$ is not cyclic, this means $m < n$; so $ps(G) \leq n$.

What about $s(G)$? There is at least one subgroup of order $p^i$ for $0 \leq i \leq n$ (just take any composition series). This gives at least $n + 1$ subgroups. But there are $p + 1$ maximal subgroups (of order $p^{n-1}$) arising from the $p + 1$ nontrivial proper subgroups of $G/\Phi(G)$. Thus $s(G) \geq n + p + 1$.

Suppose $p = 2$. If $G$ is not generalised quaternion (and by assumption $G$ is not cyclic), then $G$ has at least 3 involutions, and hence at least 3 subgroups of order 2. So, if $n > 2$, then $s(G) \geq n + 5$, meaning that $n_{ps}(G) \geq 5$, a contradiction. Thus, either $G$ is generalised quaternion or $n = 2$, which means $G \cong C_2 \times C_2$, and in this case $n_{ps}(G) = 3$. If $G$ is generalised quaternion, then $G$ has $2^{n-1} + 2$ elements of order 4, resulting in $2^{n-2} + 1$ subgroups of order 4. If $n > 3$, we get that $s(G) \geq n + 1 + 2^{n-2} \geq n + 5$. Again, this means that $n_{ps}(G) \geq 5$. Thus, $n = 3$, and then $G \cong Q_8$. Again, $n_{ps}(Q_8) = 3$.

The remaining case is $p = 3$. By Lemma 7 there are at least four subgroups of order 3 in $G$. If $n > 2$, then these are distinct from the four maximal subgroups, and so we get $s(G) \geq n + 7$. This forces $n_{ps}(G) \geq 7$, a contradiction. The only possibility is that $n = 2$. A quick check shows that $n_{ps}(C_3 \times C_3) = 4$.

Thus, $n_{ps}(G) = 3$ if and only if $G$ is $C_2 \times C_2$ or $Q_8$, and $n_{ps}(G) = 4$ if and only if $G$ is $C_3 \times C_3$. 

**Lemma 8.** Let $G$ be a finite noncyclic nilpotent group. If $G$ is not a $p$-group, then $n_{ps}(G) \geq 6$. 


Proof. Recall that a finite group is nilpotent if and only if it is the direct product of its Sylow subgroups, each of which is normal. Since $G$ is noncyclic, at least one of these Sylow subgroups is noncyclic. Let $p_1, \ldots, p_r$ be the primes dividing $|G|$, and let $P_1, \ldots, P_r$ be the respective Sylow subgroups. Assume, without loss of generality, that $P_1$ is noncyclic. Write $Q = P_2 \times \cdots \times P_r$; so $G \cong P_1 \times Q$. Since $G$ is not a $p$-group, we have that $Q \neq \{1\}$. By Lemma 4 therefore,\[nps(G) = nps(P_1)s(Q) + ps(P_1)nps(Q) \geq nps(P_1)s(Q)\]As $Q \neq \{1\}$, we have that $s(Q) \geq 2$. As $P_1$ is not cyclic, $nps(P_1) \geq 3$. Hence $nps(G) \geq 6$. \qed

Lemma 9. If $G$ is a finite non-nilpotent group such that $nps(G) \in \{3, 4\}$, then $nps(G) = 3$ and $G \cong G_{n,3} = \langle x, y : x^{2^n} = 1 = y^3, yx = xy^{-1} \rangle$, for some positive integer $n$.\[\]Proof. Suppose $G$ is finite, non-nilpotent and $nps(G) = k \in \{3, 4\}$. If $G$ had a unique Sylow $p$-subgroup for each $p$ dividing $|G|$, then $G$ would be nilpotent. So there is at least one such $p$ for which $G$ has more than one Sylow $p$-subgroup. For any such $p$, the number, $n_p$, of Sylow $p$-subgroups is congruent to 1 mod $p$. So $n_p \geq p + 1$. These groups are not normal, so are not power subgroups. Therefore, as $nps(G) \in \{3, 4\}$, we have that either $p = 2$ and $n_2 = 3$, or $p = 3$ and $n_3 = 4$. For all other primes $q$ dividing $|G|$, there must be a unique Sylow $q$-subgroup. If any subgroup of $G$, other than the Sylow $p$-subgroups, were non-normal, then it and its conjugates could not be power subgroups. Thus there would be at least two further nonpower subgroups, forcing $nps(G) \geq 5$, a contradiction. Therefore, every subgroup of $G$, other than the Sylow $p$-subgroups, is normal.

Let $P$ be one of the Sylow $p$-subgroups. Let $q_1, \ldots, q_r$ be the primes other than $p$ dividing $|G|$. Let $Q_1, \ldots, Q_r$ be the corresponding normal Sylow subgroups. Each $Q_i$ is normal and the $Q_i$ intersect trivially. Therefore, defining $H = Q_1Q_2 \cdots Q_r$, we have that $H \cong Q_1 \times Q_2 \times \cdots \times Q_r$ is a normal subgroup of $G$, with $G = PH$. Now, $P \triangleleft N_G(P)$, and setting $K = H \cap N_G(P)$, we have that $K \trianglelefteq G$ (because certainly $K$ is not a Sylow $p$-subgroup). But $P$ is normal in $N_G(P) = PK$; so $N_G(P) \cong P \times K$. Let $h \in H - N_G(P)$. Then $(PK)^h = P^hK \neq PK$. This means that $PK$ is not normal in $G$; a contradiction unless $K = \{1\}$. Therefore, $K = \{1\}$, and $P = N_G(P)$. In particular, $n_p = |G : P| = |H|$.

Suppose first that $p = 3$. Then $|H| = 4$. If $H \cong C_2 \times C_2$, then each of its cyclic subgroups would be normal, and hence the involutions they contain would be central. But that would imply that $P$ is normal in $G$, a contradiction. Therefore $H \cong C_4$. Let $z$ be a generator of $H$. We have $H \leq C_G(z) \leq G$. Thus, $|z^G| = 3^i$ for some $i$ with $0 \leq i \leq n$. But $z^G \subseteq \{z, z^{-1}\}$. The only possibility is that $z^G = \{z\}$, and $z$ is central in $G$. Again, this implies that $P$ is normal in $G$, a contradiction. Therefore, $p \neq 3$.

The remaining case is when $p = 2$. In this case, $H \cong C_3$. Let $A_1, A_2, A_3$ be the three Sylow 2-subgroups. Every proper subgroup of $P$ is not one of $A_1, A_2$ and $A_3$, so is normal in $G$ and hence contained in all of $A_1, A_2$ and $A_3$. If $P$ were cyclic, then each of its generators would generate a proper cyclic subgroup, and would hence be contained in $A_1, A_2$ and $A_3$. This implies $P \leq A_1 \cap A_2 \cap A_3$; a contradiction. Therefore, $P$ is cyclic of order $2^n$. Write $P = \langle x \rangle$ and $H = \langle y \rangle$. Certainly, $y^x \neq y$; so the only possibility is that $y^x = y^{-1}$. Therefore,

$$G = \langle x, y : x^{2^n} = 1, y^3 = 1, xy = xy^{-1} \rangle$$

for some integer $n \geq 1$. That is, $G \cong G_{n,3}$. By Theorem 1, we have $nps(G) = 3$. \qed

Theorem 3 follows immediately from Proposition 7, Lemma 8 and Lemma 9.
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References

