Multiperiod pricing theory

Multiperiod Pricing Theory

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Declaration

I confirm that this is my own work and the use of all content from other sources has been properly and fully acknowledged.

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Abstract

We extend the methodology proposed by Carr, Geman and Madan [12] for pricing and hedging in incomplete markets to more general probability spaces and prove a result, which shows the equivalence between the notion of the absence of strictly acceptable opportunities and the existence of a representative state pricing function. We also give examples of how to construct valuation test measures.

We also extend the methodology to the discrete-time setting with a finite time horizon. We specify a finite set of single-period probability measures at each non-terminating node of a tree, which are then used to generate a set of probability measures for the entire tree by pasting together these single-period measures across all the nodes. We define the concept of a strictly acceptable opportunity in this new framework and prove a result, which gives the condition that guarantees the absence of strictly acceptable opportunities.

We also consider a Lucas-type pure exchange economy (see [39]) consisting of N infinitely long-lived agents, who have access to the same information regarding the stochastic evolution of a process. However, these agents do not interpret the information in the same way. We work in a continuous-time model as discussed in Brown and Rogers [11]. Further, we assume that the agents have a homogeneous coefficient of relative risk aversion. We then give a characterisation of the equilibrium, which does not depend on any form of the utility function. Thereafter, we assume that each agent has a power utility function, and we obtain concrete results for the price of the traded asset. We also obtain an expression for the agent's wealth process and give the dynamics of the state-price density and the asset price.

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Introduction

Martingale pricing approach is a profound and powerful technique for valuing assets and contingent claims in a financial market. Dalang *et al.* [16] proved the Fundamental Theorem of Asset Pricing in the case of general probability space in discrete time with finite horizon (see also [18], [49]). This result shows the equivalence between the no-arbitrage condition and the existence of an equivalent martingale measure. However, the beauty of the approach is made manifest especially when the market is complete, that is, when there exists a portfolio of the primary assets that produces a perfect hedge for each random future payoff. In this case, the equivalent martingale measure, which exists under the assumption of no arbitrage, is unique and the price is given as the expectation of the discounted payoff under this unique equivalent martingale measure. However, the assumption of market completeness is an artefact to make the mathematical problem tractable, and models with this assumption do not represent the reality of financial markets.

Market incompleteness is an inherent part of financial markets and any model that attempts to describe the functioning of financial markets must incorporate incompleteness. When markets are incomplete, there is no martingale representation theorem to guarantee a perfect hedge for a unique price. Furthermore, the set of equivalent martingale measures is no longer a singleton. We obtain a set of equivalent martingale measures and an interval of arbitrage-free prices (see [24]). One is then faced with the problem of picking the best equivalent martingale measure, according to some criteria, to price a claim. There is also the question of finding a hedging strategy that optimises the agent's position.

There are many reasons why markets are incomplete. Relaxing any of the assumptions in the Black and Scholes model [9] will inevitably lead to market incompleteness. Another reason for market incompleteness is when the stock price is modelled by some stochastic volatility model. Furthermore, the degree of incompleteness increases as more constraints in the Black and Scholes model [9] are relaxed.

There are many approaches that have been developed to solve the problem of pricing and hedging in incomplete markets. One approach is to set a criterion and then pick a pricing measure out of the multitude of equivalent martingale measures which satisfies the criterion. There exists a substantial body of research, where authors pick pricing measures according to different optimal criteria. For example, using utility maximisation, Föllmer and Schweizer [21] introduced the minimal martingale measure; Miyahara [41] and Frittelli [25] introduced the minimal entropy martingale measure; and Bellini and Frittelli [5] introduced the minimax measure. These models solve the pricing problem, but do not tackle the hedging problem satisfactorily. In fact, some of these approaches provide no hedging strategy at all while some, like the minimal martingale measure by Föllmer and Schweizer [21], give a retrograde result in that they penalise over-hedging, which is not sensible financially.

Another approach, referred to as utility-based pricing, is based on the following: an agent attempts to price a derivative security so that the utility of their wealth, by following the optimal portfolio that includes the underlying assets only, is the same as the utility of their wealth by following the optimal portfolio that includes the underlying assets and a marginal amount of the derivative security. References include Hodges and Neuberger [34], Davis [17] and Kramkov and Schachermayer [37]. The drawback of this approach is the difficulty in specifying the input to the maximisation process.

Another approach, upon which most of this work is based, tries to extend the no-arbitrage framework to incomplete markets in a finite state, static model. Carr *et al.* [12] introduce two sets of measures: valuation test measures with the floor set to zero and stress test measures with negative floors. They then extended the set of desirable claims to become larger than arbitrage opportunities. This new extended set of claims is called the set of strictly acceptable opportunities. They defined the concept of No Strictly Acceptable Opportunities and argued that they must be eliminated from a market to ensure equilibrium. Further, they showed that, under the condition of No Strictly Acceptable Opportunities, the convex combination of the valuation test measures is a pricing measure (also called a Representative State Pricing Function). The model also provides a hedging strategy which ensures that the market is acceptably complete, even though the market is incomplete.

In the first chapter, we discuss the extension of the theory of No Strictly Acceptable Opportunities (see [12]) by considering a static model in the more general case of an infinite set of valuation test measures. We model the market by a general probability space and assume that there are a finite number of assets that are traded in the market. We assume that we have a countable number of valuation test measures defined on the probability space. We obtain the result that the condition of No Strictly Acceptable Opportunities is equivalent to the existence of a Representative State Pricing Function. We also obtain a similar result by assuming that the set of valuation test measures is uncountable, making use of results from functional analysis. Further, we give examples of how to generate valuation test measures and explain why an infinite number of such measures is sometimes necessary.

In the second chapter, we discuss the extension of the theory of No Strictly Acceptable Opportunities to a dynamic model in discrete time and with a finite horizon. We assume that there are a finite number of assets, finite number of possible states and a finite number of valuation test measures on the filtered probability space. We associate a finite number of probability measures with each single-period model, each assigning a positive weight to each possible outcome. Under the assumption that the underlying probability space is finite, we prove that the condition of No Strictly Acceptable Opportunities in each of the single-period models is equivalent to the condition of global No Strictly Acceptable Opportunities, if and only if the valuation test measures on the probability space are generated from the single-period probability measures. This will be made more precise in chapter 2. This result characterises the set of valuation test measures on the probability space that can ensure market equilibrium in this incomplete market. We obtain a further result that, by pasting together the representative state pricing function for each of the single-period models, we obtain a representative state pricing function on the probability space. It must be stated that in the multi-period case, the existence of a representative state pricing function is not a sufficient condition for the existence of No Strictly Acceptable Opportunities. This latter result is different from the result by Carr *et al.* [12] for single-period models.

In the third chapter, we consider a Lucas-type pure exchange economy (see [39]) with N infinitely long-lived agents. These agents have the same information, but differ in the way they understand and interpret the information. In essence, they do not agree on the stochastic process that models the dividend paid by the single productive asset in the economy. We work in a continuous-time model as discussed in Brown and Rogers [11]. We use a power utility function for this analysis (Brown and Rogers used a log utility function). We assume that the agents have a homogeneous coefficient of relative risk aversion. We give the characterisation of the equilibrium without any specification of a form for the utility function. To be more specific, we assume that the utility function is a power utility function and obtain concrete results for the price of the traded asset. In addition, we show the price of the asset can be expressed in terms of a generating function G, which is a function of the optimal consumption of the agents at any time t. We also obtain an expression for the individual agent's wealth process. The approach is then used to obtain an expression for the bond price and the short rate in this market. We also analyse the nature of the state price density and extend the model to the case where agents have heterogeneous coefficients of relative risk aversion.

Chapter 1

Mathematical Finance in One Period

1.1 Introduction and General Setting

The need to measure the risk of random future outcomes (or contingent claims) has led to extensive research into the construction and analysis of financial models. In certain financial markets, any random future payoff can be replicated by trading in the available risky and non-risky assets. Such payoffs are referred to as being attainable. The ability to replicate every random future payoff is equivalent to the existence of a unique probability measure (see [32, 33]), which is used for pricing random payoffs. Fundamental asset prices are the discounted expected values of random future payoffs under this unique probability measure. Markets that exhibit the above features are called complete markets (see [32, 33, 24, 42, 7]). The central market condition, upon which the analysis and pricing in complete markets is based, is the condition of absence of arbitrage opportunities (also called the no-arbitrage condition). This basically means that it should not be possible to construct an investment opportunity at zero initial cost, which yields a non-negative payoff with probability one and a positive payoff with positive probability. The idea of pricing in complete markets is well understood and the literature is full of works elucidating the no-arbitrage pricing approach. For a comprehensive summary, see [24, 42, 7, 32, 33].

However, financial markets are inherently incomplete. Market completeness is an assumption made in order to make the problem of analysing and pricing contingent claims mathematically tractable, and thus, obtain an approximation to reality. Achieving realism in pricing claims in financial markets requires that we relax the market completeness assumption. Further, it is noteworthy to state that the assumption of market completeness introduces a redundancy into our modelling framework, in that it renders the existence of attainable claims unnecessary as they can always be obtained by trading in the available tradeable assets. Relaxing this assumption will enable us to construct models that better reflect the reality of financial markets, albeit at the expense of additional complexity and the introduction of more mathematical machinery.

When the market is incomplete, it is not possible to price all contingent claims by arbitrage considerations alone. This means that it is not the case that every contingent claim can be replicated by constructing an admissible, self-financing trading strategy whose terminal payoff coincides with the target claim. Further, the pricing probability measure, which exists by the no-arbitrage condition, is no longer unique. Therefore, in the incomplete market situation, we obtain a set $\mathcal{P}$ of equivalent pricing (or martingale) measures, which then gives a price range associated with the contingent claim to be priced. Let us denote this interval of prices by $[m, M]$ and let $(\beta_t)_{t\geq 0}$ be the discount process. We also let T denote the terminal date at which $\mathbf{X}$, the contingent claim to be priced, is realised. We define the lower limit and the upper limit of the interval of prices respectively by:

$$m = \inf\{E_Q(\beta_T \mathbf{X}) : \mathbf{Q} \in \mathcal{P}\}$$

and

$$M = \sup\{E_Q(\beta_T \mathbf{X}) : \mathbf{Q} \in \mathcal{P}\}.$$

The trading strategy corresponding to the upper (resp. lower) limit of the interval of prices is called a superhedging (resp. subhedging) strategy of the contingent claim $\mathbf{X}$. This approach selects a hedging portfolio with the smallest cost that eliminates all the risk inherent in the claim $\mathbf{X}$. Therefore, from a practical viewpoint, it may be too expensive as it gives the writer full protection against any possible claim by the buyer of the claim. Furthermore, the interval is often too large to be useful for any risk pricing purposes (see [24, 18]).

Pricing of contingent claims in incomplete markets is really a conundrum. It is often not clear which of the infinitely many equivalent martingale measures to pick for pricing contingent claims. Different optimal criteria for choosing different pricing measures have been proposed. Examples of the different approaches, which require the construction of a minimal measure are discussed in [21, 41, 25, 5]. We also have other techniques with a utility maximisation perspective (see [17, 37]). It is important to note that no particular technique is without its drawbacks. The utility maximisation approach requires the specification of the investor's preference structure, in the form of their utility function, and the investor is required to trade according to these constructs. However, these constructs, in practice, cannot be easily obtained.

In the realm of risk management, the traditional technique for assessing the risk of a financial position is the Value-at-Risk method ($V@R_\alpha(\mathbf{X})$), (see [40, 24]). Let $\mathbf{X}$, modelling a financial position, be defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We specify an $\alpha \in$ (0,1) and then compute an $\alpha-$ quantile of the distribution of the random payoff $\mathbf{X}$, defined to be a real number x such that

$$\mathbf{P}[\mathbf{X} \leq x] \geq \alpha$$

and

$$\mathbf{P}[\mathbf{X} < x] \leq \alpha.$$

One introduces the lower quantile function of $\mathbf{X}$, $x_\alpha^- = \sup\{x : \mathbf{P}[\mathbf{X} < x] < \alpha\} = \inf\{x : \mathbf{P}[\mathbf{X} \leq x] \geq \alpha\}$,

and the upper quantile function

$x_\alpha^+ = \inf\{x : \mathbf{P}[\mathbf{X} \leq x] > \alpha\} = \sup\{x : \mathbf{P}[\mathbf{X} < x] \leq \alpha\}$.

We define the Value-at-Risk at level α as

$$V@R_\alpha(\mathbf{X}) = -x_\alpha^+(\mathbf{X}) = \inf\{x : \mathbf{P}[\mathbf{X} + x \leq \mathbf{0}] \leq \alpha\}.$$

The interpretation of $V@R_\alpha(\mathbf{X})$ is that it is the smallest amount of capital which, if added to the random payoff $\mathbf{X}$ and invested in the risk-free asset, keeps the probability of a negative outcome below the pre-assigned level α. In this approach, the computations are done with respect to the objective measure, so there is no need to pick an equivalent martingale measure. Furthermore, this approach has, for a long period of time, been the financial regulators' benchmark measure for measuring financial risks. However, there have been recent proposals to replace it, for regulatory purposes, by another risk measure called the Expected Shortfall.

The Expected Shortfall at level $\alpha \in (0, 1]$ of a position $\mathbf{X}$ is given by

$$ES_\alpha(\mathbf{X}) = \frac{1}{\alpha} \int_0^\alpha \mathrm{V@R}_\gamma(\mathbf{X}) \mathrm{d}\gamma.$$

The Expected Shortfall also requires the specification of a confidence level α, but takes into account the size of the losses when the Value-at-Risk is exceeded, by taking the mean of the losses (see [40, 24]).

In fact, while Value-at-Risk satisfies natural consistency properties such as monotonicity, positive homogeneity and translation invariance, it suffers from serious defects from a practical standpoint. First, it fails to satisfy the requirement of subadditivity. Subadditivity of a risk measure states that the combined risk of two financial positions should be less than the sum of their individual risks. By not satisfying subadditivity, it may fail to encourage diversification. Second, the $V@R_\alpha(\mathbf{X})$ can, in principle, expose an investor to large financial losses as it is defined only in terms of the probability of loss. It does not quantify the loss if it occurs. Both defects are repaired by the Expected Shortfall measure (see [24, 40]).

In their path-breaking set of papers, Artzner *et al.* [1, 2] proposed an axiomatic approach to the quantification of risk. They used economic insights to show the properties that a good risk measure should possess. They specified the properties of subadditivity, monotonicity, positive homogeneity and translation invariance. Risk measures which satisfy the above axioms are called coherent risk measures. Furthermore, they gave a characterisation of coherent risk measures in terms of convex sets of measures, all of which are assumed to be absolutely continuous with respect to the objective probability measure. Specifically, given a random payoff $\mathbf{X}$, the coherent risk measure of $\mathbf{X}$ is defined as the maximum expected loss evaluated under a convex set of probability measures, called generalised scenario measures. Acceptable positions are positions for which the maximum expected loss is non-positive. This means that the minimum expected worth is non-negative. This approach allows a risk measure to be defined as the minimal amount needed to make a position acceptable and, again, bypasses the need to pick one particular measure as the de facto pricing measure. It is not a priori clear, though, whether this could play the role of a price, or indeed how to define one.

The coherent risk measure idea has a strong economic justification and is an improvement of the Value-at-Risk method to quantify financial risk. However, academics and practitioners have expressed concern about the homogeneity axiom, which seems to encourage an arbitrary scaling up of risk. Also, implicit in the homogeneity axiom is the idea that diversification does not reduce the risk of a portfolio. Föllmer and Schied [22, 23, 24] suggested that market risk may increase in a non-linear way with the value of the portfolio. According to them, excessive up-scaling of a position may create liquidity risk. In light of this, they proposed that both the subadditivity and homogeneity axioms be weakened to the convexity axiom (see also Frittelli *et al.* [26, 27], Bingham and Ostaszewski [8]).

An approach that attempts to extend the arbitrage pricing theory to incomplete markets while taking into account the ideas put forward by Artzner *et al.* [1, 2] was proposed by Carr, Geman and Madan [12]. It is the intellectual progenitor for the rest of this chapter and also the next chapter on multi-period models. In their paper, these authors suggested that the pricing of contingent claims should be predicated on a finite set of measures with a floor associated to each of these measures. This set comprises the *valuation test measures*, with associated floor equal to zero, and *stress test measures*, with negative floors. Given a random payoff $\mathbf{X}$, a floor associated with a measure $\mathbf{P}$ is a non-positive real number f such that $E_{\mathbf{P}}(\mathbf{X}) \geq f$ or $E_{\mathbf{P}}(\mathbf{X}) < f$. They defined the acceptability of a random payoff representing the potential gain from a derivative position to mean that its expectation under each of the valuation test measures is non-negative, and its valuation under each stress test measure should weakly dominate the floor associated with that stress test measure. When all the floors are equal to zero, the definition of acceptability of a position in this framework coincides with the definition given in [1, 2]. It is important to note the sign differential arising from the fact that [1, 2] deal with losses, while [12] deals with gains. Further, [12] defines the concept of No Strictly Acceptable Opportunities and gives a result, which shows the equivalence between No Strictly Acceptable Opportunities and the existence of a pricing measure. This pricing measure is called a representative state pricing function and it is a convex combination of the valuation test measures.

In the following section, we extend their basic result to a single-period market model with infinitely many valuation test measures. We refer to section 1.4 below for the motivation for considering models with infinite sets of valuation test measures. These may be either countably or uncountably infinite, and our treatment of the latter will need the introduction of a further modelling component in the form of an a priori weight or measure on the set of all valuation test measures. We will specify this in section 1.5, but first treat the case of countably many valuation test measures.

1.2 Single-Period Market Models with Countably Many Valuation Test Measures

1.2.1 Acceptable Opportunities

We consider a single-period market model, that is, we allow trading only at time $t = 0$ and $t = 1$ (say) and let $S_j, j = 1 \dots N$ model the (discounted) random payoffs of the N liquidly-traded risky assets. We also assume the existence of a risk-free asset with payoff S_0 and interest rate $r = 0$. We denote the initial prices of these assets by the vector $(\pi_0 \dots \pi_N)$, and normalise the price of the risk-free asset by setting $\pi_0 = 1$.

We assume that there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ which models the possible states of the world at time $t = 1$, together with a σ-algebra $\mathcal{F}$ of economically-relevant events and their respective probabilities. We write $\mathbf{P} : \mathcal{F} \to [0, 1]$ and also assume that the state space Ω will, in general, be (uncountably) infinite.

We specify a countable collection $\Lambda = \{Q_j : j \in \mathbb{N}^* = \{1, 2 \dots\}\}$ of test measures on $(\Omega, \mathcal{F})$, where to each test measure Q_j, we associate a floor $f_j \leq 0$. We then attempt to make a decision on the basis of this collection of measures. Some of these measures have floors equal to zero, while the others have strictly negative floors. Measures with zero floors are called *valuation test measures*: these are the measures that are relevant for pricing purposes, following [12]. We denote the space of all contingent cashflows in our market model by $\mathcal{X}$: this is a linear space of measurable functions on $(\Omega, \mathcal{F})$, which contains the liquidly-traded assets S_i, $i = 0, \dots, N$. It may also contain others, such as certain Over-The-Counter derivatives written on the S_i (liquidly-traded derivatives, such as very liquid calls and puts, might already be included amongst the S_i, since they will typically have well-defined market prices, which are determined by the mechanism of supply and demand). We will assume that $E_{Q_j}(X)$ exists for each $X \in \mathcal{X}$ and each test measure Q_j, that is:

$$\mathcal{X} \subset \bigcap_{j=1}^{\infty} L^1(\Omega, \mathcal{F}, Q_j).$$

Finally, we let $\mathcal{S} = \mathrm{Span}\{S_0, \cdots, S_N\}$ be the linear span of the liquidly-traded assets, which can be interpreted as the space of portfolios $\alpha_0 S_0 + \cdots + \alpha_N S_N$ ($\alpha_i \in \mathbb{R}$) and let

$$\pi : \mathcal{S} \to \mathbb{R}$$

be a linear functional. The interpretation of this functional is that it gives the market prices of the liquidly-traded claims. We take these prices as given. One of the aims of derivative pricing is to extend this pricing functional in a reasonable way to all of $\mathcal{X}$. Such an extension is, in general, not unique. However, we will show that it is unique if the market is *acceptably complete*. We will call the quadruple $(\mathcal{X}, \mathcal{S}, \pi, \Lambda)$ a market model.

We will sometimes assume that the valuation test measures are all absolutely continuous with respect to the reference probability measure $\mathbf{P}$, but this is not necessary. Λ could be an arbitrary set of probability measures, though one would, in practice, expect and hope that these have some relationship to the objective probability $\mathbf{P}$. This will, for example, be the case when the Q_j are constructed using utility functions: see section 1.4 below. We denote the pricing functional by π and let $\mathbb{N}^*$ be the set of all natural numbers.

Definition 1.2.1 (Carr, Geman and Madan, [12]). A financial position $Y \in \mathcal{X}$ is called an *acceptable opportunity* if

(i) $\pi(Y) = 0$

.

(ii) $E_{Q_j}(Y) \geq f_j$ for all $j \in \mathbb{N}^* = \{1, 2, \ldots\}$.

As mentioned, valuation test measures have floor $f_j = 0$. The interpretation is that these measures determine when a cashflow (trade, position) X is acceptable at the margin: see also the examples of section 1.4. Measures with strictly negative floors are called *stress test measures*: their role is to prevent unlimited scaling up of a trade X which is acceptable at the margin. We want to avoid a situation where λX is acceptable for all valuation test measures, but it is not acceptable for all stress test measures: for all j for which $E_{Q_j}(X) \geq 0$, $E_{Q_j}(\lambda X) \geq 0$ for $\lambda > 0$, but $E_{Q_j}(\lambda X) \geq f_j$ is no longer true for all $f_j < 0$, although $E_{Q_j}(X) \geq f_j$ is true for all $f_j < 0$. An example of a stress test measure would be one which puts a lower bound on the expected loss, where we only take into account the future states where $X < 0$. We note that, in this case, the corresponding measure would certainly not be equivalent to $\mathbb{P}$, since it assigns zero probability to positive outcomes. Another example might be requiring a certain lower bound on the expected shortfall $ES_\alpha(X) = E_P(X|X > V@R_\alpha(X))$ for a given confidence level α close to 1.

On the other hand, the stress test measures will always be trivially satisfied on the margin if we scale down X by taking λ sufficiently small. This implies that the valuation test measures should be used to determine prices. *From now on, we will assume that all our test measures are valuation test measures with floor* $f_j = 0$. An acceptable position is an investment opportunity with zero initial cost which has a non-negative expected payoff under each of the test measures.

It is also important that for acceptability, all expected payoffs must satisfy the condition: $E_{Q_j}(X) \geq 0$, not just for some of them. According to Carr *et al.* [12]: "The central idea in our definition of acceptability is that every reasonable person would take the view that the benefits engendered by the gains adequately compensate for the costs imposed by the losses. One can regard these persons as counterparties willing to take the other side should one decide to exit after entering. By requiring that each person in a specified set finds the trade agreeable, one can enter the trade assured that there are multiple avenues for exit" . This emphasises the importance of valuation test measures as tools by which different market participants judge investment opportunities.

Definition 1.2.2 (Carr, Geman, Madan, [12]). A financial position $Y \in \mathcal{X}$ is called *strictly acceptable* if it is acceptable (according to Definition 1.2.1) and $E_{Q_j}(Y) > 0$ for at least one $j \in \mathbb{N}^*$.

We denote by $\mathcal{L}^+$ the set of positive (= non-negative) random variables on $(\Omega, \mathcal{F}, \mathbf{P})$. If Ω is finite or denumerable and $\mathcal{F}$ is the discrete σ-algebra of all subsets of Ω, $\mathcal{L}^+$ can be identified with the positive orthant of the space $\mathbb{R}^{|\Omega|}$ generated by the states in $(\Omega, \mathcal{F}, \mathbf{P})$. Furthermore, we let $\mathcal{A}^+(\Lambda)$ be the space of all strictly acceptable cashflows, where we recall that Λ is the set of all valuation test measures.

Definition 1.2.3 (Carr, Geman and Madan, [12]). A state pricing function Q is a probability measure on $(\Omega, \mathcal{F})$ such that $\pi_j = \pi(S_j) = E_Q(S_j)$ for all $j \in \{0, 1, 2, \ldots N\}$. A state pricing function Q is called a representative state pricing measure (RSPM) if there exists a set of positive weights λ_j with $\sum_{j=1}^{\infty} \lambda_j = 1$ such that

$$Q(\omega) = \sum_{j=1}^{\infty} \lambda_j Q_j(\omega)$$

for all $\omega \in \Omega$.

Definition 1.2.4 (Carr, Geman and Madan, [12]). A market model satisfies the *condition of no strictly acceptable opportunities* (NSAO) if no acceptable opportunity in $\mathcal{S}$ is strictly acceptable: we cannot construct a portfolio $\alpha = (\alpha_0, \ldots \alpha_N)$ of the assets S_i such that

(i) $\sum_{n=0}^{N} \alpha_n \pi_n = 0$

(ii) $\sum_{n=0}^{N} \alpha_n S_n \in \mathcal{A}^+(\Lambda)$.

In other words, there should not exist a zero-cost portfolio $\sum_{n=0}^{N} \alpha_n S_n$, which is a strictly acceptable opportunity. Observe that the NSAO condition only involves the liquidly-traded assets $\mathcal{S} \subset \mathcal{X}$, not all the cashflows.

Carr *et al.* [12] investigated the implications for asset pricing when there are no strictly acceptable opportunities for a finite set of valuation test measures.

An immediate observation is that if the set of valuation test measures $\Lambda = \{Q_j\}$ has the property:

$$F \in \mathcal{F}, \mathbf{P}(F) > 0 \Rightarrow \exists j : Q_j(F) > 0, \tag{1.1}$$

then an arbitrage in the classical sense would be a strictly acceptable opportunity[1]. Classical results going back to Harrison and Kreps [32] show that if there are no-arbitrage opportunities, then a pricing measure exists: see for example [24], Theorem 1.7.

We will call sets of valuation test measures satisfying (1.1) *relevant*, in the sense that together they cover all future events which can occur with positive objective probability. This will be trivial in the case where at least one Q_j is equivalent to $\mathbf{P}$, as they have the same null sets. In practice, if Ω is very large, it is not inconceivable that the set of valuation test measures used by investors may fail to cover all the events, which can occur with positive objective probability. For example, prior to the financial crisis of 2007-2008 (the "credit crunch"), market agents seemed to have failed to take into account the likelihood that defaults on house loans could be correlated across different states of the US, leading to mispricing of Collateralized Debt Obligations.

We now prove a result, which holds exclusively in the NSAO setting. The result asserts the existence of a pricing measure, which is a convex combination of the valuation test measures when there are no strictly acceptable

[1] We recall that an arbitrage opportunity is a zero-cost portfolio α such that $\sum_{n=1}^{N} \alpha_j S_j \geq 0$ with $\mathbf{P}$-probability 1 and such that this random variable is strictly positive on a set $F \in \mathcal{F}$ with strictly positive $\mathbf{P}$ -measure (see [24, 7])

opportunities. This is a first generalisation of the corresponding result of Carr, Geman and Madan [12] for finite sets of valuation test measures. We shall discuss later in this chapter why considering infinite, and even uncountable, sets of valuation test measures is natural as it opens up the opportunity for a general perspective.

Theorem 1.2.5 *If the market model* $\mathcal{M} = (\mathcal{X}, \mathcal{S}, \pi, \Lambda)$ *with a set of valuation test measures* $\Lambda = \{Q_j : j \in \mathbb{N}^*\}$ *has no strictly acceptable opportunities, then there exists a representative state pricing measure (RSPM)* $Q = \sum_{j\in\mathbb{N}^*} \lambda_j Q_j$ *with* $\sum_{j\in\mathbb{N}^*} \lambda_j = 1$ *and* $\lambda_j > 0$ *for all* j*.*

The fact that we can find a Q with $\lambda_j > 0$ for all j can be seen as an analogue of the classical fact that if there is no arbitrage, then one can find a risk-neutral pricing measure, which is equivalent to the objective measure $\mathbf{P}$. Note that if all Q_j are absolutely continuous with respect to $\mathbf{P}$, then Q is absolutely continuous with respect to $\mathbf{P}$ also. If $dQ_j = f_j d\mathbf{P}$ with $0 \leq f_j \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ of integral 1, then $dQ = f d\mathbf{P}$, with

$$f(\omega) = \sum_{j\in\mathbb{N}^*} \lambda_j f_j(\omega),$$

where the right-hand side converges for all ω (possibly to $+\infty$ for some ω). $f(\omega)$ is integrable with respect to $\mathbf{P}$ since by the monotone convergence theorem,

$$\int_\Omega f(\omega) d\mathbf{P} = \sum_{j\in N^*} \lambda_j \int_\Omega f_j(\omega) d\mathbf{P} = \sum_{j\in N^*} \lambda_j = 1.$$

If the set of valuation test measures Λ is relevant in the sense of (1.1), then the representative state pricing measure Q of Theorem 1.2.5 will be equivalent to $\mathbf{P}$ in the sense that they will have the same null sets.

In the case where we have a finite set of valuation test measures $\{Q_j : j = 1, \ldots, k\}$, the condition that $\lambda_j > 0$ for all j amounts to saying that Q lies in the relative interior[2] of the convex hull of Λ: $Q \in \mathrm{co}(\Lambda)^\circ$. For an infinite set of valuation test measures, this is less clear and will probably depend on the topology which we put on the linear space spanned by all probability measures. We have not investigated this further.

We will give two proofs of Theorem 1.2.5, the first one is based on Dieudonné's separating hyperplane theorem (see [19]) for two non-bounded closed convex sets in infinite-dimensional vector spaces, one of which is locally compact. We refer to the appendix for the precise statement and proof.

First proof of Theorem 1.2.5. Suppose that there are no strictly acceptable opportunities, and let V be the set of all zero-cost portfolios in the assets $S_0, \ldots S_N$:

$$V = \{Y = \alpha \cdot S = \sum_{k=0}^{N} \alpha_k S_k : \ \alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{R}^{N+1}, \ \sum_{k=0}^{N} \alpha_k \pi_k = 0\};$$

note that V is a finite-dimensional vector space. We will construct a map from V into $\ell^1 = \ell^1(\mathbb{N}^*)$ such that the image of V intersects the positive cone of ℓ^1 only at the point 0. Let $(w_j)_{j \geq 1}$ be a sequence of strictly positive real numbers such that

$$\sum_{j \in N^*} w_j E_{Q_j}(S_k) < \infty, \quad k = 0, 1, \ldots, N.$$

One can, for example, take

$$w_j = \frac{1}{2^j} \frac{1}{\sum_{k=0}^{N} |E_{Q_j}(S_k)|}.$$

[2]The relative interior of a convex set C is the interior of C in the smallest linear subspace which contains C (that is, the linear span of C).

For $Y \in V$ (or, more generally, in the subspace spanned by the S_k's), let

$$A(Y) = \big(w_j E_{Q_j}(Y)\big)_{j \in \mathbb{N}^*}.$$

Then $A : V \to \ell^1(\mathbb{N}^*)$, and since there are no strictly acceptable opportunities, we have that

$$A(V) \cap L_{\geq 0} = \{0\},$$

where $L_{\geq 0} = \{x = (x_j)_{j \geq 1} \in \ell^1 : x_j \geq 0\ \forall j\}$: the positive cone of ℓ^1. Hence if

$$B = \{x \in \ell^1 : x_j \geq 0, \sum_{j=1}^{\infty} x_j \geq 1\},$$

then B is a closed convex subset of ℓ^1, and $A(V) \cap B = \emptyset$. One can easily check that the recession cone (see Definition A.1.1 in the Appendix) C_B of B is equal to $L_{\geq 0}$. In contrast, $A(V)$ is equal to its recession cone. Since $A(V)$ is locally compact (it is finite dimensional) and $A(V) \cap L_{\geq 0} = \{0\}$, it follows from Theorem A.1.2 (see the Appendix) that there exists a linear functional $f \in (\ell^1)^* = \ell^\infty$, $f = (f_j)_{j \geq 1}$, and a real number γ such that

$$f(a) < \gamma \leq f(b),$$

$\forall a \in A(V)$, $\forall b \in B$. Since $0 \in A(V)$, $\gamma > f(0) = 0$, and since $A(V)$ is a linear subspace, $f = 0$ on $A(V)$: indeed, suppose that $f(a) \neq 0$ for some $a \in A(V)$. Replacing a by $-a$ if necessary, we may suppose that $f(a) > 0$, but then $f(\lambda a) = \lambda f(a) \to \infty$ as $\lambda \to \infty$, contradicting the fact that $f(\lambda a) < \gamma$ for all λ.

It follows that $f(h) > 0$ for all $h \in B$, which implies (on taking $h = e_j = \delta_{jn} = (0, \ldots, 0, 1, 0 \ldots)$) that $f_j = f(e_j) > 0$ for all j. We now define the measure $\mathbb{Q}$ by

$$\mathbb{Q} = \sum_{j \in N^*} \lambda_j Q_j; \quad \lambda_j = \frac{f_j w_j}{\sum_{\nu \in \mathbb{N}^*} w_\nu f_\nu}.$$

Then $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ and if $Y \in V$, then

$$E_{\mathbb{Q}}(Y) = \frac{\sum_{j\in N^*} f_j w_j E_{Q_j}(Y)}{\sum_{j\in N^*} f_j w_j E_{Q_j}(1)} = \frac{f(A(Y))}{f(A(1))} = 0,$$

since $f = 0$ on $A(V)$. In particular, let $Y = \alpha \cdot S$ with $\alpha = (-\pi_j, 0, \ldots, 1, 0, \ldots, 0)$ with $\alpha_j = 1$ and $\alpha_i = 0$ for $i \neq j$. Using the fact that $\pi_0 = 1$ (since we are working with discounted prices), it follows that

$$\pi_j = E_{\mathbb{Q}}(S_j),$$

showing that $\mathbb{Q}$ is a representative state pricing measure. ■

Second proof of Theorem 1.2.5. The second proof is modelled on the proof of the fundamental theorem of asset pricing for single-period models in Föllmer and Schied, [24]. It uses the following version of the separating hyperplane theorem for convex subsets of finite-dimensional spaces, where we let $v \cdot w$ denote the standard Euclidean inner product on $\mathbb{R}^p$, $p \in \mathbb{N}^*$.

Theorem 1.2.6 (Separating Hyperplane Theorem, (see [45], [42])) *Suppose that $\mathcal{C} \subset \mathbb{R}^p$ is a convex subset such that $0 \notin C$. Then there exists a $v \in \mathbb{R}^p$ such that $v \cdot x \geq 0$ for all $x \in \mathcal{C}$. Moreover, $v \cdot x > 0$ for at least one $x \in \mathcal{C}$.*

The first part of the theorem is standard. For the second part, refer to A.2 in the appendix. The example of $C = (0, \infty) \subset \mathbb{R}$ shows that 0 cannot be strictly separated from all of the points of $(0, \infty)$.

Now let

$$\mathcal{C} = \{E_{\mathbb{Q}}(S) - \pi(S) : \mathbb{Q} = \sum_{j\in N^*} \lambda_j Q_j,\ \lambda_j > 0, \sum_{j\in\mathbb{N}^*} \lambda_j = 1\} \subset \mathbb{R}^N,$$

where $E_{\mathbb{Q}}(S) - \pi(S) = (E_{\mathbb{Q}}(S_1) - \pi(S_1), \ldots, E_{\mathbb{Q}}(S_N) - \pi(S_N))$, which also equals $E_{\mathbb{Q}}(S - \pi(S)S_0)$, the Q_j being probability measures, and S_0 being equal to 1_Ω. The existence of a representative state pricing measure in $\mathrm{co}(\Lambda)^\circ$ is equivalent to 0 belonging to $\mathcal{C}$. We will argue by contradiction, and therefore suppose that $0 \notin \mathcal{C}$. By the separating hyperplane theorem, there exists a $v \in \mathbb{R}^N$, $v \neq 0$, such that

$$v \cdot x \geq 0, \quad \forall x = E_{\mathbb{Q}}(S) - \pi(S) \in \mathcal{C},$$

which is equivalent to

$$\sum_{j \in N^*} \lambda_j \big(E_{Q_j}(v \cdot S) - v \cdot \pi(S)\big) \geq 0, \quad \forall \lambda_j > 0 \text{ such that } \sum_{j \in N^*} \lambda_j = 1.$$

We show that this implies that $X = v \cdot S - v \cdot \pi(S) S_0$ is acceptable, that is, $E_{Q_j}(X) \geq 0$, for all j. After renumbering, we may assume, without loss of generality, that $j = 1$. By taking $\lambda_\varepsilon := (1 - \varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2^2}, \ldots)$, we find that

$$(1-\varepsilon) E_{Q_1}\left(v \cdot S - v \cdot \pi(S) S_0\right) + \sum_{j=2}^{\infty} \frac{\varepsilon}{2^{j-1}} E_{Q_j}\left(v \cdot S - v \cdot \pi(S) S_0\right) \geq 0,$$

and letting $\varepsilon \to 0$, we arrive at the desired conclusion.

Since $X = v_1 S_1 + \cdots + v_N S_N - v \cdot \pi(S) S_0$ has zero initial cost, therefore it is an acceptable opportunity. To show that it is a strictly acceptable opportunity, we use the fact that there exists an $x_0 \in \mathcal{C}$ for which $v \cdot x_0 > 0$. This means that there exist $\lambda_j^0 > 0$, $\sum_{j \in \mathbb{N}^*} \lambda_j^0 = 1$ such that

$$\sum_{j \in N^*} \lambda_j^0 \big(E_{Q_j}(v \cdot S) - v \cdot \pi(S) S_0\big) > 0,$$

which implies that there exists at least one Q_j such that

$$E_{Q_j}(X) = E_{Q_j}(v \cdot S) - v \cdot \pi(S) > 0.$$

Hence, X is a strictly acceptable opportunity, which contradicts the hypothesis that no such opportunities exist. ■

Remark 1.2.7 We note that the converse of Theorem 1.2.5 is also true: if there exists a representative state pricing measure $\mathbb{Q} = \sum_{j=1}^{\infty} \lambda_j Q_j$ with $\lambda_j > 0$ for all j and $\sum_{j=1}^{\infty} \lambda_j = 1$, then there exists no strictly acceptable opportunities. Suppose $\alpha \cdot S = \sum_{i=0}^{N} \alpha_i S_i$ is an acceptable opportunity. Then $\alpha \cdot E_{\mathbb{Q}}(S) = \alpha \cdot \pi(S) = 0$, or

$$\sum_{j \in N^*} \lambda_j E_{Q_j}(\alpha \cdot S) = 0.$$

Since $\mathbb{E}_{Q_j}(\alpha \cdot S) \geq 0$ and $\lambda_j > 0$ for all j, this then implies that $E_{Q_j}(\alpha \cdot S) = 0$ for all j, so no acceptable opportunity can be strictly acceptable. Note that it is important here that all "weights" λ_j are strictly positive.

Once we have constructed a representative state pricing measure $\mathbb{Q} = \sum_{j \in N^*} \lambda_j Q_j$, we can use it to value claims X in $\mathcal{X}$ which are not in $\mathcal{S}$ by taking $\mathbb{E}_{\mathbb{Q}}(X)$ as the price of X. Of course, in case there are more than one representative state pricing measure, then we are faced with a similar problem as the one in the introduction, when there are more than one risk-neutral measure. However, for the set of claims we will examine in the next section, this price will be unique.

1.3 Acceptable Completeness in Financial Market Models

In a complete market model, it is possible to replicate all contingent claims and this results in the existence of a unique martingale measure, which is effectively the pricing measure. Furthermore, the residual risk after hedging a short position in a contingent claim is zero. However, in an incomplete market, the situation is much more complicated as there are many possible martingale measures to choose from, and not all contingent claims can be replicated exactly. This section attempts to provide concepts analogous to those in the complete market model framework and characterise the new concepts in terms of the martingale measures. Our discussion is a reformulation of the result in the paper by Carr, Geman and Madan, [12], allowing for some generalisation and simplification. Let $\Lambda = \{Q_k : 1 \leq k \leq K\}$ be a set of valuation test measures. We will see below that, contrary to the previous section, this set will have to be finite if we want the representative measure Q to be unique in a strong sense. We continue to work on a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The key definition is the following[3]:

Definition 1.3.1 (Carr, Geman and Madan, [12]) The market model $(\mathcal{X}, \mathcal{S}, \pi, \Lambda)$ is called *acceptably complete* if, for all $X \in \mathcal{X}$, there exist constants $\alpha_n \in \mathbb{R}$, $n = 0, 1 \ldots N$ such that, for all $Q_k \in \Lambda$,

$$E_{Q_k}\left(-X + \sum_{n=0}^{N} \alpha_n S_n\right) = 0.$$

We will then say that X can be acceptably hedged by the portfolio $(\alpha_0, \alpha_1, \ldots \alpha_N)$.

Therefore, in this framework, the requirement is that the excess of the expected payoff of the residual over the floor should be zero for each test measure. This replaces the requirement in the classical framework that the residual error after hedging should be equal to zero.

Definition 1.3.2 (Carr, Geman and Madan, [12]). A representative state pricing measure (RSPM) $\mathbb{Q}$ is *strongly unique* if there exists at most

[3]Recall that we are only considering valuation test measures; for the general case, replace 0 on the right-hand side by the floor f_k.

one set of positive weights, w_k for for each Q_k such that

$$\mathbb{Q}(\omega) = \sum_{Q_k \in \Lambda} w_k Q_k(\omega)$$

is a state pricing function. We will call $\mathbb{Q}$ *weakly unique*, relative to the space of claims $\mathcal{X}$, if any other RSPM has the same expectations on $\mathcal{X}$, that is, if it is unique as a linear functional on $\mathcal{X}$.

We now show that the RSPM is unique if all contingent claims can be acceptably hedged. If $X \in \mathcal{X}$, we let

$$E_\Lambda(X) = (E_{Q_1}(X), \ldots, E_{Q_K}(X)) \in \mathbb{R}^K$$

be the vector of its expected values under the valuation test measures. We will call this the *evaluation vector* of X with respect to Λ. Acceptable completeness means that $E_\Lambda(\mathcal{X})$ be contained in the span of $E_\Lambda(S_i)$, $i = 0, \ldots, N$. In particular, we should have that $N + 1 \geq \dim(E_\Lambda(\mathcal{X}))$, so there should be sufficiently many tradeable 'hedging instruments' S_i in the market.

With this definition, we now have the following extension of the classical uniqueness result in complete markets. We continue to assume that the market does not admit strictly acceptable opportunities so that at least one representative state pricing measure $\mathbb{Q} = \sum_{k=1}^{K} \lambda_k Q_k$ exists, with $\lambda_k > 0$ for all k.

Theorem 1.3.3 *The market model is acceptably complete if and only if there exists a weakly unique representative state pricing function, in the sense that two representative state pricing functions give identical expectations on* $\mathcal{X}$.

Proof. Suppose that the market model is acceptably complete, and let $\mathbb{Q}$ and $\mathbb{Q}'$ be two representative state pricing functions in $\mathrm{co}(\Lambda)$. We show that

$E_{\mathbb{Q}}(X) = E_{\mathbb{Q}'}(X)$ for all $X \in \mathcal{X}$. Indeed, given $X \in \mathcal{X}$, there exists a 'hedge' $(\alpha_0, \ldots, \alpha_N)$ such that

$$E_{Q_k}(X) = \sum_{i=0}^{N} \alpha_i E_{Q_k}(S_i).$$

Hence if $\mathbb{Q} = \sum_{k=1}^{K} \lambda_k Q_k$, then

$$\begin{aligned} E_{\mathbb{Q}}(X) &= \sum_{k=1}^{K} \lambda_k E_{Q_k}(X) \\ &= \sum_{k=1}^{K} \sum_{i=0}^{N} \lambda_k \alpha_i E_{Q_k}(S_i) \\ &= \sum_{i=0}^{N} \alpha_i E_{\mathbb{Q}}(S_i), \end{aligned}$$

which only depends on the α_i and the expectations of the S_i under $\mathbb{Q}$. If $\mathbb{Q}'$ is another representative state pricing function, then $E_{\mathbb{Q}'}(S_i) = \pi_i = E_{\mathbb{Q}}(S_i)$ for all i, and it follows that $E_{\mathbb{Q}'}(X) = E_{\mathbb{Q}}(X)$, as claimed.

We note in passing that we do not claim that if $\mathbb{Q} = \sum_{k=1}^{K} \lambda_k Q_k$ and $\mathbb{Q}' = \sum_{k=1}^{K} \lambda'_k Q_k$ are two representative state pricing measures, then $\lambda_k = \lambda'_k$ for all k: after all the Q_k might be linearly dependent. The following theorem gives a condition that guarantees uniqueness.

To show that weak uniqueness of a representative state pricing measure on $\mathcal{X}$ implies acceptable completeness, let

$$\mathcal{V} = \left\{ E_\Lambda(Y) : Y = \sum_{i=0}^{N} \alpha_i S_i : \alpha_i \in \mathbb{R} \right\} = \left\{ \big(\sum_{i=0}^{N} \alpha_i E_{Q_k}(S_i) \big)_{k=1}^{K} \right\},$$

and suppose that $\mathcal{V}$ is a proper subspace of $\{E_\Lambda(X) : X \in \mathcal{X}\}$. We then can find a $Z \in \mathcal{X}$ such that $E_\Lambda(Z)$ is orthogonal to $\mathcal{V}$, that is, the Euclidean inner product

$$E_\Lambda(Z) \cdot E_\Lambda(S_i) = \sum_{k=1}^{K} E_{Q_k}(Z)\, E_{Q_k}(S_i) = 0, \quad i = 0, \ldots, N.$$

Let $z = E_\Lambda(Z) \in \mathbb{R}^K$. If we choose the portfolio $\alpha_0 = 1$ and $\alpha_j = 0$ for all $j = 1,2 \ldots N$, we find that the K-vector $E_\Lambda(Y) = (1, 1, \ldots 1)$ is orthogonal to z, that is

$$\sum_{k=1}^{K} z_k = E_\Lambda(Z) \cdot \underline{1} = 0.$$

After scaling, we can assume that $|z_k| < \lambda_k$ for all k. We claim that

$$\mathbb{Q}^{'} = \sum_{k=1}^{K} (\lambda_k + z_k) Q_k = \mathbb{Q} + \sum_{k=1}^{K} z_k Q_k,$$

is a representative state pricing function. It is a probability measure since it is a convex combination of the probability measures Q_k: each $\lambda_k + z_k > 0$ and

$$\sum_{k=1}^{K} (\lambda_k + z_k) = \sum_{k=1}^{K} \lambda_k + \sum_{k=1}^{K} z_k = 1,$$

and

$$\begin{aligned} \sum_{\omega} Q^{'}(\omega) &= \sum_{\omega} Q(\omega) + \sum_{\omega} \sum_{k=1}^{K} z_k Q_k(\omega) \\ &= 1 + \sum_{k=1}^{K} z_k = 1. \end{aligned}$$

Further, for any asset $S_i, i = 0, 1, 2, \ldots N$, we obtain

$$\begin{aligned} \sum_{k=1}^{K} (\lambda_k + z_k) E_{Q_k}(S_i) - \sum_{k=1}^{K} \lambda_k E_{Q_k}(S_i) &= \sum_{k=1}^{K} z_k E_{Q_k}(S_i) \\ &= E_\Lambda Z \cdot E_\Lambda S_i = 0. \end{aligned} \tag{1.2}$$

Therefore, $\mathbb{Q}'$ is another representative state pricing function. Finally, it differs from $\mathbb{Q}$ on $\mathcal{X}$, since

$$E_{\mathbb{Q}'}(Z) - E_{\mathbb{Q}}(Z) = \sum_{k=1}^{K} z_k^2 \neq 0,$$

so lack of acceptable completeness implies non-uniqueness of the representative state pricing measure. ■

Theorem 1.3.4 *The representative state pricing measure is strongly unique if and only if* $\mathrm{Span}\{E_\Lambda(S_i) : i = 0, \cdots, N\} = \mathbb{R}^K$. *In this case, the market is automatically acceptably complete.*

Proof. If we let $u \cdot v = \sum_{j=1}^{K} u_j v_j$ denote the Euclidean inner product on $\mathbb{R}^K$, then $\mathbb{Q} = \sum_{j=1}^{K} \lambda_j Q_j$ is a representative state pricing measure iff $\lambda = (\lambda_1, \ldots, \lambda_K)^{transpose}$ solves the system of equations

$$E_\Lambda(S_i) \cdot \lambda = \pi_i, \quad i = 0, \ldots, N, \tag{1.3}$$

with $\pi_i = \pi(S_i)$ the price of S_i. If the $E_\Lambda(S_i)$ span $\mathbb{R}^K$, these equations determine λ uniquely[4].

For the converse statement, if the $E_\Lambda(S_i)$ do not span $\mathbb{R}^K$, then we can find a $z \in \mathbb{R}^K$ which is perpendicular to the $E_\Lambda(S_i)$. Arguing as in the proof of the previous theorem, we can construct another representative state pricing measure.

Alternatively, one can observe that if the solution of the system (1.3) is

[4] For example, each basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ can be written as a sum of the $E_\Lambda(S_i)$, and $\lambda_i = \lambda \cdot e_i$

unique, then the linear map from $\mathbb{R}^K$ to $\mathbb{R}^{N+1}$ with matrix

$$C := \left(\mathbb{E}_{Q_j}(S_i)\right)_{i,j} = \begin{pmatrix} E_{Q_1}(S_0) & E_{Q_2}(S_0) & \dots & E_{Q_K}(S_0) \\ E_{Q_1}(S_1) & E_{Q_2}(S_1) & \dots & E_{Q_K}(S_1) \\ \vdots & \vdots & \ddots & \vdots \\ E_{Q_1}(S_N) & E_{Q_2}(S_N) & \dots & E_{Q_K}(S_N) \end{pmatrix}$$

has kernel 0. This implies that $\mathrm{Im}(C^t) = \mathbb{R}^K$, where C^t is the transpose. Since the columns of C^t are precisely the vectors $\mathbb{E}_\Lambda(S_i)$, this means that the latter span $\mathbb{R}^K$. ■

Remark 1.3.5 (i) Note that the proof of Theorem 1.3.3 shows that if a claim X can be acceptably hedged by the portfolio $(\alpha_0 \cdots \alpha_N)$, then its price under the RSPM $\mathbb{Q}$ is

$$E_{\mathbb{Q}}(X) = \sum_{i=0}^{N} \alpha_i \pi_i,$$

where $\pi_i = \pi(S_i)$ is the price of S_i.

(ii) Note that strong uniqueness of the RSPM implies that $K \leq N+1$, since the rank of the $(N+1) \times K$-matrix C has to be K.

(iii) Carr, Geman and Madan [12], in their discussion of uniqueness of the RSPM, introduce a condition which they called *undertesting*[5] and which, in our notation, is equivalent to $\{E_\Lambda(X) : X \in \mathcal{X}\} = \mathbb{R}^K$. They showed that when the undertesting condition is satisfied, strong uniqueness of the RSPM is equivalent to acceptable completeness. This is equivalent to our Theorem 1.3.3, since undertesting plus acceptable completeness is the same as saying

[5] They defined undertesting when the underlying probability space of the model is finite, $\Omega = \{\omega_1, \dots, \omega_L\}$, by requiring that if B is the matrix $B = (Q_j(\omega_i))_{i,j}$, then the linear map defined by the transpose $B^t : \mathbb{R}^L \to \mathbb{R}^K$ should be surjective. Here, $\mathcal{X}$ is implicitly taken to be the space of all measurable functions on Ω provided with the discrete σ-algebra, that is, the space of all functions of Ω into the set of real numbers.

that the $\mathbb{E}_\Lambda(S_i)$ span $\mathbb{R}^K$.

Undertesting is not necessary for weak uniqueness: at first sight one might think that if the span of the $E_\Lambda(X)$'s is strictly smaller than $\mathbb{R}^K$, then one can construct a new representative state pricing measure $\mathbb{Q}'$ by picking a vector $z \in \mathbb{R}^K$, which is orthogonal to this span, as in the proof above. However, the two measures would agree on $\mathcal{X}$, and can be considered identical for pricing purposes.

1.4 Examples of Economically-Motivated Valuation Measures

We review two methods for constructing valuation test measures, one based on expected marginal utility, and the other using the Sharpe ratio to identify what might be called good deals. The latter is related to, but not quite the same as, the work by Cochrane and Saá-Requejo [13] on good-deal bounds on the pricing measure, which was based on the Hansen-Jagannathan inequality [29]. Both methods will show that it is natural and, in certain cases, even inevitable to consider market models with infinitely (including uncountably) many valuation test measures. For example, to take into account the unknown portfolio holdings and risk preferences of the various market participants, or when the convex cone of acceptable opportunities is not polyhedral.

1.4.1 Utility-based Valuation Measures

This section further elaborates the ideas behind the second economy example in section 2 of Carr, Geman and Madan [12] in this more general framework. We assume that interest rates are equal to zero. Suppose that an investor with a utility function of wealth U, time 0 wealth w and time T random wealth W_T is presented with an investment opportunity X at a unit price of $\pi(X)$. If the agent decides to invest in a small quantity ϵ of X, then their expected utility becomes

$$E_{\mathbf{P}}\left(U(W_T - \epsilon\pi(X) + \epsilon X)\right),$$

since the agent would have spent $\epsilon\pi(X)$ at time 0 to acquire the payoff of ϵX at time T (see Davis [17]). The agent will consider this opportunity acceptable if it, at least, does not decrease their expected utility compared to the corresponding case when $\epsilon = 0$ (when not buying X), so we should have that

$$\frac{d}{d\epsilon} E_{\mathbf{P}}\left(U(W_T - \epsilon\pi(X) + \epsilon X)\right)\Big|_{\epsilon=0} \geq 0,$$

or

$$E_{\mathbf{P}}\left(U'(W_T)(X - \pi(X)\right) \geq 0.$$

By rearranging, we find that

$$E_{\mathbf{P}}\left(X \frac{U'(W_T)}{E_{\mathbf{P}}(U'(W_T))}\right) \geq \pi(X).$$

We can define the expectation of X under the new probability measure Q_U given by[6]

$$dQ_U = \frac{U'(W_T)}{E_{\mathbf{P}}(U'(W_T))} d\mathbf{P}.$$

Utility functions are, by definition, increasing: $U'(W_T)$ is a positive random variable. In particular, if X has zero initial cost, $(\pi(X) = 0)$, then X is acceptable from the viewpoint of expected utility maximisation if

$$E_{Q_U}(X) \geq 0.$$

This investor, therefore, can use Q_U as a valuation test measure.

[6] Q_U also depends on time T wealth W_T, though we suppress this from the notation for simplicity.

Now suppose we have several agents with time T wealth $W_T^i,\ i \in I$ and utility functions U_i. To be more specific, we can assume that

$$W_T^i = e^i + \alpha^i \cdot S_T,$$

where e^i and $\alpha^i = (\alpha_0^i, \alpha_1^i, \ldots, \alpha_N^i)$ are agent i's endowment and investment decision, respectively at time 0. Then, this will give us a family of valuation test measures Q_i with Radon-Nikodym derivatives

$$dQ_i = \frac{U_i'(e^i + \alpha^i \cdot S_T)}{E_{\mathbf{P}}(U_i'(e^i + \alpha^i \cdot S_T)} d\mathbf{P}, \quad i \in I.$$

As a further generalisation, we might allow each agent i to use their own subjective probability measure $\mathbf{P}_i$ to compute this expectation, where $\mathbf{P}_i$ may or may not all be absolutely continuous with respect to $\mathbf{P}$ (that is, agents not necessarily agreeing on which are the null events). This situation of *heterogeneous beliefs* will be further examined in the third chapter of this thesis.

Remark 1.4.1 One might naively think that if $\mathbb{E}_{U_i}(X) > 0$, then market participant i would accept the opportunity X, regardless of the views of the other participants, who may all assign a negative marginal utility to X. However, as stressed by [12], being able to exit a position is as important as taking the position. Therefore, the criterion for acceptability is indeed that $\mathbb{E}_{U_i}(X) \geq 0$ for all i, and not for some i.

In practice, we are unlikely to know the utility functions, the initial endowments and investment decisions of the individual investors in a financial market. To make this into a workable model, we can leave the last two as free parameters: $e \in \mathbb{R}$ or $(0, \infty)$ and $\alpha \in \mathbb{R}^{N+1}$ respectively, and choose a parametrized family of utility functions U_γ, where γ ranges over some subset Γ of some Euclidean space. This naturally leads to a model with an uncountable number of valuation test measures depending on parameters (e, α, γ).

As a utility function, one can, for example, take an exponential utility,[7] a constant relative risk aversion (CRRA) utility,[8] or the two-parameter class of hyperbolic absolute risk aversion (HARA) utility functions (see [24]). The latter includes the previous two as special cases.

With respect to the utility functions, there is a technical point which needs clarification: we need a utility function defined on all of $\mathbb{R}$ since the portfolio value might become negative with positive probability, but what is the utility of negative wealth? If we use the exponential utility $U(w) = 1 - e^{-w}$ for negative values of w, then $U(w) < 0$, $U'(w) > 1$ if $w < 0$, while $U'(w) < 1$ if $w > 0$. This would mean that marginal utility can become much bigger in states of negative wealth than in states of positive wealth, which seems somewhat counter-intuitive.

Venter [52] mentions that one sometimes requires that $U'(w) = 0$ for all $w < 0$ since, in finance, negative wealth corresponds to a state of default or bankruptcy, and bankruptcy laws do not differentiate between financial entities based on their level of bankruptcy. Venter suggested that we should take $U(w) = -U(-w)$ for negative wealth, where the utility U is initially defined for positive w, but did not develop this further. In the case of exponential utility, Venter's prescription gives $U(w) = e^{w} - 1$ for negative w, and the marginal utility $U'(w) = e^{w}$ will be less than one, as for positive wealth.

[7] $U_\gamma(w) = 1 - e^{-\gamma w}$, with absolute constant risk aversion parameter $\gamma > 0$

[8] $U_\gamma(w) := (w^{1-\gamma} - 1)/1 - \gamma$, $\gamma > 0$, initially defined for $w > 0$ and interpreted as $\log w$ if $\gamma = 1$.

1.4.2 Valuation test measures constructed from convex cones

We start with a general observation: given a set Λ of valuation test measures, the set

$$\mathcal{A} = \mathcal{A}(\Lambda) = \{X \in \mathcal{X} : E_Q(X) \geq 0, \forall\, Q \in \Lambda\}$$

is a convex cone of $\mathcal{X}$. If $\mathcal{X}$ is equipped with a norm such that expectations $X \mapsto E_Q(X)$ are continuous linear functionals with respect to this norm, then $\mathcal{A}$ is a closed convex cone. We will call $\mathcal{A} = \mathcal{A}(\Lambda)$ the cone of acceptable investments associated to Λ. An acceptable opportunity is an element of $\mathcal{A} \cap \ker(\pi) \cap \mathcal{S}$ (where we recall that $\mathcal{S}$ is the space of portfolios in the liquid assets) and the NSAO condition means that this intersection is the singleton $\{0\}$.

Conversely, assume that $\mathcal{X}$ is a real Banach space of measurable functions on $(\Omega, \mathcal{F})$. Given a cone $\Gamma \subset \mathcal{X}$, its *dual cone* is defined by

$$\Gamma^* = \{f \in \mathcal{X}^* : f(x) \geq 0, \forall x \in \Gamma\},$$

where $\mathcal{X}^*$ is the dual of $\mathcal{X}$. This is a closed convex cone in $\mathcal{X}^*$ provided with the weak-* topology, even if Γ is an arbitrary subset of $\mathcal{X}$. The bipolar theorem (see [45]) states that if $\Gamma \subset \mathcal{X}$ is a closed convex cone[9], then

$$\Gamma = \Gamma^{**} = \{x \in \mathcal{X} : f(x) \geq 0, \forall f \in \Gamma^*\}. \tag{1.4}$$

More generally, if $E \subset \mathcal{X}$ is an arbitrary subset, then E^{**} is the smallest closed convex cone containing E. If one can show that, for a given closed convex cone Γ, the elements of Γ^* can be interpreted as positive finite measures, then we can take the measures of mass 1 as valuation test measures. The bipolar theorem then implies that Γ can be interpreted as the cone of

[9] closed with respect to the norm topology or, equivalently (since Γ is convex), closed for the weak topology.

acceptable investments for these valuation test measures. To be able to interpret elements of Γ^* as positive measures, Γ should contain the cone of all non-negative bounded measurable functions on Ω.

The point is that for an arbitrary such convex cone Γ, its dual Γ^* will be an infinite set. Therefore, infinitely many valuation measures may be necessary to interpret Γ as the set of acceptable opportunities for these valuation test measures. This can be seen in the case where the sample space Ω is a finite set. Suppose $\Omega = \{\omega_1, \ldots, \omega_p\}$ is provided with the discrete σ-algebra $\mathcal{F}_{\text{disc}}$, then the set of measurable functions on Ω can be identified with $\mathbb{R}^p$. The set of positive measures can also be identified with the set of linear functionals on $\mathbb{R}^p$ which are positive on $\mathbb{R}^p_+ := \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p : x_i \geq 0, i = 1, \ldots, p\}$ which, identifying $\mathbb{R}^p$ with itself via the standard Euclidean inner product, is again equal to $\mathbb{R}^p_+$. If we take for Γ, a rotationally symmetric cone properly containing $\mathbb{R}^p_+$, and $p \geq 3$, then Γ cannot be described as the dual cone of a finite set of positive linear functionals. This is because the latter are polyhedral in that their boundary is piecewise flat (a union of finitely many polyhedral contained in hyperplanes), and the boundary of Γ is not. We now consider a concrete example which is relevant for financial practice.

Example 1.4.2 *Acceptance sets related to Sharpe ratios (see [13], [24]).* Assume that $(\Omega, \mathcal{F})$ is provided with a reference probability measure $\mathbf{P}$, interpreted as the objective probability. Investors might decide that an opportunity is acceptable if its expectation is large with respect to its variance. This suggests introducing the set

$$\Gamma_\alpha = \{X : s_X \geq \alpha\} = \{X : E(X) \geq \alpha\sigma(X)\}, \quad \alpha > 0, \tag{1.5}$$

where the expectation is taken with respect to $\mathbf{P}$, and where $\sigma(X)^2 = E(X - E(X))^2$ is the variance of X, also computed using $\mathbf{P}$. One checks that this is a convex cone: indeed, letting $||\cdot||_2$ be the L^2-norm, we have that $\sigma(X) =$

$||X - E(X)||_2$, and therefore

$$\begin{aligned} & \sigma(X_1 + X_2) = ||\, X_1 + X_2 - (E(X_1) + E(X_2))\, ||_2 \\ \leq \;& ||X_1 - E(X_1)||_2 + ||X_2 - E(X_2)||_2 \\ = \;& \sigma(X_1) + \sigma(X_2), \end{aligned}$$

which implies that $X_1 + X_2$ belongs to Γ_α if X_1 and X_2 belong to Γ_α. Also, $\lambda X \in \Gamma_\alpha$ if $X \in \Gamma_\alpha$ and $\lambda \geq 0$ since both sides in the defining inequality for Γ_α are positively homogeneous of degree 1: $\sigma(\lambda X) = \lambda\sigma(X)$ and similarly for $E(\lambda X)$. We note, for further use below, that since $\sigma(X)^2 = E(X^2) - E(X)^2$,

$$E(X) \geq \alpha\sigma(X) \Leftrightarrow E(X) \geq \gamma\sqrt{E(X^2)} = \gamma||X||_2, \tag{1.6}$$

where

$$\gamma = \gamma(\alpha) = \sqrt{\frac{\alpha^2}{1+\alpha^2}};$$

observe that $0 < \gamma < 1$ and that $\gamma \to 1$ if $\alpha \to \infty$.

Unfortunately, it is not always the case that the cone of positive L^2-functions $L^2_+(\Omega) \subset \Gamma_\alpha$, even if Ω is finite and the L^2-spaces are finite-dimensional. If $\Omega = \{\omega_1, \ldots, \omega_L\}$, we can identify $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbf{P})$ with the vector space $\mathbb{R}^L$ (identifying the random variable X with the vector $x = (X(\omega_1), \ldots, X(\omega_L)))$, provided with the inner product

$$(x, y) = \sum_i p_i x_i y_i,$$

where $p_i = \mathbf{P}(\{\omega_i\})$. Now, by convexity and homogeneity, the cone of positive functions $L^2_+(\Omega) = \mathbb{R}^L_+ \subset \Gamma_\alpha$ if the basis functions $e_i = (0, \ldots, 0, 1, 0 \ldots 0)$ $\in \Gamma_\alpha$ for $i = 1, \ldots, L$ which, by (1.6), is equivalent to

$$p_i = E(e_i) \geq \gamma||e_i||_2 = \gamma\sqrt{p_i},$$

or $\gamma \leq \sqrt{p_i}$. This forces γ to be small if L is large, for example, $\gamma \leq 1/\sqrt{L}$ if we take for $\mathbf{P}$ the uniform probability (all outcomes are equally likely). Since Sharpe ratios for US stocks are historically situated around $\alpha = 0.5$ or $\gamma^2 = 0.2$, this would limit L to be ≤ 5, which would not make for a very realistic model. The easiest way to repair this is to "manually" add the cone of positive functions to the acceptance set, and therefore define an acceptance set by

$$\mathcal{A}_\alpha = \overline{L^2_+(\Omega) + \Gamma_\alpha}, \tag{1.7}$$

where the bar denotes taking the closure (which is not necessary in finite dimension).

It is easy to describe the dual cone of Γ_α: since $E(X) = (X, \mathbf{1}_\Omega)$, where $\mathbf{1}_\Omega$ is the function which is identically equal to 1 on Ω, and since

$$||X||_2 = \sup_{||Y||_2 \leq 1} (X, Y),$$

the inner product being that of $L^2(\Omega, \mathcal{F}, \mathbf{P})$, we see that $X \in \Gamma_\alpha$ if and only if $(X, \mathbf{1}_\Omega) \geq (X, \gamma Y)$. Therefore, $(X, \mathbf{1}_\Omega - \gamma Y) \geq 0$, for all Y , $||Y||_2 \leq 1$. Putting $G = \mathbf{1}_\Omega - \gamma Y$ and observing that this belongs to the closed ball $B_\gamma = B(\mathbf{1}_\Omega, \gamma)$ of radius γ centred at $\mathbf{1}_\Omega$, we see that

$$\Gamma_\alpha = \{X : (X, G) \geq 0, \ \ \forall\, G \in B_\gamma\}.$$

If we let $B_{\gamma,+}$ be the set of a.s. non-negative functions in B_γ, then

$$\mathcal{A}_\alpha = \{X : (X, G) \geq 0, \ \forall\, G \in B_{\gamma,+}\},$$

and we may finally take as our set of valuation measures

$$\Lambda_\alpha = \left\{ \frac{G}{E(G)} : G \in B_{\gamma,+} \setminus 0 \right\}.$$

No Strictly Acceptable Opportunities then means that no zero-cost position $\sum_{i=0}^{N} w_i S_i$ has only positive outcomes or a Sharpe ratio exceeding α.

We briefly comment on the relationship of our construction with the Sharpe ratio, which is defined in terms of returns instead of prices. The return of a portfolio $X \in \mathcal{S}$ with non-zero price $\pi(X)$ is $r(X) := (X - \pi(X))/\pi(X)$, and its Sharpe ratio equals

$$s_X = \frac{E(r(X)) - r_f}{\sigma(r(X))},$$

where for clarity, we have reinstated the risk-free rate r_f. Using the fact that $\sigma((X - a)/b) =: \sigma(X)/|b|$ if $a, b \in \mathbb{R}$ with $a = b = \pi(X)$, we find that

$$s_X = \frac{E(X) - \pi(X)(1 + r_f)}{\sigma(X)},$$

an expression, which also makes sense if $\pi(X) = 0$. Taking $r_f = 0$ as before, the set of investments with a Sharpe ratio bigger than or equal to α is then precisely

$$\{X \in \mathcal{S} : X - \pi(X) \in \Gamma_\alpha\},$$

which has the same zero-cost opportunities as Γ_α. The difficulty in defining the acceptance set using the Sharpe ratio is that the pricing functional π is a priori only defined on $\mathcal{S}$, not on all of $L^2(\Omega)$. Indeed, if it were, we would basically be dealing with a complete market.

The geometry of the sets $\mathcal{A}_\alpha$ can be quite complicated, even in finite dimension. Geometrically, Γ_α is the set of vectors $x \in \mathbb{R}^L$, $x = (x_1, \ldots, x_L) = (X(\omega_1), \ldots, X(\omega_L))$, which make an angle $\varphi < \arccos(\gamma(\alpha))$ with the vector $\mathbf{1}_\Omega$ (which, incidentally, equals our risk-free asset S_0). This is the case because $E(X) = (X, \mathbf{1}_\Omega) = ||X||_2 \cos\varphi$, the norm of $\mathbf{1}_\Omega$ being 1. However, when converting this into a picture, we must remember that the metric here is not the standard Euclidean metric, but the one associated to $\mathbf{P}$: $(x, y) =$

$\sum_{i=1}^{L} p_i x_i y_i$. To convert this to standard Euclidean geometry, we introduce vectors

$$\widetilde{x} = (\sqrt{p_1}x_1, \ldots, \sqrt{p_L}x_L)\,, \quad \widetilde{e} = (\sqrt{p_1}, \ldots, \sqrt{p_L})\,.$$

Then $x \in \Gamma_\alpha$ if $(\widetilde{x}, \widetilde{e})_e \geq \gamma ||\widetilde{x}||_e$, where $(u, v)_e = \sum_{i=1}^{L} u_i v_i$ is now the standard Euclidean inner product on $\mathbb{R}^L$ and $|| \cdot ||_e$ its associated norm. So Γ_α (or more precisely, its image under the map $x \to \widetilde{x}$) is the cone of vectors making an angle of at most $\arccos(\gamma)$ with $\widetilde{e}$. Here $\widetilde{e}$ is a vector on the unit sphere $\{v : ||v||_e = 1\}$ with positive components, since the p_i sum to 1, being probabilities. Depending on the location of this vector, that is, the choice of measure $\mathbf{P}$ and the size of γ, this cone may either contain the positive cone $\mathbb{R}_+^L$, be contained in it, or have a non-zero intersection with its complement.

In the limiting case of $\alpha \to \infty$ or $\gamma \to 1$, Γ_α will shrink to just $\mathbb{R}_{\geq 0} \cdot \mathbf{1}_\Omega$, and $\mathbb{A}_\infty$ will, for large α, consist of just the positive functions, which can be defined by a finite set of measures in Ω. For finite state spaces at least, the construction then does not give any new acceptable opportunities beyond the classical no-arbitrage opportunities. If $\gamma = 0$, then $\Gamma_0 = \mathcal{A}_0$ is the half-space $\{X : E(X) \geq 0\}$, defined by just one measure.

If the state space is infinite, and if $\mathcal{F}$ contains events A of arbitrarily small positive probability, then Γ_α will never contain $L_+^2(\Omega)$: indeed, $X = \mathbf{1}_A \in \Gamma_\alpha$ iff $\mathbf{P}(A) \geq \gamma ||\mathbf{1_A}||_\mathbf{2}$ or $\mathbf{P}(A) \geq \gamma^2$. Conversely, Γ_α cannot be contained in $L_+^2(\Omega)$ either, for any $\alpha > 0$, however big. Suppose $B \in \mathcal{F}$ such that the probability of its complement $B^c = \Omega \setminus B$ satisfies $0 < \mathbf{P}(B^c) < 1 - \gamma^2$, with $\gamma = \gamma(\alpha)$. Then $\gamma^2 < \mathbf{P}(B) < 1$ will be arbitrarily close to 1 if α is large. Now consider

$$X = \mathbf{1}_B - \lambda \mathbf{1}_{B^c},$$

where $\lambda \geq 0$. Then $X \in \Gamma_\alpha$ if

$$\mathbf{P}(B) - \lambda \mathbf{P}(B^c) > \gamma \sqrt{\mathbf{P}(B) + \lambda^2 \mathbf{P}(B^c)}.$$

Since this is true for $\lambda = 0$, by the choice of B, it will remain true for small $\lambda > 0$, by continuity, and since $X \notin L^2_+(\Omega)$ then, this shows that Γ_α will contain cashflows which are not a.s. positive.

1.5 Infinite sets of Valuation Test Measures: the uncountable case

Suppose now that we have an infinite set of valuation test measures

$$\Lambda = \{Q_a : a \in A\},$$

on $(\Omega, \mathcal{F})$, where the index set A may now be uncountable, for example an open subset of some $\mathbb{R}^p$. The examples of the previous section showed that this can occur naturally in practice. To be able to derive a representation theorem, we will make an additional assumption on the set A that it comes equipped with the structure of a finite measure space: a σ-algebra of measurable sets $\mathfrak{A}$ and a finite positive measure ν on $\mathfrak{A}$. For example, in the case of the utility-based valuation test measures of the previous section, ν might be some a priori Borel measure on the parameter set $\{(e, \alpha, \gamma)\}$. We will no longer assume that $(\Omega, \mathcal{F})$ comes equipped with a probability measure $\mathbf{P}$, which is interpreted as the objective probability; in some sense, we will look at future market outcomes uniquely through the eyes of the agents, via the different valuation test measures.

We will suppose that for all $X \in \mathcal{X}$, $E_{Q_a}(|X|) < \infty$ for all $a \in A$ and that, moreover, $a \to E_{Q_a}(X)$ is a measurable map from $(A, \mathfrak{A})$ to $\mathbb{R}$ provided with the Borel σ-algebra. The condition of NSAO will be adapted to this new situation as follows.

Definition 1.5.1 An investment outcome $X \in \mathcal{X}$ is an Acceptable Opportunity if $\pi(X) = 0$ and if $E_{Q_a}(X) \geq 0$ for ν-almost all $a \in A$. An acceptable opportunity is Strictly Acceptable if there exists a $B \in \mathfrak{A}$ with $\nu(B) > 0$ such that $E_{Q_b}(X) > 0$ for all $b \in B$.

Examples 1.5.2 (i) If $A \subset \mathbb{N}^*$ is a finite or countable set, we can take for $\mathfrak{A}$ the σ-algebra of all subsets of A (the discrete σ-algebra) and for ν, the measure which assigns 2^{-j} to $\{j\}$. We are then in the situation of section 1.2.

(ii) If $(\Omega, \mathcal{F}, \mathbf{P})$ is a general probability space of market outcomes, we can take $A = \Omega$ and

$$\Lambda = \{\delta_\omega : \omega \in \Omega\},$$

where δ_ω is the Dirac measure on Ω: $E_{\delta_\omega}(X) := X(\omega)$. If we then take $\mathfrak{A} = \mathcal{F}$ and $\nu = \mathbf{P}$, then a Strictly Acceptable Opportunity is the same as a classical Arbitrage Opportunity.

Theorem 1.5.3 *Suppose the space $\mathcal{S} = \{\sum_{j=0}^N \alpha_j S_j : \alpha_j \in \mathbb{R}\}$ has no Strictly Acceptable Opportunities in the sense of Definition 1.5.1. Then there exists a positive ν-integrable function w on A which is strictly positive ν-almost everywhere, such that*

$$\pi(S_j) = \int_A E_{Q_a}(S_j) w(a) d\nu(a), \tag{1.8}$$

for $j = 0, \ldots, N$.

We note that when $j = 0$, $w(a)d\nu(a)$ is a probability measure on A.

Proof. Either of the two proofs of Theorem 1.2.5 can be generalised to this situation. We give yet another proof which reduces the theorem to the classical fundamental theorem of asset pricing for the single-period case. Define functions $\widetilde{S}_j$ on A by

$$\widetilde{S}_j(a) = E_{Q_a}(S_j),$$

and normalise ν to have total mass 1. Then the No Strictly Acceptable Opportunities condition translates into the classical no-arbitrage condition for the "assets" $\widetilde{S}_j$ on A, and there exists a ν-a.e. positive integrable function w on A such that $\widetilde{S}_j$ is integrable with respect to $wd\nu$ and such that

$$\pi(S_j) = \int_A \widetilde{S}_j(a)w(a)d\nu(a),$$

which is (1.8). ∎

Remark 1.5.4 By replacing the measure ν by $gd\nu$, where g is a ν-a.e. positive function whose integral is finite, we do not need to suppose that ν is a finite measure. Similar to the objective probability $\mathbf{P}$ in the classical no-arbitrage theory, the role of ν is to decide which set measures Q_a can be discarded when defining acceptable and strictly acceptable opportunities.

The proof, while simple, is nevertheless interesting from an interpretational point of view. By replacing S_j by $\widetilde{S}_j$, we are somehow replacing uncertainty of the outcome of the investment S_j by uncertainty about what the different agents making up the market expect this outcome to be. The notion of an objective probability measure $\mathbb{P}$ has disappeared, except to the extent that the individual Q_a might depend on such a probability, as when the valuation test measures are given by marginal expected utilities. However, even in that situation, there may be no agreement on what the objective probability is. Prices are now "explained" by the aggregate expectations of the market.

We next verify that the right-hand side of (1.8) defines a probability measure Q on $(\Omega, \mathcal{F})$, via

$$E_Q(X) = \int_A E_{Q_a}(X)w(a)d\nu(a),$$

where X is a positive random variable on $(\Omega, \mathcal{F})$. In particular, if $E \in \mathcal{F}$, then

$$Q(E) = \int_A E_{Q_a}(1_E)w(a)d\nu(a).$$

Then Q is a positive set function on $\mathcal{F}$ such that $Q(\Omega) = 1$, since the integral of w with respect to $d\nu$ is 1. To check that Q is σ-additive, we can use the monotone convergence theorem: if $E_j \in \mathcal{F}$ are disjoint, $j \in \mathbb{N}$, then

$$\begin{aligned}
Q\left(\cup_{j=1}^{\infty} E_j\right) &= \int_A Q_a\left(\cup_{j=1}^{\infty} E_j\right) w(a)d\nu(a) \\
&= \int_A \left(\sum_{j=1}^{\infty} Q_a(E_j)\right) w(a)d\nu(a) \\
&= \int_A \lim_{k\to\infty} \sum_{j=1}^{k} Q_a(E_j)w(a)d\nu(a) \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \int_A Q_a(E_j)w(a)d\nu(a) \\
&= \sum_{j=1}^{\infty} Q(E_j).
\end{aligned}$$

We verify next that Q is in the closure of the convex hull of $\Lambda = \{Q_a : a \in A\}$ with respect to a certain natural weak topology. Let $B = B(\Omega, \mathcal{F})$ be the linear space of all bounded measurable functions on Ω and let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{F})$ be the vector space of all bounded (signed) measures. These form a dual pair of vector spaces with the pairing

$$\langle \cdot, \cdot \rangle : B \times \mathcal{M} \to \mathbb{R}, \quad \langle F, \mu \rangle = \int_\Omega F(\omega)d\mu(\omega) = E_\mu(F).$$

We can then provide $\mathcal{M}$ with the $\sigma(\mathcal{M}, B)$-topology, which is the weakest topology which makes all linear applications $\mu \to \int_\Omega F d\mu$ continuous, F ranging over the space B. This topology makes $\mathcal{M}$ into a locally convex vector space. We then prove the result below.

Proposition 1.5.5 *The representative state pricing measure Q is in $\overline{\mathrm{co}(\Lambda)}^{\sigma(\mathcal{M},B)}$, the closure of the convex hull of Λ with respect to the $\sigma(\mathcal{M},B)$-topology.*

Proof. It is known (see [48]) that the dual of $(\mathcal{M},\sigma(\mathcal{M},B))$ is B. If $Q \notin \overline{\mathrm{co}(\Lambda)}^{\sigma(\mathcal{M},B)}$, by the hyperplane separation theorem for locally convex vector spaces, there exists an $F \in B$ and a real number α such that

$$\langle Q, F\rangle < \alpha < \langle Q_a, F\rangle, \quad \text{for all } a \in A\ ,$$

or

$$E_Q(F) < \alpha < E_{Q_a}(F), \quad a \in A.$$

The second inequality and the definition of Q imply that

$$E_Q(F) = \int_A E_{Q_a}(F)w(a)d\nu(a) > \alpha \int_A w(a)d\nu(a) = \alpha,$$

which is a contradiction. Hence Q is in the closed convex hull. ■

Remark 1.5.6 The space of finite measures can be provided with the variation norm defined by

$$||\mu|| = \sup\left\{\sum_\nu \mu(E_\nu) : \Omega = \cup_\nu E_\nu, E_\nu \cap E_{\nu'} = \emptyset, \nu \neq \nu'\right\},$$

making it into a Banach space. It is not clear whether Q is in the norm-closed convex hull of Λ. For a proof along the same lines, we would need a concrete description of the dual space of $\mathcal{M}$ with the norm-topology, which is a complicated object. On the other hand, if we assume that all Q_a are absolutely continuous with respect to a probability measure $\mathbf{P}$, then Λ can be identified with a subset of $L^1(\mathbf{P}) := L^1(\Omega,\mathcal{F},\mathbf{P})$. If $dQ_a = f_a d\mathbf{P}$ with $(a,\omega) \to f_a(\omega)$ measurable on $A \times \Omega$ with respect to the product σ-algebra of $\mathfrak{A}$ and $\mathcal{F}$, then an application of Fubini's theorem (see [47]) shows that

$dQ = fd\mathbb{P}$ with Radon-Nikodym derivative (see [42], [7])

$$f(\omega) = \frac{dQ}{d\mathbf{P}}(\omega) = \int_A f_a(\omega)w(a)d\nu(a).$$

Since $L^1(\mathbf{P})^* = L^\infty(\mathbf{P})$, the proof above now shows that $f \in \overline{\mathrm{co}(\{f_a : a \in A\})}^{||\cdot||_1}$, or

$$Q \in \overline{\mathrm{co}(\Lambda)}^{||\cdot||},$$

the closure in the variation norm.

Chapter 2

Dynamic Models in Discrete Time

2.1 Theory of No Strictly Acceptable Opportunities

In this chapter, we discuss the problem of extending the theory of No Strictly Acceptable Opportunities (henceforth called NSAO's, see [12]) to models that have a discrete number of possible trading dates. We will limit our study to finite market models in which there are a finite number of financial assets, a finite probability space and finite number of probability measures defined on this space. We will also assume that each of the probability measures assigns a positive probability to each of the outcomes in the space. We further specify a fixed time horizon T over which all trading activities take place. We give the specification of the market model below.

2.2 The Model

We make the following assumptions:

1. We denote the set of trading dates by $\mathcal{T} = \{0, 1, 2, \dots T\}$.

2. There are N risky assets $S^1 \dots S^N$ and one risk-free asset S^0, where S_t^j is the price of asset j at time t. We suppose that all these prices have been discounted to their $t = 0$ prices; in particular, $S_t^0 = 1$ for all $t \in \mathcal{T}$.

3. The financial market is modelled by a multinomial tree in which there are K possible outcomes[1] at each non-terminating node of the tree, which we will denote by the set $A = \{a_1 \dots a_K\}$. We call A the set of possible future states of the economy at each non-terminating node of the tree.

4. Our sample space Ω thus consists of all sequences $\omega = (a_1 \dots a_T)$ with $a_j \in A$ for $j = 1 \dots T$. Equivalently, $\Omega = A^T$. Each sequence $\omega \in \Omega$ corresponds to a full history of the process, that is, a trajectory of realised outcomes over the period $[0, T]$.

5. In order to model the flow of information in our model, we use a filtration of σ-algebras $\mathcal{F}_0 \subset \dots \subset \mathcal{F}_t \subset \dots \subset \mathcal{F}_T$ with $\mathcal{F}_0 = \{\phi, \Omega\}$ and $\mathcal{F}_t$ is defined as follows: if

$$\mathrm{pr}_t : \Omega = A^T \to \Omega_t = A^t \tag{2.1}$$

is the projection map which sends a full trajectory $\omega = (a_1, \dots, a_T)$ to its first t coordinates $(a_1, \dots, a_t)$, then we let

$$\mathcal{F}_t = \{\mathrm{pr}_t^{-1}(B), B \subset \Omega_t\}$$

[1]For much of what we do in this Chapter, A might be allowed to be countably infinite.

be the σ-algebra generated by subsets of Ω_t. Observe that $\mathcal{F}_T$ is the discrete σ-algebra on $\Omega = A^T$, that is, the set of all subsets of Ω. We will, as usual, assume that our price process $(S_t)_{t \in \mathcal{T}}$ is adapted with respect to this filtration.

A sequence $\omega' = (a_1, \dots, a_t)$ of length $t < T$ is called a partial history, and we will denote the set of all partial histories by

$$\Omega' = \cup_{t<T} \Omega_t,$$

where we note that the union is one of disjoint sets.

We can represent our market model in the form of a K-branched tree, with the nodes of our tree being given by the partial or full histories $\omega' = (a_1, \dots, a_t)$, $t \leq T$. We adopt the convention that $\omega' = \emptyset$ if $t = 0$ (corresponding to the initial node). Two nodes ω' and ω'' are said to be connected by an edge if they are of the form $\omega' = (a_1, \dots, a_t)$, $\omega'' = (a_1, \dots, a_{t+1})$ for some $t < T$ and $a_1, \dots, a_{t+1} \in A$. We will, in this case, sometimes write $\omega'' = \omega' * (a_{t+1})$ and say that ω'' is obtained from ω' by the *concatenation* with the one-element sequence (a_{t+1}) (we can extend this in the obvious way to the concatenation with sequences of length larger than one).

The σ-field $\mathcal{F}_t$ is generated by the sets

$$F(\omega') = \{\omega \in \Omega : \mathrm{pr}_t(\omega) = \omega'\}, \tag{2.2}$$

where ω' runs over all partial histories of length t. $F(\omega')$ is the set of all full histories which have the same first t components equal to those of ω'. An alternative expression for $F(\omega^{'})$ is

$$F(\omega') = \{\omega' * \eta : \eta \in A^{T-t}\}.$$

In this tree picture, $\omega^{'} * \eta$ corresponds to the subtree which "fans out" from the node ω'.

6. As in [46], we associate with each node of our tree, that is, with each partial history ω', a finite collection of single-period probability measures

$$\mathcal{P}^s(\omega') = \{q_i^{\omega'} : 1 \leq i \leq n(\omega')\} \tag{2.3}$$

on A, but which we will interpret as single-period *valuation test measures*, in the sense of [12]. Here $n(\omega') \in \mathbb{N}^*$ and $q_i^{\omega'} : A \to \mathbb{R}_{\geq 0}$ are such that $\sum_{a \in A} q_i^{\omega'}(a) = 1$. These probability measures reflect the beliefs of the community of agents regarding the profitability of the different investments over the single-period ahead of ω'. Note that they are allowed to depend on the partial history ω' of the process up till $t = \ell(\omega^{'})$. We assume each of these probability measures assigns a positive probability to each of the possible states in A.

Remark 2.2.1 More generally and following [12], we can associate to each partial history ω', a set of ordered couples $\{(q_i^{\omega'}, f_i^{\omega'}) : 1 \leq i \leq n(\omega')\}$, where $q_i^{\omega'}$ is again a probability measure on A and the $f_i^{\omega'}$ is a non-positive real number called a *floor*. An investment over the single period ahead of ω' would then be considered to be acceptable if the expected value, with respect to $q_i^{\omega'}$, of its profit weakly exceeds $f_i^{\omega'}$ $\forall i,\ 1 \leq i \leq n(\omega^{'})$. Valuation test measures are those for which the corresponding floor is zero; those with negative floors are called *stress test measures* [12]. We will only consider valuation test measures.

7. Starting from the collection of single-period measures $\mathcal{P}^s(\omega')$, $\omega^{'} \in \Omega^{'}$, we can generate a set of probability measures defined on $(\Omega, \mathcal{F}_T)$ by the procedure of *pasting* the single-period measures. For each partial history $\omega' \in \Omega'$, choose an element $q^{\omega'} \in \mathcal{P}^s(\omega')$: specifically, $q^{\omega'} = q_i^{\omega'}$ for some $i = i(\omega') \in \{1, 2, \ldots, n(\omega')\}$. We then define a probability measure $P = P^q$

on $(\Omega, \mathcal{F}_T)$ as follows: if $\omega = (a_1, \ldots, a_T) \in \Omega = A^T$, then

$$\begin{aligned} P^q(\{\omega\}) &= \prod_{i=0}^{T-1} q^{(a_1,\ldots,a_i)}(a_{i+1}) \qquad (2.4) \\ &= q^{\emptyset}(a_1) q^{(a_1)}(a_2) q^{(a_1,a_2)}(a_3) \cdots q^{(a_1,\cdots,a_{T-1})}(a_T), \end{aligned}$$

where the term with $i = 0$ corresponds to the empty sequence $\emptyset \in \Omega'$. See [46, 4].

It is not difficult to verify that P^q defines a probability measure on Ω: indeed, since each $q^{\omega'}$ is a probability measure on A,

$$\begin{aligned} \sum_{\omega} P^q(\omega) &= \sum_{a_1 \in A} \cdots \sum_{a_T \in A} q^{\emptyset}(a_1) q^{(a_1)}(a_2) q^{(a_1,a_2)}(a_3) \cdots q^{(a_1,\cdots,a_{T-1})}(a_T) \\ &= \sum_{a_1 \in A} \cdots \sum_{a_{T-1} \in A} q^{\emptyset}(a_1) q^{(a_1)}(a_2) q^{(a_1,a_2)}(a_3) \cdots q^{(a_1,\cdots,a_{T-2})}(a_{T-1}) \\ &= \cdots \\ &= \sum_{a_1 \in A} q^{\emptyset}(a_1) \\ &= 1. \end{aligned}$$

We will let $\mathcal{P}^{\text{paste}}$ be the set of all such global measures P^q, where q ranges over the set of all maps $q : \omega' \to q^{\omega'}$ of Ω' into the disjoint union $\cup_{\omega'} \mathcal{P}^s(\omega') \times \{\omega'\}$ of all single-period measures such that $q^{\omega'} \in \mathcal{P}^s(\omega')$.

More generally, if $\omega' \in \Omega_t$ is a partial history of length t, then we define $\mathcal{P}(\omega')$ to be the set of probability measures on $F(\omega')$ (provided with the discrete σ-algebra). These are constructed in the same way from single-period measures in $\mathcal{P}^s(\omega'')$ with $\omega'' \in \cup_{t \le s < T} \Omega_s$ such that $\text{pr}_{s,t}(\omega'') = \omega'$, $\text{pr}_{s,t} : A^s \to A^t$ being the projection on the first t coordinates. Thus, if $q^{\omega''} \in \mathcal{P}^s(\omega'')$ for all such ω'', and if $*$ denotes *concatenation* of sequences[2],

[2]Specifically, $(a_1, \ldots, a_i) * (b_1, \ldots, b_j) = (a_1, \ldots, a_i, b_1, \ldots, b_j)$.

then we define the measure $P^q(\cdot|\omega')$ on $F(\omega')$ by

$$P^q(\omega|\omega') = q^{\omega'}(a_{t+1})\, q^{\omega' * (a_{t+1})}(a_{t+2}) \cdots q^{\omega' * (a_{t+1},\dots,a_{T-1})}(a_T) \qquad (2.5)$$

if $\omega = \omega' * (a_{t+1}, \dots, a_T)$. We will sometimes simply write $P^q(a_{t+1}, a_{t+2}, \dots a_T \mid \omega')$ (note that $F(\omega')$ identifies with the set A^{T-t}).

Example 2.2.2 The following elementary example will be useful in order to connect the concept of absence of strictly acceptable opportunities, which we will introduce below with that of the usual absence of arbitrage. If we take

$$\mathcal{P}^s(\omega') = \{\delta_{a_1}, \dots, \delta_{a_K}\},$$

where δ_{a_i} is the Dirac measure concentrated at a_i: $\delta_{a_i}(a_j) = \delta_{ij} = 1$ if $j = i$ and 0 if $j \neq i$, then $\mathcal{P} = \{\delta_\omega : \omega \in \Omega\}$, is the set of Dirac measures on Ω.

Remark 2.2.3 We can provide a geometric picture of this multiple beliefs model by introducing the disjoint union of the different $\mathcal{P}^s(\omega')$,

$$\mathcal{P}^s = \bigcup_{\omega' \in \Omega'} \mathcal{P}^s(\omega') \times \{\omega'\},$$

together with the map $\pi : \mathcal{P}^s \to \Omega'$ defined by $\pi((q, \omega')) = \omega'$ for $q \in \mathcal{P}^s(\omega')$. This corresponds to a discrete fibre bundle over Ω' (see [51]), with the fibres being the different sets of single-period valuation test measures: $\pi^{-1}(\omega') = \mathcal{P}^s(\omega')$, which one might call the bundle of single-period valuation test measures. A map $\omega' \to q^{\omega'} \in \mathcal{P}^s(\omega')$ then corresponds to a section of this bundle, and the pasting procedure described above defines a map from the set $\Gamma(\Omega', \mathcal{P}^s)$ of all global sections to the set of probability measures on $(\Omega, \mathcal{F}_T)$. We may call the latter the set of global probability measures (to distinguish them from the single-period valuation test measures, which are only locally defined on the tree). We will not use this terminology, but the picture might help us to visualise the model.

Remark 2.2.4 If P^q is the global probability measure obtained from the pasting of $\omega' \to q^{\omega'}$, we can recover the latter as a conditional probability:

$$q^{\omega'}(a) = \frac{P^q(F(\omega' * a))}{P^q(F(\omega'))} = P^q(F(\omega' * a)|\omega').$$

More generally, any probability measure P on $(\Omega, \mathcal{F})$ will, in this way, give rise to a family of single-period measures. For any random variable X on $(\Omega, \mathcal{F})$, the conditional expectation $E_P(X|\mathcal{F}_t)$ can be identified with the function

$$\omega' \to E_P(X|\omega') := \frac{E_P(X\mathbf{1}_{F(\omega')})}{P(F(\omega'))}, \tag{2.6}$$

and we then define the single-period valuation test measures $q^{\omega'}$ by

$$q^{\omega'}(a) = q(P)^{\omega'}(a) = E_P(\mathbf{1}_{F(\omega' * a)}|\omega').$$

If we only have a single probability measure P, pasting of the conditional single-period valuation test measures will return the original measure P, as is easily verified. However, if we start off with a collection $\mathcal{P} = \{P_i\}$ of globally-defined probability measures, we obtain collections of single-period valuation test measures $q_i^{\omega'} = q(P_i)^{\omega'}$ at each node of the tree, and pasting these together in all possible ways will give rise to a new set of globally-defined measures which will, in general, be larger than the original one. Returning to P^q, we note the following formula for conditional expectations.

Lemma 2.2.5 If X is a random variable on $(\Omega, \mathcal{F}_T)$ and P^q is a pasted measure as above. For all partial histories ω',

$$E_{P^q}(X|\omega') = E_{q^{\omega'}}\left[E_{P^q}(X|\omega' * a)\right], \tag{2.7}$$

where $\omega' * a$ is the concatenation of ω' with the one-element sequence (a), $a \in A$.

Proof. This is a version of the tower property of conditional expectations, in view of formula (2.6). It is equally easy to verify directly: if $\omega' = (a_1, \ldots, a_t)$, then it follows from the definition of P^q that

$$P^q(F(\omega')) = q^{\emptyset}(a_1) q^{(a_1)}(a_2) \cdots q^{(a_1,\ldots,a_{t-1})}(a_t).$$

Hence, using (2.6) twice,

$$\begin{aligned}
& E_{P^q}(X|\omega') \\
= & \sum_{a_{t+1},\ldots,a_T} q^{(a_1,\ldots,a_t)}(a_{t+1}) \cdots q^{(a_1,\ldots,a_t,a_{t+1}\cdots a_{T-1})}(a_T) X(a_1,\ldots,a_T) \\
= & \sum_{a_{t+1}} q^{(a_1,\ldots,a_t)}(a_{t+1}) \left(\sum_{a_{t+2},\ldots,a_T} q^{(a_1,\ldots,a_{t+1})}(a_{t+2}) \cdots q^{(a_1,\ldots,a_t,a_{t+1}\cdots a_{T-1})}(a_T) X(a_1,\ldots,a_T) \right) \\
= & \sum_{a_{t+1}} q^{(a_1,\ldots,a_t)}(a_{t+1}) E_{P^q}(X|(a_1,\ldots,a_{t+1})) \\
= & E_{q^{(a_1,\ldots,a_t)}} \left[E_{P^q}(X|(a_1,\ldots;a_t) * (a)) \right],
\end{aligned}$$

which proves the lemma. ■

2.3 Strictly Acceptable Opportunities

We start by reviewing the basic concept of self-financing dynamic trading strategies in discrete-time markets.

Definition 2.3.1 (see [7, 24, 42, 18]) A *dynamic trading strategy* is a $\mathbb{R}^{N+1}$-valued vector stochastic process $\varphi = (\varphi(t))_{t=1}^T$ which is predictable: for each t, $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots \varphi_N(t))_{t=1}^T$ is $\mathcal{F}_{t-1}$-measurable for t $\geq$ 1.

Intuitively, this means that φ represents a portfolio that can change randomly over time, but whose composition at time t is based on the available information before time t. The components $\varphi_i(t)$, $i = 1, 2, \dots N$ denote the number of units of asset i that are held in the portfolio between the times $t-1$ and t. In concrete terms, φ_t is predictable which means that φ_t, as a function on A^T, only depends on the first $t-1$ coordinates: $\varphi_t(a_1, \dots, a_T) = \varphi_t(a_1, \dots, a_{t-1})$.

The space of all dynamic strategies is complex and too large for any meaningful economic analysis. From this space, we single out the subspace of self-financing dynamic strategies.

If $v, w \in \mathbb{R}^{N+1}$ are two vectors, we will denote their Euclidean scalar product by $v \cdot w$

$$v \cdot w := \sum_{i=1}^{N+1} v_i w_i.$$

Definition 2.3.2 (see [7, 24, 42, 18]) A dynamic strategy φ is *self-financing* if

$$\varphi(t) \cdot S(t) = \varphi(t+1) \cdot S(t) \tag{2.8}$$

for $t = 1 \dots T-1$. We will denote the subspace of all self-financing trading strategies by Φ.

The self-financing condition (2.8) means the following: to change the portfolio from $\varphi(t)$ to $\varphi(t+1)$, an agent can only use the total wealth available in the portfolio at time t. Therefore, any change in the agent's wealth comes from changes in the market prices of assets and changes in interest rates, and no injection of extra capital or withdrawal of capital is allowed. It is, therefore, a restriction on the flow of capital in and out of the portfolio. Associated with each dynamic strategy is the wealth process which gives the value of the dynamic strategy at each time $t \in \mathcal{T}$:

$$V_{\varphi}(t) = \varphi(t) \cdot S(t) = \sum_{i=0}^{N} \varphi_i(t) S_i(t) \tag{2.9}$$

for $t = 1, \ldots, T$. If the strategy is self-financing, then $V_{\varphi}(t)$ can be written as

$$V_{\varphi}(t) = V_{\varphi}(0) + \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta S(\tau), \tag{2.10}$$

where $\Delta S(\tau) = S(\tau) - S(\tau - 1)$, $\tau > 0$ denotes the increment resulting from market price changes at time τ. Indeed, $V_{\varphi}(t)$ can be written as a telescoping sum,

$$V_{\varphi}(t) = V_{\varphi}(0) + \sum_{\tau=1}^{t} \left(V_{\varphi}(\tau) - V_{\varphi}(\tau - 1) \right),$$

and $V_{\varphi}(\tau) - V_{\varphi}(\tau - 1) = \varphi(\tau) \cdot S(\tau) - \varphi(\tau) \cdot S(\tau - 1) = \varphi(\tau) \Delta S(\tau)$ by the self-financing condition. The sum $\sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta S(\tau)$ represents the total change in the value of the dynamic strategy.

In the sequel, we consider the subspace $\Phi_0 \subset \Phi$ consisting of self-financing dynamic strategies with zero initial cost: $V_{\varphi}(0) = 0$. Traditionally, one imposes on a financial market model the condition of absence of arbitrage: for a given probability measure on $(\Omega, \mathcal{F})$, interpreted as the objective probability, there exists no trading strategy $\varphi \in \Phi_0$ for which $V_{\varphi}(T)$ is non-negative a.s. and strictly positive with a non-zero probability. Carr *et al.* [12] replaced this condition, for single-period financial markets, with that of absence of strictly acceptable opportunities relative to a set of valuation test measures. We first recall their definition for the static case and then explain how we can generalise it to the case of dynamic (multi-period) market models.

Definition 2.3.3 (see [12]): Consider a single-period market model on a measurable space $(A, \mathcal{G})$, $\mathcal{G}$ a σ-algebra of subsets of A. Suppose that there are N risky assets and one risk-free asset available for trading in this market model, and there are M single-period probability measures[3] $q_1 \dots q_M$ defined on $(A, \mathcal{G})$ with associated expectation E_{q_i}. A zero-cost portfolio $\alpha = (\alpha_0, \dots, \alpha_N)$ is a strictly acceptable opportunity if

$$E_{q_i}(\sum_{j=0}^{N} \alpha_j S_j) \geq 0 \tag{2.11}$$

for all $q_i, i = 1 \dots M$ and there exists at least one probability measure $q \in \{q_1, q_2, \dots q_M\}$ such that

$$E_q(\sum_{j=0}^{N} \alpha_j S_j) > 0. \tag{2.12}$$

In effect, a strictly acceptable opportunity is a zero-cost portfolio which has a non-negative valuation under all probability measures and a positive valuation under at least one probability measure. Since strictly acceptable opportunities give the community of investors an expectation of a positive return, such opportunities should be eliminated from the market to ensure a kind of equilibrium. Further, Carr *et al.* [12] showed that if there are no strictly acceptable opportunities in a single-period model with finitely many probability measures, it is possible to form a pricing measure by taking a convex combination of the available valuation test measures. This pricing measure is called a representative state pricing measure. It is unique provided that the number of traded assets weakly exceeds the number of valua-

[3] As mentioned already, Carr *et al.* [12] considered two classes of probability measures for assessing acceptable opportunities. The first class, called valuation test measures is used for pricing, while the second class called stress test measures is used to limit the upward scaling of payoffs. In this chapter, all our probability measures will be valuation test measures.

tion test measures: see also Chapter 1 of this thesis.

The extension of this theory to the multi-period setting is not entirely straight-forward. We can take a "top-down" approach and start off with a set of globally-defined valuation test measures defined on $(\Omega, \mathcal{F}_T)$ (that is, measures defined on the space of all paths) and use these to define what would be a strictly acceptable trading strategy. Alternatively, we can follow a "bottom-up" procedure and take a collection of single-period valuation test measures, defined at each non-terminating node of our tree, as the primitive of our theory. We start with the former, and point out a difficulty with this approach, following Roorda *et al.* [46].

Definition 2.3.4 Let φ be a zero-cost self-financing dynamic strategy and let $\mathcal{P}$ be a collection of probability measures on the space $(\Omega, \mathcal{F}_T)$. We will say that φ is a strictly acceptable opportunity with respect to $\mathcal{P}$ if

$$E_P(V_\varphi(T)) \geq 0$$

for all $P \in \mathcal{P}$ and there exists at least one probability measure $P_0 \in \mathcal{P}$ such that

$$E_{P_0}(V_\varphi(T)) > 0.$$

Absence of strictly acceptable opportunities then means that for all $\varphi \in \Phi_0$,

$$E_P(V_\varphi(T)) \geq 0, \forall P \in \mathcal{P} \Rightarrow E_P(V_\varphi(T)) = 0, \forall P \in \mathcal{P}.$$

Example 2.3.5. If $\mathcal{P} = \{\delta_\omega : \omega \in \Omega\}$, then absence of strictly acceptable opportunities is equivalent to the classical notion of absence of arbitrage. Indeed, a strictly acceptable portfolio in this case is a portfolio φ in Φ_0 for which $V_\varphi(T)(\omega) \geq 0$ for all $\omega \in \Omega$, and strictly positive for at least one such

ω. If $(\Omega, \mathcal{F}_T)$ comes with a reference measure $\mathbb{P}$ such that $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$, this is equivalent to $\mathbb{E}_{\mathbb{P}}(V_\varphi(T)) > 0$, that is, φ represents an arbitrage opportunity on $(\Omega, \mathcal{F}_T, \mathbb{P})$. As we have seen in Example 2.2.2, the $\mathcal{P}$ considered here arises as the set of pasted measures associated to the single-period measures $\{\delta_a : a \in A\}$.

In Definition 2.3.4, the collection $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F}_T)$ is, in principle, arbitrary. We can strengthen the absence of acceptable opportunities condition by requiring that for $t \in \mathcal{T}$,

$$E_P(V_\varphi(T) \mid \mathcal{F}_t) \geq 0 \;\forall P \in \mathcal{P} \Longrightarrow E_P(V_\varphi(T) \mid \mathcal{F}_t) = 0 \;\forall P \in \mathcal{P}. \tag{2.13}$$

We would like this property to be equivalent to absence of single-period strictly acceptable opportunities with respect to $\mathcal{P}^s(\omega')$ at each non-terminating node ω' of our tree. This places a constraint on the collection $\mathcal{P}$, which must be large enough to make it impossible for a zero-cost portfolio to be strictly acceptable at any time $t \in \mathcal{T}$. It is important to realise that if such a collection of probability measures $\mathcal{P}$ is not large enough, then it is possible to form a zero-cost self-financing trading strategy, which is strictly acceptable. The following example is given by Roorda *et al.* [46].

Example 2.3.6 ([46]). Consider the following two-period economy with two possible future states $A = \{u, d\}$ consisting of two traded assets: a risk-free bond with time $t = 0$ price of 100, and a stock with time $t = 0$ price of 100. We assume that the risk-free rate $r = 0$ and that the evolution of the stock is given by

$$S_1(u) = 110, \;\; S_1(d) = 90$$

at time $t = 1$, and

Table 2.1: Probability of the states under the different measures

probability measures	states	
	u	d
P_1	0.6	0.4
P_2	0.4	0.6

$$S_2(uu) = 120,\ S_2(ud) = S_2(du) = 100,\ S_2(dd) = 80,$$

at time $t = 2$. At the initial node $\omega' = \emptyset$ and at the two nodes $\omega' = (u)$ and $\omega' = (d)$ we take the same two probability measures, P_1 and P_2 on A, one with the probability of an 'up' move of 0.6 and the probability of a 'down' move of 0.4. The other probability measure reverses the above probability assignments. Table 2.1 gives the assignments in the different states.
Finally, as our set of global measures, we take the two product measures: [4] $\mathcal{P} = \{P_1 \otimes P_1, P_2 \otimes P_2\}$ on $\Omega = A \times A = \{(uu), (ud), (du), (dd)\}$.

We first show that each of the single-period models comprising this two-period model allows no strictly acceptable opportunities. There is essentially only one zero-cost portfolio containing the stock, the one consisting of shorting one bond at time 0 and buying one stock; all others are simply multiples of this portfolio. We renormalise prices by deducting the price of the stock at time t ($t = 0, 1$) from its time $t+1$ ($t = 0, 1$) prices to obtain single-period models with initial price 0 and final prices 10 in the up state u and -10 in the down state d. We note that for the bond, its price after this normalisation is zero at all times t, $t = 0, \ldots, 2$.

The expected values of the stock under probability measures P_1 and P_2 are:

$$E_{P_1}(S) = (0.6)(10) + (0.4)(-10) = 2,$$

[4] $P_i \otimes P_i((a_1, a_2)) = P_i(a_1)P_i(a_2)$ for $(a_1, a_2) \in A \times A$.

$$E_{P_2}(S) = (0.4)(10) + (0.6)(-10) = -2.$$

Therefore, there are no strictly acceptable opportunities in each of the single-period models.

The next step is to construct a self-financing trading strategy φ comprising the two available assets, which will result in a strictly acceptable opportunity over the two periods. Our trading portfolio will be contingent on the state of the world at time t=1. This is specified as follows:

At time $t = 0$, we do nothing.
At time $t = 1$ and in the up state u, we buy 1 unit of the stock and borrow 1.1 units of bond.
At time $t = 1$ and in the down state d, we sell 1 unit of the stock and lend 0.9 unit of the bond.

The value of the portfolio at time $t = 1$ in the two states u and d is equal to zero. It is therefore a self-financing strategy. However, the values of the portfolio at time $t = 2$ are:

$$\begin{aligned}
V_\varphi(2)(uu) &= (1)(120) + (-1.1)(100) = 10, \\
V_\varphi(2)(ud) &= (1)(100) + (-1.1)(100) = -10, \\
V_\varphi(2)(du) &= (-1)(100) + (0.9)(100) = -10, \\
V_\varphi(2)(dd) &= (-1)(80) + (0.9)(100) = 10.
\end{aligned}$$

We next compute the expected value of this portfolio under the two probability measures in $\mathcal{P}$ to obtain

$$E_{P_1 \otimes P_1}(V_\varphi(2)) = (0.6)[(0.6)(10)+(0.4)(-10)]+(0.4)[(0.6)(-10)+(0.4)(10)] = 0.4,$$

$$E_{P_2 \otimes P_2}(V_\varphi(2)) = (0.4)[(0.4)(10)+(0.6)(-10)]+(0.6)[(0.4)(-10)+(0.6)(10)] = 0.4.$$

Table 2.2: Probability of the states under the different measures

probability measures	states			
	uu	ud	du	dd
$\mathbb{P}_1$	0.36	0.24	0.24	0.16
$\mathbb{P}_2$	0.36	0.24	0.16	0.24
$\mathbb{P}_3$	0.24	0.36	0.24	0.16
$\mathbb{P}_4$	0.24	0.36	0.16	0.24
$\mathbb{P}_5$	0.24	0.16	0.36	0.24
$\mathbb{P}_6$	0.24	0.16	0.24	0.36
$\mathbb{P}_7$	0.16	0.24	0.36	0.24
$\mathbb{P}_8$	0.16	0.24	0.24	0.36

This portfolio has a positive evaluation under both probability measures and is, therefore, a strictly acceptable opportunity.

The reason for this situation of a zero-cost portfolio resulting in a strictly acceptable opportunity is that we have not chosen a collection of global measures $\mathcal{P}$, which is diverse enough to take into account all of the market views of the agents over each of the single periods. One solution to this anomaly is to enlarge $\mathcal{P}$ to all of the probability measures which can be obtained from pasting the two given single-period probability measures. This involves using all the possible probability assignments for the branches in the tree and then collecting all ensuing probability measures. Table 2.2 gives the collection one obtains for this example.

It is clear that under this improved collection, our portfolio strategy φ is no longer a strictly acceptable opportunity. For example, the expected value of $V_\varphi(2)$ with respect to the probability measure $\mathbb{P}_7$ gives:

$$E_{\mathbb{P}_7}(X) = (0.16)(10) + (0.24)(-10) + (0.36)(-10) + (0.24)(10) = -2.$$

In the single-period case, it is established in Carr *et al.* [12] that the mathematical concept of the existence of a representative state pricing function is equivalent to the economic concept of no strictly acceptable opportunities. This example shows that in the multi-period case, the two concepts are not necessarily equivalent. Further conditions need to be imposed on the collection of probability measures to arrive at this equivalence. However, there is a representative state pricing function associated with this example even when the equivalence does not hold. For example, by pasting together the representative state pricing function for each of the single-period models, we obtain the two-period state pricing function

$$\mathbb{Q}(\omega) = 0.25 \ for \ all \ possible \ paths.$$

The example above shows that, whereas each of the single-period models admits no strictly acceptable opportunities, the two-period model is not free of strictly acceptable opportunities. Therefore, in order to have absence of strictly acceptable opportunities in the multi-period setting, we require a collection of valuation test measures defined on the entire tree, which is large and robust enough to handle all dynamic strategies, which are not strictly acceptable in the static case. This then raises the question of how to construct the collection of global valuation test measures to be used in the definition of no strictly acceptable opportunities. We will take this collection to be the set of all valuation test measures, which are obtained by pasting the single-period valuation test measures, and show that this will lead to a satisfactory theory. It is an interesting problem whether there exist smaller sets of global valuation test measures for which absence of single-period strictly acceptable opportunities is equivalent to absence of global strictly acceptable opportunities. Henceforth, we will assume that our set of global valuation test measures is given by

$$\mathcal{P} = \mathcal{P}^{\text{paste}} = \{P^q : q = (q^{\omega'})_{\omega' \in \Omega'} \text{ with } q^{\omega'} \in \mathcal{P}^s(\omega'), \forall \omega' \ \}, \tag{2.14}$$

where P^q is as defined in (2.4).

The first result we shall show is that absence of strictly acceptable opportunities with respect to $\mathcal{P}^{\text{paste}}$ in the multi-period market model implies absence of strictly acceptable opportunities in each of the single-period market models.

Recall that we are given a vector process of asset prices $(S_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^{N+1}$, adapted to our filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, where $S_t = (S_t^0, S_t^1, \ldots, S_t^N)$ with $S_t^0 = 1$; equivalently, the prices of the risky assets $S_t^1, \ldots, S_t^N$ are discounted prices. The components $\varphi_1(t), \ldots, \varphi_N(t)$ of our self-financing trading strategy can be chosen arbitrarily (provided they are predictable), and $\varphi_0(t)$ can be determined so as to make the strategy self-financing (assuming, as usual, that unlimited borrowing and lending is allowed).

The total gain $G = G(\varphi) = V_T(\varphi) - V_0(\varphi)$ of the portfolio over the period $[0, T]$ can be written as

$$G_T = G_T(\varphi) = \sum_{\tau=1}^{T} \varphi(\tau) \cdot \Delta S(\tau),$$

with

$$G_t = G_t(\varphi) = \sum_{\tau=1}^{t} \varphi(\tau) \cdot \Delta S(\tau),$$

where $\Delta S_t^i = S_t^i - S_{t-1}^i$, and where we note that neither $\varphi_0(t)$ nor S_t^0 appears in this expression. The random variable G represents the accumulated gains/losses resulting from following the trading strategy φ: holding $\varphi_i(t)$ units of S_t^i at time t, decided upon at time $t-1$ and financed from profits of the preceding trading period plus borrowing if necessary. We note that G_t is $\mathcal{F}_t$-measurable.

With these notations, we can reformulate the notions of single-period and multi-period strictly acceptable opportunities for a given portfolio strategy as follows:

Definition 2.3.7 A collection of single-period measures $(\mathcal{P}^s(\omega'))_{\omega'\in\Omega'}$ defines single-period strictly acceptable opportunities if there exist an $\omega' \in \Omega'$, $\ell(\omega') = t-1$ $(t \leq T)$ and a vector $h \in \mathbb{R}^N$ such that the single-period gains process $G_t(\cdot|\omega')$, defined on A by

$$G_t(a|\omega') = h \cdot (S_t(\omega' * (a)) - S_{t-1}(\omega')) \,,$$

satisfies

$$0 \leq \min_{q\in\mathcal{P}^s(\omega')} E_q\left(G_t(\cdot|\omega')\right) < \max_{q\in\mathcal{P}^s(\omega')} E_q\left(G_t(\cdot|\omega')\right).$$

Therefore, absence of single-period strictly acceptable opportunities means that for all h, the non-negativity of $E_q(G_t(\cdot|\omega'))$ for all $q \in \mathcal{P}^s(\omega')$ implies they are all equal to zero.

We can similarly reformulate existence and absence of global strictly acceptable opportunities with respect to a set of reference measures $\mathcal{P}$ (for which we will below take the set of all pasted measures obtained from the bundle of single-period measures).

Definition 2.3.8 Let $\mathcal{P}$ be a set of probability measures on Ω. A trading strategy $\varphi = (\varphi_t)_{1\leq t\leq T}$ is a strictly acceptable opportunity with respect to $\mathcal{P}$ if its total gain $G = G(\varphi)$ over $[0,T]$ satisfies

$$0 \leq \min_{P\in\mathcal{P}} E_P(G) < \max_{P\in\mathcal{P}} E_P(G).$$

Equivalently, there are no strictly acceptable opportunities relative to $\mathcal{P}$ if for all trading strategies,

$$E_P(G) \geq 0 \ \forall P \in \mathcal{P} \Longrightarrow E_P(G) = 0 \ \forall P \in \mathcal{P}.$$

An important property that a collection of probability measures should possess is the property of relevance.

Definition 2.3.9 We will say that the collection of single-period measures $(\mathcal{P}^s(\omega'))_{\omega' \in \Omega'}$ is *relevant* if for each partial history $\omega^{'} \in \Omega^{'}$ and for all $a \in A$, there exists a $q \in \mathcal{P}^s(\omega')$ such that $q(a) \neq 0$.

Relevance means that no single state which is "one-period ahead" is ever neglected by *all* of the agents, though it may be by some of them. Recall that the set of global pasted measures is defined by

$$\mathcal{P}^{\text{paste}} = \{P^q : q : \omega' \to q^{\omega'} \in \mathcal{P}^s(\omega^{'})\}.$$

Relevance easily implies that for each $\omega \in \Omega = A^N$, there exists at least one probability measure $P \in \mathcal{P}^{\text{paste}}$ such that $P(\omega) \neq 0$.

The following is the main result of this section.

Theorem 2.3.10 (cf. [46], Theorem 3.2) Suppose that the collection of single-period measures $(\mathcal{P}^s(\omega'))_{\omega' \in \Omega'}$ is relevant. Then there are no global strictly acceptable opportunities relative to $\mathcal{P}^{\text{paste}}$ if and only if there are no single-period strictly acceptable opportunities.

Another way of formulating this is that absence of single-period strictly acceptable opportunities over all of the single periods of the model is equivalent to absence of dynamic, T-period strictly acceptable opportunities relative to the set of pasted measures. As indicated, this theorem is due to Roorda, Schumacher and Engwerda [46], whose proof was based on Stiemke's lemma. We provide a different proof which, we believe, has some chance of being generalisable to models with general state space A.

Proof. Suppose there exists a single-period strictly acceptable opportunity at $\omega_0' \in \Omega'$, with corresponding trading strategy $h \in \mathbb{R}^N$. We define a global portfolio strategy by only trading at the node ω_0' according to h and doing nothing at the other nodes:

$$\varphi(t)(\omega') = \begin{cases} h, & \omega' = \omega_0' \\ 0, & \text{otherwise.} \end{cases}$$

We then verify that this is a global strictly acceptable opportunity with respect to $\mathcal{P}^{\text{paste}}$. Indeed, with the notations of Definition 2.3.7 and letting $\ell(\omega_0') = t_0 - 1$, the total gain of this portfolio equals (using the notation of Definition 2.3.7)

$$G(\omega) = \begin{cases} G_{t_0}(a|\omega_0'), & \omega \in F(\omega_0' * (a)), a \in A \\ 0, & \text{otherwise .} \end{cases}$$

If $\omega_0' = (a_1^0, \ldots, a_{t_0-1}^0)$, it then follows that for any choice of measures $q^{\omega'} \in \mathcal{P}^s(\omega')$,

$$E_{P^q}(G) = \sum_{\omega \in F(\omega_0')} q^{\emptyset}(a_1^0) q^{(a_1^0)}(a_2^0) \cdots q^{(a_1^0, \cdots, a_{t_0-2})}(a_{t_0-1}) E_{q^{\omega_0'}}(G_{t_0}(\cdot|\omega_0')),$$

which, since h is a single-period strictly acceptable opportunity and since $\mathcal{P}^s$ is relevant, is non-negative for all q and strictly positive for at least one q. Hence, $\varphi(t)$, $t \leq T$, is a dynamic strictly acceptable opportunity.

In order to prove the converse, we first make the following observation.

Lemma 2.3.11 Suppose that the bundle $(P^s(\omega'))_{\omega' \in \Omega'}$ of single-period measures admits no single-period strictly acceptable opportunities. Let $G_t = \sum_{\tau=1}^{t} \varphi(\tau) \cdot (S(\tau) - S(\tau - 1))$ for some portfolio strategy $\varphi(t)$, $t \leq T$, and $G_t(\cdot|\omega')$ be the restriction of G_t to the single-period subtree starting at node ω'. Assume that

$$E_t = \{\omega' \in \Omega_{t-1} : E_q(G_t(\cdot|\omega')) \geq 0 \ \forall q \in \mathcal{P}^s(\omega')\},$$

and $\widetilde{E}_t = \mathrm{pr}_{t-1}^{-1}(E_t) = \{(a_1, \ldots, a_T) \in A^T : (a_1, \ldots, a_{t-1}) \in E_t\}$. Then $E_{P^q}\left(G_t \mathbf{1}_{\widetilde{E}_t}\right) = 0$ for all $P^q \in \mathcal{P}^{\text{paste}}$.

Proof. It will be convenient to introduce the following notation: if P^q is the measure pasted from the collection of single-period measures $q^{\omega'} \in \mathcal{P}^s(\omega')$, then

$$P^q(a_1, \ldots, a_t) = P^q\left(F(a_1, \ldots, a_t)\right), \quad (a_1, \ldots, a_t) \in \Omega_t = A^t;$$

concretely,

$$P^q(a_1, \ldots, a_t) = q^{\emptyset}(a_1) q^{(a_1)}(a_2) q^{(a_1,a_2)}(a_3) \cdots q^{(a_1,\cdots,a_{t-1})}(a_t),$$

as is easily verified.

Since there are no single-period strictly acceptable opportunities, we have that if $\omega' \in E_t$, $E_q(G_t(\cdot|\omega')) \geq 0$ for all single-period measures $q \in \mathcal{P}^s(\omega')$, then $E_q(G_t(\cdot|\omega')) = 0$ for all single-period measures $q \in \mathcal{P}^s(\omega')$. This is then also true for all product measures P^q as one of the single-period measures that generates P^q is contained in $\mathcal{P}^s(\omega')$. Specifically,

$$E_{P^q}\left(G_t \mathbf{1}_{\widetilde{\mathbf{E}}_\mathbf{t}}\right) = \sum_{a_1 \ldots a_t} P^q(a_1, a_2, \ldots a_{t-1}, a_t) G_t(a_1, a_2, \ldots a_{t-1}, a_t) \mathbf{1}_{E_t}(a_1, a_2, \ldots a_{t-1})$$

$$= \sum_{a_1, a_2, \ldots a_t} P^q(a_1, a_2, \ldots a_{t-1}) q^{(a_1,\ldots,a_{t-1})}(a_t) G_t(a_1, a_2, \ldots a_{t-1}, a_t) \mathbf{1}_{E_t}(a_1, a_2, \ldots a_{t-1})$$

$$= \sum_{(a_1,a_2,\ldots a_{t-1}) \in E_t} P^q(a_1, a_2, \ldots a_{t-1}) E_{q^{(a_1,\ldots,a_{t-1})}}\left(G_t(\cdot|(a_1, a_2, \ldots a_{t-1}))\right) \geq 0.$$

This implies that $E_{P^q}\left(G_t \mathbf{1}_{\widetilde{\mathbf{E}}_\mathbf{t}}\right) = 0$ for all multi-period measures $P^q \in \mathcal{P}^{paste}$. ■

To prove the other implication of Theorem 2.3.10, suppose that there are no single-period strictly acceptable opportunities and that $G = \sum_{t=1}^{T} G_t$ is the total gain associated with a trading strategy for which $E_{P^q}(G) \geq 0$ for all $P^q \in \mathcal{P}^{\text{paste}}$. We will show by induction that

$$E_{P^q}(G_1 + G_2 + \ldots + G_t) \geq 0 \;\forall P^q \Longrightarrow E_{P^q}(G_1 + G_2 + \ldots + G_t) = 0 \;\forall P^q,$$

for all $t \in \{1, \ldots, T\}$. If $t = 1$, this follows immediately from the absence of strictly acceptable opportunities over the first trading period. Suppose that

$$E_{P^q}(G_1 + G_2 + \ldots + G_{t-1}) \geq 0 \;\forall\; P^q \Longrightarrow E_{P^q}(G_1 + G_2 + + \ldots G_{t-1}) = 0 \;\forall P^q,$$

with $t > 1$, and that $E_{P^q}(G_1 + \ldots G_t) \geq 0 \;\forall P^q \in \mathcal{P}^{\text{paste}}$. Suppose first that there exists a product-type measure $P^{q_0} \in \mathcal{P}^{\text{paste}}$ for which $E_{P^{q_0}}(G_1 + G_2 + \ldots G_{t-1}) < 0$. Since $E_{P^q}(G_1 + \cdots + G_{t-1}) = E_{P^{q_0}}(G_1 + \cdots + G_{t-1})$ for all q such that $q^{\omega'} = q_0^{\omega'}$ for all $\omega' \in \Omega_{t-2}$ (that is, of length $\ell(\omega') \leq t - 2$), it follows that $E_{P^q}(G_t) > 0$ for all such q. On the other hand, by Lemma 2.3.11, this expectation is equal to $E_{P^q}(G_t \mathbf{1}_{\widetilde{F}_t})$, where $F_t = \Omega_{t-1} \backslash E_t$. If F_t is not empty, then there exists for each $\omega^{'} \in F_t$ at least one $q^{\omega'} \in \mathcal{P}^s(\omega^{'})$ such that $E_{q^{\omega'}}(G_t(\cdot|\omega')) < 0$. Pasting these with the single-period measures of the restriction of q_0 to Ω_{t-2}, we can construct a product-type probability measure $P \in \mathcal{P}^{\text{paste}}$ which coincides with P^{q_0} on Ω_{t-2} such that $E_P(G_t) \leq 0$, giving a contradiction.

We therefore conclude that $F_t = \emptyset$ and that $\Omega_{t-1} = E_t$. By Lemma 2.3.11, we have that $E_P(G_t) = 0$ for all $P \in \mathcal{P}^{\text{paste}}$ and, consequently, $E_P(G_1 + \ldots + G_{t-1}) \geq 0$ for all $P \in \mathcal{P}^{\text{paste}}$, contradicting the existence of P^{q_0}. Hence $E_P(G_1 + G_2 + \ldots + G_{t-1}) \geq 0$ for all $P \in \mathcal{P}^{\text{paste}}$ and, by the induction hypothesis, $E_P(G_1 + \cdots + G_{t-1}) = 0$ for all such P. Hence $E_P(G_t) \geq 0$ for all P. Using Lemma 2.3.11 again, we have that $E_P(G_t \mathbf{1}_{\widetilde{F}_t}) \geq 0)$. But from the definition of F_t and using the relevance of the single-period measures, it is easy to construct a product-type measure for which this expectation is strictly negative. It follows, once more, that F_t is the empty set, and that

$E_P(G_t) = E_P(G_t \mathbf{1}_{\tilde{E}_t}) = 0$ for all $P \in \mathcal{P}^{paste}$. The same is, therefore, true for $E_P(G_1 + \cdots + G_t)$, which proves the induction step and, thereby, the theorem.■

2.4 Representative State Pricing Measures

A representative state pricing measure, also called a martingale measure, is a probability measure on $(\Omega, \mathcal{F})$ for which the (vector-valued) price process $(S_t)_{0 \le t \le T}$ is a martingale (with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$, see [12]). Carr *et al.* [12] showed that, in the situation of the single-period model of section 1.2, absence of strictly acceptable opportunities relative to a set of valuation test measures $\mathcal{P}^s = \{q_1, \ldots, q_M\}$ implies that there exists a measure ρ which is a convex combination of the valuation test measures:

$$\rho = \sum_{i=1}^{M} w_i q_i, \quad 0 < w_i < 1, \ \sum_{i=1}^{M} w_i = 1, \tag{2.15}$$

such that

$$E_\rho(S_1) = S_0,$$

that is, $E_\rho(S_1^j) = S_0^j$ for $j = 1, \ldots, N$. It is convenient, especially for the multi-period framework we will deal with below, not to use indices to designate individual members of $\mathcal{P}^s$ but to express ρ as

$$\rho = \sum_{q \in \mathcal{P}^s} w(q) q, \tag{2.16}$$

where $w : \mathcal{P}^s \to [0, \infty)$ with $\sum_{q \in \mathcal{P}^s} w(q) = 1$. The set of all such convex combinations is, by definition, the convex hull $\mathrm{co}(\mathcal{P}^s)$ of $\mathcal{P}^s$.

If $\rho \in \mathrm{co}(\mathcal{P}^s)$ is such a representative state pricing measure, given by (2.15) or (2.16), then there are no strictly acceptable opportunities with respect to the set[5] $S(\rho) = \{q \in \mathcal{P}^s : w(q) > 0\} = \{q_i : w_i > 0, i = 1 \dots M\}$. If $h \in \mathbb{R}^N$ is a single-period portfolio strategy for which $E_q(h \cdot \Delta S) \geq 0$ for all $q \in S(\rho)$, then[6]

$$0 = E_\rho(h \cdot \Delta S) = \sum_{q \in \mathcal{P}^s} w(q) E_q(h \cdot \Delta S)$$

implies that $E_q(h \cdot \Delta S) = 0$ for all $q \in S(\rho)$. Note that this argument does not tell us anything about the valuation test measures which are not "seen" by the representative state pricing measure ρ, but for which there might well exist strictly acceptable opportunities. However, these valuation test measures lack distinction in a market, which determines its asset prices through ρ.

We now turn to the existence of representative state pricing measures in multi-period models, which do not admit strictly acceptable opportunities. Roorda *et al.* [46] claim that it follows from Theorem 2.3.10 that the result in Carr *et al.* [12], showing the equivalence of the absence of strictly acceptable opportunities and the existence of a representative state pricing function, can be generalised to the multi-period setting, but they provided no details. Further, it is not a priori clear what form the representative state pricing measure would take, nor do we know what its relation would be with either the collection of single-period valuation test measures or with $\mathcal{P}^{paste}$. As we will now argue, the relationship between the absence of (multi-period) strictly acceptable opportunities and the existence of a representative state pricing measure is slightly subtle.

In our multi-period market model, we specify a set of single-period valuation test measures $\mathcal{P}^s(\omega')$ at each non-terminating node ω' of our tree. One

[5] One might call this the spanning set of $\rho \in \mathrm{co}(\mathcal{P}^s)$, except that the representation (2.16) isn't necessarily unique. If it isn't, it would be natural to take the union of such spanning sets over the different representations of ρ as a convex combination, but we will ignore this point below, and always work with a given convex combination.

[6] using that $E_\rho(\Delta S) = 0$

can apply the result in Carr *et al.* [12] showing the equivalence between the absence of strictly acceptable opportunities and the existence of a representative state pricing function in the static case at each non-terminating node ω', and paste together the resulting single-period state pricing measures. This leads to the following theorem.

Theorem 2.4.1 Suppose that there are no single-period strictly acceptable opportunities for any of the $\mathcal{P}^s(\omega')$, $\omega' \in \Omega'$. Then there exist $\rho^{\omega'} \in \mathrm{co}(\mathcal{P}^s(\omega'))$ such that the pasted measure $R = P^\rho$ (defined by (2.4) with $q^{\omega'}$ replaced by $\rho^{\omega'}$) is a representative state pricing measure for the tree.

Proof. We associate with each partial history $\omega^{'}$, the single-period representative state pricing function $\rho^{\omega^{'}} \in \mathrm{co}(\mathcal{P}^s(\omega'))$ by Carr *et al.* [12]. We then define R to be the multi-period probability measure on $(\Omega, \mathcal{F})$ generated by pasting together these single-period representative state pricing functions associated with the partial histories. We verify that $E_R(\Delta S_{t+1}|\mathcal{F}_t) = 0$, that is, $(S_t)_{0\le t\le T}$ is a martingale with respect to R. This is equivalent to showing that if $\ell(\omega') = t < T$, then $E_R(\Delta S_{t+1}|\omega') = 0$. But by Lemma 2.2.5 and the definition of R as a pasted measure of the $\rho^{\omega'}$ (and the fact that ΔS_{t+1} is $\mathcal{F}_{t+1}$-measurable),

$$E_R(\Delta S_{t+1}|\omega') = E_{\rho^{\omega'}}\left[S_{t+1}(\omega * .) - S_t(\omega')\right] = 0,$$

since $\rho^{\omega'}$ is a single-period representative state pricing function. ■

Observe that the theorem above did not need absence of dynamic strictly acceptable opportunities, it only required that there should be no strictly acceptable opportunities over each of the single periods of our tree. The next theorem should clarify the relationship between the former and representative state pricing measures.

Theorem 2.4.2 Suppose that $R = P^\rho$ is a representative state pricing measure as in Theorem 2.4.1, and for each $\omega' \in \Omega'$, let $S_R(\omega') = S(\rho^{\omega'})$ be the subset of elements in $\mathcal{P}^s(\omega')$, which occur with a strictly positive coefficient in $\rho^{\omega'}$. Equivalently,

$$S_R(\omega') := \{q \in \mathcal{P}^s(\omega') : q \ll \rho^{\omega'}\},$$

the set of single-period valuation test measures which are absolutely continuous with respect to $\rho^{\omega'}$. Then there are no global strictly acceptable opportunities with respect to the set of measures

$$S^{\text{paste}} = \{P^q : q = (q^{\omega'})_{\omega' \in \Omega'}, q^{\omega'} \in S_R(\omega')\}$$

which are pasted from measures in spanning sets of the different $\rho^{\omega'}$.

Proof. The fact that R is a representative state pricing measure implies that for each ω',

$$S_t(\omega') = \sum_{q \in \mathcal{P}^s(\omega')} w(q) E_q(S_{t+1}(\omega' * .)|\omega').$$

This implies that there are no single-period strictly acceptable opportunities relative to $\{q \in \mathcal{P}^s(\omega') : w(q) > 0\} = S(\rho^{\omega'})$. We can then invoke Theorem 2.3.10 with $\mathcal{P}^s$ replaced by S_R and $\mathcal{P}^{paste}$ replaced by $\mathcal{S}^{paste}$ to conclude. ■

As a further remark, we note that, since the operation of pasting is multilinear (linear in each of the $q^{\omega'}$), one can write the representative state pricing measure R of Theorem 2.4.1 as a (huge) convex combination of the valuation test measures in $\mathcal{P}^{paste}$. Conversely, if R is such a convex combination, then the associated single-period valuation test measures

$$a \to R(F(\omega' * a)|\omega'), \quad a \in A,$$

can be written as convex combinations of elements of $\mathcal{P}^s(\omega')$ with coefficients which only depend on ω', and R can be obtained by pasting these together.

We finally note that if our market model satisfies the necessary and sufficient condition of Theorem 1.3.4 at each node ω' (which is a condition on both $\mathcal{P}^s(\omega')$ and on $S_{t+1}|\omega'$, where $t = \ell(\omega')$), then the representative state pricing function R of Theorem 2.4.1 is unique; equivalently, there is a unique convex combination of valuation test measures from $\mathcal{P}^{paste}$ under which the price process $(S_t)_{t\geq 1}$ is a martingale.

Chapter 3

Diverse Beliefs-Pure Exchange Economy

3.1 Introduction and Main Results

In this chapter, we consider a Lucas-type pure exchange economy (see [39]) with N infinitely long-lived agents, who have access to the same information, but have different interpretations of the information. Specifically, they have divergent views on the stochastic evolution of the dividend paid by the economy's unique productive asset. We work with a continuous-time model of the type exposited by Brown and Rogers [11]. As it is usual for this type of economy, there is only one productive asset whose produce or dividend δ_t is, at each point t in time, completely consumed by the aggregate of all agents. Therefore, there is no possibility of storage. We will assume that the agents have time-separable constant relative risk aversion (CRRA) utility functions with identical coefficient of relative risk aversion γ across all agents. However, we will allow different subjective discount rates for the utility of consumption for the different agents. Brown and Rogers [11] gave a detailed account for the special case of log utility, which corresponds to $\gamma = 1$. For that case, it is possible to have general stochastic processes for the dividend and for the likelihood-ratio martingales, which model the different beliefs.

In this context, we will model both the dividend process $(\delta_t)_{t\geq 0}$ and the various belief processes by geometric Brownian motion. Specifically, we will assume that

$$d\delta_t = \kappa\delta_t dt + \eta\delta_t dW_t. \tag{3.1}$$

where k and η are constants, $(W_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}^0)$, and $\mathbf{P}^0$ is some reference probability measure. Following the idea in Brown and Rogers [11], we assign a probability measure to each agent, which corresponds to their idiosyncratic interpretation of the common information. Therefore, different agents j, $j = 1 \dots N$ use different probability measures $\mathbf{P}^j$ to assess the likelihoods of future events and to determine their individual optimal consumption plans. We will require that each $\mathbf{P}^j$ is locally equivalent to the reference measure $\mathbf{P}^0$. One way to achieve this is to express $\mathbf{P}^0$ as

$$\mathbf{P}^0 = \sum_{j=1}^{N} \omega_j \mathbf{P}^j, \tag{3.2}$$

where $\omega_j > 0,\ j = 1 \dots N$ and $\sum_{j=1}^{N} \omega_j = 1$; more generally, the ω_j can be taken as random here, with the sum of their expectations being 1.

However, $\mathbf{P}^0$ can also be interpreted as the objective probability measure and $\mathbf{P}^j$ as agent j's perception or estimate of $\mathbf{P}^0$ which, in the model we will consider in this chapter, will translate in a different perception or estimate for the drift κ. Note that this would not be possible anymore if investors disagree on the volatility η, since this would lead to mutually singular probability measures, at least, in a Brownian-motion framework (see [10], [35]).

Further, corresponding to each probability measure $\mathbf{P}^j, j = 1 \dots N$ is a likelihood process Λ^j given by

$$\Lambda_t^j = \frac{d\mathbf{P}^j}{d\mathbf{P}^0}|_{\mathcal{F}_t}. \tag{3.3}$$

In section 3.2, we will explain how the different beliefs are modelled. In order to keep the technicalities to a minimum, we will limit ourselves to the case where all agents agree on the dividend's volatility η, but have different views on the mean rate of return. In this case, we have that agent j, $j = 1 \dots N$ believes that the dividend δ_t has a mean rate of return $k + \alpha_j \eta$ with $\alpha_j \in \mathbf{R}$. Therefore, agent j's model for the dividend process is

$$d\delta_t = (k + \alpha_j \eta)\delta_t dt + \eta \delta_t dW_t^j, \tag{3.4}$$

where $(W_t^j)_{t\geq 0}$ is the Brownian motion with respect to the probability measure $\mathbf{P}^j$. We can call agents with $\alpha_j > 0$ optimistic while those with $\alpha_j < 0$ are called pessimistic.

Agents consume part of the dividend and hold stock in the productive asset. We assume that there is a market for all contingent claims on the dividend and that there is total agreement on the prices of such claims amongst the different agents. Brown and Rogers [11] showed that, given a vector of initial consumption pattern $c_0 = (c_0^1 \dots c_0^N)$ of the agents at time $t = 0$, the market-clearing condition implies the existence of a unique state-price density or pricing kernel (relative to the reference measure $\mathbf{P}^0$) for the prices of these claims. We will recall their argument in section 3.2. We note that this result is not limited to uniform CRRA utility functions, but holds in great generality with idiosyncratic utility functions for different agents. For a CRRA utility function with uniform γ, one can find a closed-form expression for the state-price density.

The following are the quantities we shall be interested in:

1. The price S_t at time $t > 0$ of a stock in the productive asset.

2. The prices $P_{t,T}$ of zero-coupon bonds of various maturities and the short rate r_t.

3. The individual agent's wealth w_t at time t, expressed in units of the unique consumption good.

We will show that all of these quantities can be expressed as deterministic functions of the agents' optimal consumptions c_t^j, $j = 1 \dots N$. For example, we will prove the existence of a deterministic function

$$G : \mathbf{R}_{\geq 0}^N \to \mathbf{R}_{\geq 0},$$

where the domain of G is the space of consumption patterns such that the following holds.

Theorem 3.1.1 Assume that the agents have a time-separable preference with a common constant coefficient of relative risk aversion $\gamma > 0$. Let c_t^j be agent j's consumption at time t. Then the stock price at time t is given by

$$S_t = \delta_t^{1-\gamma} G(c_t^1 \dots c_t^N), \tag{3.5}$$

while agent j's wealth, defined as the value at time t of the total future optimal consumption, is given by

$$w_t^j = \frac{1}{\gamma} \delta_t^{1-\gamma} D_j G(c_t^1 \dots c_t^N), \tag{3.6}$$

where $D_j G(c_t^1 \dots c_t^N) = c_j \frac{\partial G(c_t^1 \dots c_t^N)}{\partial c_j}$.

Remark 3.1.2 (i) We will see that G is a homogeneous function of degree γ. Using equation (3.4) and Euler's Theorem on differentiation of homogeneous functions, we will then obtain that $\sum_{j=1}^{N} w_t^j = S_t$.

(ii) Given the homogeneity of G, equation (3.5) can be written in the equivalent form

$$S_t = \delta_t G(\frac{c_t^1}{\delta_t} \dots \frac{c_t^N}{\delta_t}), \tag{3.7}$$

where we note that $\frac{c_t^j}{\delta_t}$ is the proportion of the total output that is consumed at time t by agent j.

The function $G(c)$ can be identified with the stock price at time $t = 0$, assuming that the economy starts off with an initial consumption pattern given by $c = (c_1 \dots c_N)$ with $\delta_0 = 1$. We will show that G satisfies a partial differential equation (or PDE) of order 2, which becomes a constant-coefficient second order degenerate heat equation when expressed in log-consumption coordinates $x_j = \log(c_j)$.

Other results we will obtain are a coupled non-linear system of stochastic differential equations (or SDEs) for the optimal consumption patterns c_t^j as well as an SDE for the pricing kernel, which underlies the theory from which one finds an, again non-linear, expression for the risk-free rate in terms of the consumptions.

3.2 The Brown and Rogers' Diverse Beliefs Model

We briefly review the Diverse Beliefs model discussed in Brown and Rogers [11]. They adopted a continuous-time model describing a market for a single risky asset with a unit supply, whose dividends are consumed by the N economic agents. The N agents are assumed to be heterogeneous, both in their preferences as well as their beliefs as to the unique asset's future productivity. The dividend of the asset is an adapted process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}^0)$. As usual, $\mathcal{F}_t$ represents the information available to the agents at time t. In this chapter, we will assume that the filtration is generated by the dividend process. However, in more general cases, the filtration could be associated to some vector-valued process, one of whose components is the dividend process δ_t and the other components represent other economically-relevant variables.

The N agents of the economy, which will be indexed by $j \in \{1, 2, \dots N\}$, all share the same information at time t, but have different views as to the future evolution of the dividend process. In order to compute the probabilities of future events, agent j uses his own idiosyncratic probability measure $\mathbf{P}^j$. We assume all $\mathbf{P}^j$ to be equivalent, on the filtration, to some reference probability measure $\mathbf{P}^0$. We can achieve this equivalence if we set $\mathbf{P}^0$ equal to some non-degenerate convex combination of the $\mathbf{P}^j$'s. We interpret equivalence to mean that all agents agree on what the null events in the economy are. Therefore, the adapted process $(\Lambda_t^j)_{t\geq 0}$ defined by

$$\Lambda_t^j = \frac{d\mathbf{P}^j}{d\mathbf{P}^0}|_{\mathcal{F}_t}$$

is a strictly positive martingale with respect to $\mathbf{P}^0$.

The notion of agents having different probability measures is not as outlandish as it might seem at first sight. After all, investors routinely use their own, sometimes proprietary, time series models to model and predict the economic variables of interest. Furthermore, even if the same model is used by all investors, different statistical estimation procedures, based on different samples, would inevitably lead to different calibrations of the model, and therefore, to different probability distributions of the future values of the economic variables of interest. Different probability measures also arise from agents' persistent beliefs and confirmation bias where each agent clings to hard-wired beliefs about the market evolution and shows an unwillingness to change in light of new market information.

In the sequel, we assume that each agent has their own personal utility function $U^j(c,t)$ and their probability measure $\mathbf{P^j}$, $j = 1 \dots N$ where c denotes the agent's consumption. These utility functions are used to determine their optimal consumption plans as follows. Consider a claim giving the right to a payoff of Y units of the consumption good at some future time $t > 0$. To determine the price $\pi_0(Y)$ agent j would be willing to pay for (a small multiple of) this claim, the agent will balance the loss of present consumption against the expected gain of future consumption. By equating, as usual, the marginal utility loss at time $t = 0$ resulting from delayed consumption with the expected value (with respect to the measure $\mathbf{P^j}$) of the gain in marginal utility arising from additional consumption at time t, we obtain,

$$\pi_0(Y) \cdot U_j^{'}(c_0^j, 0) = E^j(U_j^{'}(c_t^j, t) \cdot Y). \tag{3.8}$$

Therefore, the price that agent j is willing to pay for this asset is given by

$$\pi_0(Y) = E^j(Y \frac{U_j^{'}(c_t^j, t)}{U_j^{'}(c_0^j, 0)}), \tag{3.9}$$

where $E^j, j = 1 \dots N$ stands for the expectation with respect to the probability measure $\mathbf{P^j}$, and $U^{'}(c,t)$ denotes the derivative of the utility function $U(c,t)$ with respect to consumption. Suppose we take the expectation with respect to the reference measure $\mathbf{P^0}$. Then the price $\pi_0(Y)$ becomes

$$\pi_0(Y) = E^0(Y \frac{U_j^{'}(c_t^j, t)\Lambda_t^j}{U_j^{'}(c_0^j, 0)}). \tag{3.10}$$

By setting $\zeta_t^j = \frac{U_j'(c_t^j,t)\Lambda_t^j}{U_j'(c_0^j,0)}$, the pricing formula becomes

$$\pi_0(Y) = E^0(\zeta_t^j Y). \tag{3.11}$$

Suppose that we are now in a complete market in which all $\mathcal{F}_t$-measurable European claims are traded. We further assume that all agents agree on the prices so that $\zeta_t^j = \zeta_t$ for all $j = 1 \dots N$. In this case, we can compute the price by using a common kernel ζ_t relative to the reference measure $\mathbf{P^0}$. Therefore, the expression for the kernel is given by

$$\zeta_t = \upsilon_j^{-1} U_j'(c_t^j, t)\Lambda_t^j \tag{3.12}$$

for all $j \in \{1, 2, \dots N\}$, where $\upsilon_j = U_j'(c_0^j, 0)$ is the marginal utility of consumption at time $t = 0$. More generally, the price agent j will be willing to pay for an asset that pays off Y_u at time $u > t$ is given by

$$\pi_t(Y_u) = E^j(Y_u \frac{U_j'(c_u^j, u)}{U_j'(c_t^j, t)} \mid \mathcal{F}_t)$$

$$= E^0(Y_u \frac{U_j'(c_u^j, u)\Lambda_u^j}{U_j'(c_t^j, t)\Lambda_t^j} \mid \mathcal{F}_t). \tag{3.13}$$

Using equation (3.12) we have the pricing formula

$$\pi_t(Y_u) = E^0(Y_u \frac{\zeta_u}{\zeta_t} \mid \mathcal{F}_t) \tag{3.14}$$

for a claim paying off Y_u at time $u > t$. Assuming that the utility function U_j satisfies the Inada conditions: $U_j(c, t)$ is increasing and strictly concave with $U_j'(c, t) \to \infty$ as $c \to 0$ and $U_j'(c, t) \to 0$ as $c \to \infty$, then the marginal utility function $U_j'(., t)$ can be inverted. By denoting the inverse of the marginal utility function by $I_j(., t)$, we can express agent j's optimal consumption plan in terms of the (as yet undetermined) state-price density ζ_t and the constants υ_j as

$$c_t^j = I_j(\frac{\upsilon_j \zeta_t}{\Lambda_t^j}, t). \tag{3.15}$$

Market-clearing condition leads to the equation

$$\sum_{j=1}^{N} I_j(\frac{\upsilon_j \zeta_t}{\Lambda_t^j}, t) = \delta_t. \tag{3.16}$$

Equation (3.16) is, in general, a transcendental (that is, non-algebraic) equation for $\zeta_t \geq 0$ which yields the latter as a function of the dividend δ_t and the initial consumption pattern $(c_0^1 \ldots c_0^N)$ through the constants υ_j. Since the inverse of the marginal utility function is strictly decreasing, the above equation is invertible, so the equation can be solved for ζ_t.

It is, in general, not possible to solve equation (3.16) explicitly for ζ_t. However, there are a few cases where an explicit expression for ζ_t can be computed: Brown and Rogers [11] discussed in detail the case of a log utility function. In this chapter, we study the case where agents have time-separable power utility functions with the same coefficient of relative risk aversion. However, we allow agents to have different subjective discount factors for their utility of consumption.

The state-price density allows us to value the unique productive asset of the economy. The price S_t of the asset is the total expected value of the discounted dividend stream

$$S_t = E^0(\int_t^\infty \frac{\delta_u \zeta_u}{\zeta_t} du \mid \mathcal{F}_t), \tag{3.17}$$

where Fubini's Theorem (see [47]) permits the interchange of the integral and expectation operators. Other relevant economic variables are the agents' wealth processes, which are defined by

$$w_t^j = \zeta_t^{-1} \int_t^\infty E^0(c_u^j \zeta_u \mid \mathcal{F}_t) du \tag{3.18}$$

$j = 1 \dots N$, which are the time t values of their respective lifetime optimal consumptions. By the market-clearing condition, the total wealth at time t is

$$\sum_{j=1}^N w_t^j = S_t. \tag{3.19}$$

The pricing kernel or state-price density ζ_t depends on the initial consumption vector $(c_0^1 \dots c_0^N)$, which can be interpreted as a way of specifying the initial state of the economy. We will show that the vector $(c_0^1 \dots c_0^N)$ is related to the initial wealth distribution $w_0 = (w_0^1 \dots w_0^N)$ through a system of (in general transcendental) equations of gradient type. If we can show that this system has a unique solution, then another way of specifying the initial state of the economy will be by prescribing the initial vector $w_0 = (w_0^1 \dots w_0^N)$. An interesting question in the present context of diverse belief models is to study how the wealth distribution $w_t = (w_t^1/S_t \dots w_t^N/S_t)$ will stochastically evolve over time. We now turn to the special case analysed in this chapter.

3.3 Diverse Beliefs Equilibrium under Power Utility

We will suppose that agent j, $j = 1 \dots N$ has a time-separable power utility function given by

$$U_j(c,t) = e^{-\rho_j t} \frac{c^{1-\gamma} - 1}{1-\gamma}, \tag{3.20}$$

where $\gamma > 0$. When $\gamma = 1$, we obtain the log utility case discussed in Brown and Rogers [11]. We assume that all agents have a common coefficient of relative risk aversion γ, but we allow the agents to have idiosyncratic discount factors ρ_j (see Hara [30]).

3.3.1 The State-Price density

We note that $\frac{\partial}{\partial c}U_j(c,t) = e^{-\rho_j t}c^{-\gamma}$. Therefore,

$$c = I_j(\frac{\partial}{\partial c}U_j(c,t),t) = (e^{\rho_j t}\frac{\partial}{\partial c}U_j(c,t))^{-\frac{1}{\gamma}}. \tag{3.21}$$

From equation (3.12),

$$\frac{\partial}{\partial c}U_j(c,t) = \frac{\upsilon_j \zeta_t}{\Lambda_t^j}. \tag{3.22}$$

Market-clearing condition implies that

$$\sum_{j=1}^{N}(\frac{\upsilon_j \zeta_t e^{\rho_j t}}{\Lambda_t^j})^{-\frac{1}{\gamma}} = \delta_t, \tag{3.23}$$

so that

$$\zeta_t = \delta_t^{-\gamma}(\sum_{j=1}^{N}(\upsilon_j^{-1}e^{-\rho_j t}\Lambda_t^j)^{\frac{1}{\gamma}})^{\gamma}. \tag{3.24}$$

For ease of notation, we introduce the discounted beliefs

$$\tilde{\Lambda}_t^j = e^{-\rho_j t}\Lambda_t^j, \tag{3.25}$$

where the discounting is done using each agent's subjective discount rate. Using the fact that $\upsilon_j^{-\frac{1}{\gamma}} = c_0^j$, which is agent j's initial consumption, we obtain a simplified version for the state-price density

$$\zeta_t = \zeta_t(c_0^1 \dots c_0^N) = \delta_t^{-\gamma}(\sum_{j=1}^{N} c_0^j(\tilde{\Lambda}_t^j)^{\frac{1}{\gamma}})^{\gamma}. \tag{3.26}$$

We observe that $\zeta_0 = 1$: by applying the market-clearing condition at time $t = 0$ and using the fact that $\widetilde{\Lambda}_0^j = E_{\mathbf{P}_0}(d\mathbf{P}^j/\mathbf{P}^0) = 1$. Further, applying equation (3.26) and the market-clearing condition, we deduce that

$$c_t^j = c_0^j(\tilde{\Lambda}_t^j)^{\frac{1}{\gamma}}\zeta_t^{-\frac{1}{\gamma}}. \tag{3.27}$$

This relates optimal consumption with initial consumption, discounted belief and state-price density.

3.3.2 Dividend and Belief Processes

In order to make the model concrete, we will assume that both the dividend process δ_t and the belief processes $\Lambda_t^j, j = 1 \ldots N$ follow a geometric Brownian motion

$$d\delta_t = (\kappa + \alpha_j\eta)\delta_t dt + \eta\delta_t dW_t^j \tag{3.28}$$

with respect to $\mathbf{P}^j$, while

$$d\Lambda_t^j = \alpha_j\Lambda_t^j dW_t, \tag{3.29}$$

where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P^0})$ and η is a constant volatility parameter for the dividend process. We thus assume that the belief processes are driven by the common Brownian motion, and the Girsanov Theorem (see [7, 42]) then implies that $dW_t^j = -\alpha_j dt + dW_t$ and, consequently,

$$d\delta_t = \kappa\delta_t dt + \eta\delta_t dW_t$$

with respect to $\mathbf{P}^0$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ in this model is generated by the Brownian motion in the dividend (and belief) processes. From the theory of stochastic calculus, we can obtain the solutions to the above stochastic differential equations as

$$\delta_t = \delta_0 e^{(\kappa - 0.5\eta^2)t + \eta W_t} = \delta_0 e^{(\kappa_j - 0.5\eta^2)t + \eta W_t^j}, \quad \Lambda_t^j = \Lambda_0^j e^{-0.5\alpha_j^2 t + \alpha_j W_t}, \tag{3.30}$$

where $\kappa_j = \kappa + \alpha_j\eta$ is agent j's value of the dividend's mean rate of growth. Each agent j, $j = 1 \ldots N$ has a probability measure $\mathbf{P^j}$ and the dividend process under $\mathbf{P}^j$ has mean κ_j and volatility η. The interpretation of the belief process thus relies on the Girsanov Theorem (see [7, 42]) where the usual assumption is that each $\mathbf{P}^j$ is absolutely continuous with respect to the reference measure $\mathbf{P^0}$. Furthermore, Λ_t^j is a martingale with respect to $\mathbf{P}^0$.

One possible interpretation of the model is that $\mathbf{P}^0$ is the true objective probability measure, which is imperfectly observable by the agents. Agent $j,\ j = 1 \dots N$, believes it to be $\mathbf{P}^j = \Lambda^j \mathbf{P^0}$, either because of estimation errors or on the basis of their a priori convictions on what the drift of δ_t should be. We note, though, that all agents agree on δ_t's volatility η. Another interpretation of the model would be to interpret α_j as δ_t's market price of risk (the extra return required per unit of volatility) according to agent j. Moreover, different agents have different views on what this price of risk is or should be.

From equation (3.29) and by applying Itô's lemma (see [42], [7]), the dynamics of the process $(\tilde{\Lambda}_t^j)$ are given by

$$d\tilde{\Lambda}_t^j = \tilde{\Lambda}_t^j(-\rho_j dt + \alpha_j dW_t). \tag{3.31}$$

We can interpret the parameter κ as the true mean of the process under the objective probability measure $\mathbf{P}^0$. This parameter is not perfectly observable and a statistical approach is used to estimate its value. However, even with this approach, it is usually very difficult to achieve an estimate within a reasonably tight confidence interval. In light of this, different agents find different values for κ, which we have denoted by $\kappa + \alpha_j \eta, j = 1 \dots N$. They might, in practice, update their estimates as new information becomes available e.g. using Bayesian methods (see [44]) alongside classical estimators. However, we will assume that these estimates are constant on any specified time horizon.

3.3.3 Asset Prices

In this subsection, we show that the price-earnings ratio, S_t/δ_t, can be written as a deterministic function of the current consumption-output vector, $\delta_t^{-1}c_t = (c_t^1/\delta_t \dots c_t^N/\delta_t)$. This is a function of the initial consumption vector $(c_1 \dots c_N)$, and we will call it the generating function of the economy. As we will see, it is simply the asset price at time $t = 0$ assuming that $\delta_0 = 1$. Let δ_t^0 be the solution of the stochastic differential equation

$$d\delta_t^0 = \kappa\delta_t^0 dt + \eta\delta_t^0 dW_t \tag{3.32}$$

with initial value $\delta_0^0 = 1$. We note that $\Lambda_0^j = 1$ for all $j = 1 \dots N$ since we assume that $\mathcal{F}_0$ is the trivial $\sigma-$algebra.

Definition 3.3.1 The generating function of our diverse beliefs economy as specified by the utility functions (3.20), the dividend process (3.28) and the belief processes (3.29) is the function $G : \mathbf{R}_{\geq 0}^N \to \mathbf{R}_{\geq 0}$ defined by

$$G(c_1 \dots c_N) = \int_0^\infty E^0((\delta_t^0)^{1-\gamma}(\sum_{j=1}^N c_j(\tilde{\Lambda}_t^j)^{1/\gamma})^\gamma)dt. \tag{3.33}$$

The generating function G is defined for values of the parameters κ, η, ρ_j and α_j for which the integral is finite. The motivation for this definition is that we can express important characteristics of the economy, such as asset prices, individual agent's wealth process etc. in terms of G and its derivatives. We note that the function G is homogeneous of degree γ and G is therefore completely determined by its restriction to the $(N-1)$-simplex $\triangle_{N-1} \subset \mathbf{R}_{\geq 0}^N$ defined by

$$\triangle_{N-1} = \{(x_1 \dots x_N) \in \mathbf{R}^N : x_j \geq 0, \ \sum_{j=1}^N x_j = 1\}. \tag{3.34}$$

Below, we will evaluate G at the vector $(c_t^1/\delta_t, \ldots, c_t^N/\delta_t)$ which belongs to $\triangle_{N-1}$ since $\sum_{j=1}^N c_t^j/\delta_t = 1$.

Further, by substituting δ_t/δ_0 for δ_t^0 in equation (3.32) and setting $c_j = c_0^j$ for $j = 1 \ldots N$, we obtain

$$G(c_0^1 \ldots c_0^N) = \int_0^\infty E^0((\delta_t/\delta_0)^{1-\gamma}(\sum_{j=1}^N c_0^j(\tilde{\Lambda}_t^j)^{1/\gamma})^\gamma)dt \tag{3.35}$$

$$= \delta_0^{\gamma-1}\int_0^\infty E^0((\delta_t)^{1-\gamma}(\sum_{j=1}^N c_0^j(\tilde{\Lambda}_t^j)^{1/\gamma})^\gamma)dt$$

$$= \delta_0^{\gamma-1}\int_0^\infty E^0(\delta_t\zeta_t(c_0^1 \ldots c_0^N)dt$$

$$= \delta_0^{\gamma-1}S_0. \tag{3.36}$$

Therefore, $S_0 = \delta_0^{1-\gamma}G(c_0^1 \ldots c_0^N)$. We extend this to arbitrary times below, but our first result is to determine the conditions on the parameters under which the function G is finite. We first introduce some new notations.

Let $Z_t^j = (\delta_t^0)^{\frac{1}{\gamma}-1}(\tilde{\Lambda}_t^j)^{\frac{1}{\gamma}}$. Then $G(c_1 \ldots c_N)$ can be rewritten as

$$G(c_1 \ldots c_N) = \int_0^\infty E^0[(\sum_{j=1}^N c_j Z_t^j)^\gamma]\,dt. \tag{3.37}$$

Note that Z_t^t is a geometric Brownian motion. From equation (3.30), we have that

$$Z_t^j = e^{a_j t + b_j W_t} \tag{3.38}$$

with

$$a_j = (\frac{1}{\gamma}-1)(\kappa - \frac{1}{2}\eta^2) - \frac{1}{\gamma}(\rho_j + \frac{1}{2}\alpha_j^2);\ b_j = (\frac{1}{\gamma}-1)\eta + \frac{\alpha_j}{\gamma}. \tag{3.39}$$

We now state and prove the following lemma.

Lemma 3.3.2 Recall that $\kappa_j = \kappa + \alpha_j \eta$ is agent j's value of the dividend's mean growth rate. Then, $G(c_1 \ldots c_N) < \infty$ for all $(c_1 \ldots c_N) \in \mathbf{R^N}$ if and only if

$$(1-\gamma)(\kappa_j - \frac{1}{2}\gamma\eta^2) < \rho_j \tag{3.40}$$

for all $j = 1 \ldots N$.

Proof. There exist constants $c = c(\gamma, N) > 0$ and $C = C(\gamma, N) > 0$ such that for all $(x_1, \ldots, x_N) \in \triangle_{N-1}$, we have that

$$c(x_1^\gamma + \ldots + x_N^\gamma) \leq (x_1 + \ldots + x_N)^\gamma \leq C(x_1^\gamma + \ldots + x_N^\gamma).$$

To show this, simply note that both sides of the two inequalities are homogeneous of order γ and that $\frac{(x_1+\ldots+x_N)^\gamma}{(x_1^\gamma+\ldots+x_N^\gamma)}$ is continuous and non-zero on the $(N-1)$-simplex $\triangle_{N-1}$, which is compact. Therefore, $G(c_1, \cdots, c_N) < \infty$ for all $(c_1, \ldots, c_N)$ if and only if $\int_0^\infty E^0((Z_t^j)^\gamma)dt < \infty$ for all j. But

$$\int_0^\infty E^0[(Z_t^j)^\gamma]dt = \int_0^\infty E^0[e^{\gamma a_j t + b_j \gamma W_t}]dt = \int_0^\infty e^{\gamma a_j t + \frac{1}{2} b_j^2 \gamma^2 t} dt$$

for all $j = 1, \ldots, N$. The expression $\int_0^\infty e^{\gamma a_j t + \frac{1}{2} b_j^2 \gamma^2 t} dt < \infty$ if and only if $\gamma a_j + \frac{1}{2} b_j^2 \gamma^2 < 0$. By substituting $(\frac{1}{\gamma} - 1)(\kappa - \frac{1}{2}\eta^2) - \frac{1}{\gamma}(\rho_j + \frac{1}{2}\alpha_j^2)$ and $(\frac{1}{\gamma} - 1)\eta + \gamma^{-1}\alpha_j$ for a_j and b_j respectively, we obtain

$$(1-\gamma)(\kappa - \frac{1}{2}\eta^2) - (\rho_j + \frac{1}{2}\alpha_j^2) + \frac{1}{2}((1-\gamma)\eta + \alpha_j)^2 < 0.$$

Upon simplification, the result follows. ■

Now that we have shown that the generating function is finite-valued, the next step is to express the stock price at any time t in terms of the generating function. This is the purpose of the following theorem.

Theorem 3.3.3 The asset price at time t can be expressed as

$$S_t = \delta_t^{1-\gamma} G(c_t^1 \dots c_t^N), \tag{3.41}$$

where c_t^j is agent j's optimal consumption at time t.

Remark 3.3.4 In view of the homogeneity of G, equation (3.41) can be rewritten as a relationship between price-dividend ratio of the asset at time t and the contemporaneous share of output vector:

$$\frac{S_t}{\delta_t} = G(\frac{c_t^1}{\delta_t} \dots \frac{c_t^N}{\delta_t}), \tag{3.42}$$

which is a relation between dimensionless quantities, given that prices are denominated in units of the unique consumption good.

Proof. From equations (3.17) and (3.26), we have

$$S_t = \frac{1}{\zeta_t} E^0 \left(\int_t^\infty \delta_u \zeta_u du \mid \mathcal{F}_t \right)$$

$$= \frac{1}{\zeta_t} E^0 \left(\int_t^\infty \delta_u (\delta_u^{-\gamma} (\sum_{j=1}^N c_0^j (\tilde{\Lambda}_u^j)^{\frac{1}{\gamma}})^\gamma) du \mid \mathcal{F}_t \right)$$

$$= \frac{1}{\zeta_t} \int_t^\infty E^0 (\delta_u^{1-\gamma} (\sum_{j=1}^N c_0^j (\tilde{\Lambda}_u^j)^{\frac{1}{\gamma}})^\gamma) \mid \mathcal{F}_t) du.$$

Since a geometric Brownian motion can be written as a product of other geometric Brownian motion-type processes, we therefore have that

$$\delta_u = \delta_t \cdot (\delta_u / \delta_t); \quad \tilde{\Lambda}_u^j = \tilde{\Lambda}_t^j \cdot (\tilde{\Lambda}_u^j / (\tilde{\Lambda}_t^j).$$

Further, by equation (3.27) $c_0^j (\tilde{\Lambda}_t^j)^{\frac{1}{\gamma}} = c_t^j \zeta_t^{\frac{1}{\gamma}}$.

By substituting for δ_u, $\tilde{\Lambda}_u^j$ and $c_0^j (\tilde{\Lambda}_t^j)^{\frac{1}{\gamma}}$, we obtain

$$S_t = \frac{1}{\zeta_t}\delta_t^{1-\gamma}\int_t^\infty E^0((\frac{\delta_u}{\delta_t})^{1-\gamma}(\sum_{j=1}^N c_t^j \zeta_t^{\frac{1}{\gamma}}(\tilde{\Lambda}_u^j/(\tilde{\Lambda}_t^j)^{1/\gamma})^\gamma \mid \mathcal{F}_t)du$$

$$= \delta_t^{1-\gamma}\int_t^\infty E^0((\frac{\delta_u}{\delta_t})^{1-\gamma}(\sum_{j=1}^N c_t^j(\tilde{\Lambda}_u^j/(\tilde{\Lambda}_t^j)^{1/\gamma})^\gamma \mid \mathcal{F}_t)du. \tag{3.43}$$

Using the fact that δ_u/δ_t (resp. $\tilde{\Lambda}_u^j/\tilde{\Lambda}_t^j$) has the same distribution as δ_{u-t}^0 (resp. $\tilde{\Lambda}_{u-t}^j$) (it is equal to 1 for $u = t$), equation (3.43) becomes

$$\delta_t^{1-\gamma}\int_t^\infty E^0((\delta_{u-t}^0)^{1-\gamma}(\sum_{j=1}^N c_t^j(\tilde{\Lambda}_{u-t}^j)^{1/\gamma})^\gamma \mid \mathcal{F}_t)du.$$

By changing the variable $u - t$ to u, we arrive at the stated result:

$$S_t = \delta_t^{1-\gamma}\int_0^\infty E^0((\delta_u^0)^{1-\gamma}(\sum_{j=1}^N c_t^j(\tilde{\Lambda}_u^j)^{1/\gamma})^\gamma \mid \mathcal{F}_t)du = \delta_t^{1-\gamma}G(c_t^1 \dots c_t^N). \blacksquare$$

3.3.4 Agents' Wealth Processes

The next step is to express agent j's wealth in terms of the generating function G. In order to do this, we shall introduce a new notation to represent certain derivatives of G. In the sequel, we shall let $\partial_j^H G = c_j \frac{\partial G}{\partial c_j}$. The following is the statement of the result.

Theorem 3.3.5 The process which models agent j's wealth is given by

$$w_t^j = \frac{1}{\gamma}\delta_t \partial_j^H G(c_t^1/\delta_t \dots c_t^N/\delta_t). \tag{3.44}$$

Remark 3.3.6 We note that

$$\sum_{j=1}^N w_t^j = \sum_{j=1}^N \frac{1}{\gamma}\delta_t \partial_j^H G(c_t^1/\delta_t \dots c_t^N/\delta_t)$$

$$= \sum_{j=1}^{N} \frac{1}{\gamma} \delta_t c_j \frac{\partial G}{\partial c_j} (c_t^1/\delta_t \dots c_t^N/\delta_t)$$

$$= \frac{1}{\gamma} \delta_t \sum_{j=1}^{N} c_j \frac{\partial G}{\partial c_j} (c_t^1/\delta_t \dots c_t^N/\delta_t)$$

$$= \frac{1}{\gamma} \delta_t \gamma G(c_t^1/\delta_t \dots c_t^N/\delta_t)$$

$$= \delta_t G(c_t^1/\delta_t \dots c_t^N/\delta_t) = \delta_t^{1-\gamma} G(c_t^1 \dots c_t^N) = S_t.$$

Therefore, we have used the Euler's identity for homogeneous functions and the fact that G is homogeneous to show that the total wealth at time t is equal to the asset price. This would, of course, also follow from the equation (3.18) of w_t^j together with the market-clearing condition $\sum_j c_t^j = \delta_t$.

Proof. Recall that E_t^0 denotes the conditional expectation with respect to $\mathbf{P^0}$. By using (3.26) and (3.27), we can write w_t as

$$w_t^j = \frac{1}{\zeta_t} \int_t^\infty E_t^0 (c_u^j \zeta_u) du$$

$$= \frac{1}{\zeta_t} \int_t^\infty E_t^0 [c_0^j (\tilde{\Lambda}_u^j)^{1/\gamma} \zeta_u^{1-1/\gamma}] du$$

$$= \frac{1}{\zeta_t} \int_t^\infty E_t^0 [c_0^j (\tilde{\Lambda}_u^j)^{1/\gamma} (\delta_u^{-\gamma} (\sum_{k=1}^{N} c_0^k (\tilde{\Lambda}_u^k)^{1/\gamma})^\gamma)^{1-1/\gamma}] du$$

$$= \frac{1}{\zeta_t} \int_t^\infty E_t^0 [\delta_u^{1-\gamma} c_0^j (\tilde{\Lambda}_u^j)^{1/\gamma} (\sum_{k=1}^{N} (c_0^k (\tilde{\Lambda}_u^k)^{1/\gamma})^{\gamma-1}] du$$

$$= \frac{\delta_t^{1-\gamma}}{\zeta_t} \int_t^\infty E_t^0 [(\frac{\delta_u}{\delta_t})^{1-\gamma} c_0^j (\tilde{\Lambda}_t^j)^{1/\gamma} (\frac{\tilde{\Lambda}_u^j}{\tilde{\Lambda}_t^j})^{1/\gamma} (\sum_{k=1}^{N} c_0^k (\tilde{\Lambda}_t^k)^{1/\gamma} (\frac{\tilde{\Lambda}_u^k}{\tilde{\Lambda}_t^k})^{1/\gamma})^{\gamma-1}] du$$

$$= \frac{\delta_t^{1-\gamma}}{\zeta_t} \int_t^{\infty} E_t^0 [(\delta_{u-t}^0)^{1-\gamma} c_0^j (\tilde{\Lambda}_t^j)^{1/\gamma} (\tilde{\Lambda}_{u-t}^j)^{1/\gamma} (\sum_{k=1}^{N} c_0^k (\tilde{\Lambda}_t^k)^{1/\gamma} (\tilde{\Lambda}_{u-t}^k)^{1/\gamma})^{\gamma-1}] du, \tag{3.45}$$

where, as before, δ_u^0 is the normalised dividend process with initial value 1. Let $c_k = c_0^k (\tilde{\Lambda}_t^k)^{1/\gamma}$ and $s = u - t$. Then equation (3.45) becomes

$$\frac{\delta_t^{1-\gamma}}{\zeta_t} \int_0^{\infty} E^0 [(\delta_s^0)^{1-\gamma} c_j (\tilde{\Lambda}_s^j)^{1/\gamma} (\sum_{k=1}^{N} c_k (\tilde{\Lambda}_s^k)^{1/\gamma})^{\gamma-1}] ds \tag{3.46}$$

$$= \frac{\delta_t^{1-\gamma}}{\zeta_t \gamma} c_j G_j(c_1, \ldots, c_N),$$

where G_j is a deterministic function defined by

$$G_j(c_1 \ldots c_N) = \int_0^{\infty} E^0 [(\delta_s^0)^{1-\gamma} \gamma (\tilde{\Lambda}_s^j)^{1/\gamma} (\sum_{k=1}^{N} c_k (\tilde{\Lambda}_s^k)^{1/\gamma})^{\gamma-1}] ds,$$

with δ_t^0 the solution of the equation (3.30) such that $\delta_0^0 = 1$. By equation (3.33)

$$G(c_1 \ldots c_N) = \int_0^{\infty} E^0 [(\delta_t^0)^{1-\gamma} (\sum_{k=1}^{N} c_k (\tilde{\Lambda}_t^k)^{1/\gamma})^{\gamma}] dt.$$

By differentiating G with respect to c_j and multiplying the result by c_j, we obtain

$$c_j \frac{\partial G}{\partial c_j} = E^0 \int_0^{\infty} [(\delta_t^0)^{1-\gamma} \gamma c_j (\tilde{\Lambda}_t^j)^{1/\gamma} (\sum_{k=1}^{N} c_k (\tilde{\Lambda}_t^k)^{1/\gamma})^{\gamma-1}] dt.$$

Further simplification gives

$$\frac{1}{\gamma} c_j \frac{\partial G}{\partial c_j} = E^0 \int_0^{\infty} [(\delta_t^0)^{1-\gamma} c_j (\tilde{\Lambda}_t^j)^{1/\gamma} (\sum_{k=1}^{N} c_k (\tilde{\Lambda}_t^k)^{1/\gamma})^{\gamma-1}] dt. \tag{3.47}$$

By equations (3.46) and (3.47) we finally obtain

$$w_t^j = \frac{1}{\gamma}\frac{\delta_t^{1-\gamma}}{\zeta_t}c_j\frac{\partial G}{\partial c_j}(c_1 \ldots c_N).$$

Using the fact that $c_k = c_0^k(\tilde{\Lambda}_t^k)^{1/\gamma}$, the homogeneity of G and equation (3.27), we can rewrite w_t^j as

$$\begin{aligned} w_t^j &= \frac{1}{\gamma}\delta_t^{1-\gamma}c_j\frac{\partial G}{\partial c_j}(c_0^1(\tilde{\Lambda}_t^1)^{1/\gamma}\zeta_t^{-1/\gamma} \ldots c_0^N(\tilde{\Lambda}_t^N)^{1/\gamma}\zeta_t^{-1/\gamma}) \\ &= \frac{1}{\gamma}\delta_t c_j\frac{\partial G}{\partial c_j}(\frac{c_t^1}{\delta_t} \ldots \frac{c_t^N}{\delta_t}) \\ &= \frac{1}{\gamma}\delta_t\partial_j^H G(\frac{c_t^1}{\delta_t} \ldots \frac{c_t^N}{\delta_t}). \ \blacksquare \end{aligned}$$

3.3.5 Bond Prices and the short rate

In this section, we extend this asset pricing method via a generating function to the pricing of bonds and the determination of the short rate. We recall that the price (in units of consumption good) at time t of a zero-coupon bond which matures and pays off 1 unit of consumption good at time T is given by

$$P_{t,T} = E^0(1 \cdot \frac{\zeta_T}{\zeta_t} \mid \mathcal{F}_t). \tag{3.48}$$

By using arguments similar to those in section 3.3.3, we define a deterministic function of the consumptions at time t. Let the function $B : \mathbf{R}_{>0}^N \to \mathbf{R}_{>0}$ be given by

$$B(c_1 \ldots c_N, \tau) = E^0[(\delta_\tau^0)^{-\gamma}(\sum_{j=1}^N c_j(\tilde{\Lambda}_\tau^j)^{1/\gamma})^\gamma] \tag{3.49}$$

where $\tau = T - t$. The following proposition gives the representation of bond prices in terms of the above generating function.

Proposition 3.3.7 The bond price at time t equals

$$P_{t,T} = (\delta_t)^{-\gamma} B(c_t^1 \dots c_t^N, \tau), \tag{3.50}$$

where c_t^j is agent j's optimal consumption at time t and $\tau = T - t$ is time-to-maturity.

Proof. By (3.26)

$$\begin{aligned}
P_{t,T} &= \zeta_t^{-1} E^0[\zeta_T \mid \mathcal{F}_t] \\
&= \zeta_t^{-1} E^0[(\delta_T)^{-\gamma} (\sum_{j=1}^{N} c_0^j (\tilde{\Lambda}_T^j)^{1/\gamma})^{\gamma} \mid \mathcal{F}_t] \\
&= \zeta_t^{-1} E^0[(\delta_T/\delta_t)^{-\gamma} (\delta_t)^{-\gamma} (\sum_{j=1}^{N} c_0^j (\tilde{\Lambda}_T^j/\tilde{\Lambda}_t^j)^{1/\gamma} (\tilde{\Lambda}_t^j)^{1/\gamma})^{\gamma} \mid \mathcal{F}_t].
\end{aligned}$$

Using the fact that δ_T/δ_t (resp. $\tilde{\Lambda}_T^j/\tilde{\Lambda}_t^j$) is independent of $\mathcal{F}_t$ and equidistributed with δ_{T-t}^0 (resp. $\tilde{\Lambda}_{T-t}^j$) and by equation (3.27), we obtain

$$\begin{aligned}
P_{t,T} &= \zeta_t^{-1} (\delta_t)^{-\gamma} E^0[(\delta_\tau^0)^{-\gamma} (\sum_{j=1}^{N} c_t^j \zeta_t^{1/\gamma} (\tilde{\Lambda}_\tau^j)^{1/\gamma})^{\gamma} \mid \mathcal{F}_t] \\
&= \zeta_t^{-1} (\delta_t)^{-\gamma} \zeta_t E^0[(\delta_\tau^0)^{-\gamma} (\sum_{j=1}^{N} c_t^j (\tilde{\Lambda}_\tau^j)^{1/\gamma})^{\gamma} \mid \mathcal{F}_t] \\
&= (\delta_t)^{-\gamma} E^0[(\delta_\tau^0)^{-\gamma} (\sum_{j=1}^{N} c_t^j (\tilde{\Lambda}_\tau^j)^{1/\gamma})^{\gamma} \mid \mathcal{F}_t] \\
&= (\delta_t)^{-\gamma} B(c_t^1 \dots c_t^N, \tau)
\end{aligned}$$

as was to be shown. ■

Remark 3.3.8 Forward rates can be defined in the usual manner:

$$f_{t.T} = -\frac{\partial}{\partial T} \log P_{t,T}.$$

In terms of the function B, the forward rates are given as functions of the optimal consumption patterns by

$$f_{t,T} = -\frac{\partial_\tau B(c_t^1, \dots, c_t^N, \tau)}{B(c_t^1, \dots, c_t^N, \tau)}, \quad \tau = T - t.$$

The short rate is given by $\lim_{T\to t} f_{t.T} = r_t$. An alternative approach is to read off r_t from the dynamics for ζ_t:

$$dr_t = -E^0(\frac{d\zeta_t}{\zeta_t}),$$

in equation (3.62). See section 3.3.9 below for the dynamics of ζ_t.

3.3.6 General Remarks

In this subsection, we give some remarks about the mathematical structure of ζ and offer possible abstract interpretations of the object. We will also look at two limiting cases.

The state price density (3.26) can be interpreted as the l_p-norm of the random vector $\delta_t^{-\gamma}((c_0^1)^\gamma \tilde{\Lambda}_t^1 \dots (c_0^N)^\gamma \tilde{\Lambda}_t^N) \in \mathbf{R^N}$. The following is a slightly different, but related, interpretation. Let $\Omega_N = \{1, 2, \dots N\}$ be the space of agents in our economy and introduce a probability measure $\mathbf{P_c}$ on Ω_N such that $\mathbf{P_c}(j) = \frac{c_0^j}{\sum_{k=1}^N c_0^k} = \frac{c_0^j}{\delta_0}$. We note that the probability measure depends on the initial consumption levels of the agents. For a fixed time t, we associate with agent j, a function $\tilde{\Lambda}_t^j$ defined on Ω_N. Therefore, we can rewrite (3.26) as

$$\zeta_t = (\delta_t/\delta_0)^{-\gamma} ||\tilde{\Lambda}_t||_{L^{1/\gamma}, \mathbf{p_c}}, \tag{3.51}$$

where the norm on the right-hand side of (3.51) is associated with the discrete probability measure $\mathbf{P_c}$. On this occasion, ζ_t depends on the fractions c_0^j/δ_0 of the total initial consumption rather than c_0^j. When $\gamma \leq 1$, we obtain a genuine norm and it is a convex function of discounted beliefs. However, when $\gamma > 1$, ζ_t is a quasinorm as it satisfies subadditivity up to a multiplicative γ-dependent constant $C(\gamma) \geq 1$. That is,

$$||\tilde{\Lambda}_t^1 + \tilde{\Lambda}_t^2||_{L^{1/\gamma},\mathbf{P_c}} \leq C(\gamma)(||\tilde{\Lambda}_t^1||_{L^{1/\gamma},\mathbf{P_c}} + ||\tilde{\Lambda}_t^2||_{L^{1/\gamma},\mathbf{P_c}}).$$

In the case where there is a continuum of agents in the economy, we define a density function $f(j)$ on the space of agents Ω_N = [0,1]. We interpret $\int_a^b f(j)dj$ as the fraction of the total output at time t=0 consumed by agents $j \in [a, b]$. Agents' beliefs will now be a function $j \to (\Lambda_t(j))_{t\geq 0}$ on [0,1] with values in a space of strictly positive integrable martingales on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$. Under suitable measurability and integrability conditions, we can write the state-price density as

$$\zeta_t = (\delta_t/\delta_0)^{-\gamma}(\int_0^1 e^{-\rho_j t/\gamma}\Lambda_t(j)^{1/\gamma} f(j)dj)^{\gamma}.$$

In the case where $\gamma = 0$, the marginal utility $\partial_c U_j(c, t)$ equals $e^{-\rho_j t}$ so the utility function is affine. Further, equation (3.12) does not have a solution unless in the case where all discount factors are identical and all beliefs are identical: $\rho_j = \rho$ and $\Lambda_t^j = \Lambda_t$ for all $j = 1 \dots N$. Here, optimal consumptions c_t^j are undetermined and are only constrained by the market-clearing condition. This is consistent with the fact that affine utility is equivalent to risk-neutrality, and that for risk-neutral investors, the state-price density is simply the discounted probability, $e^{-\rho t}\Lambda_t$. We therefore conclude that heterogeneous beliefs are not compatible with linear utility.

We now explore the limiting case of $\gamma \to 0$ where, for a positive γ, the diverse belief equilibrium has a well-defined unique state-price density ζ_t and optimal consumptions c_t^j with different ρ_j and Λ_t^j. Using the fact that

$\upsilon_j = U'_j(c_0^j, 0) = (c_0^j)^{-\gamma} \to 1$ and that $||.||_{1/\gamma} \to ||.||_\infty$ as $\gamma \to 0$, we find that

$$\zeta_t = ||\tilde{\Lambda}_t|| = \max_j ||\tilde{\Lambda}_t^j|| = \max_j e^{-\rho_j t}\Lambda_t^j.$$

We now analyse the behaviour of the optimal consumption components c_t^j given by equation (3.27) when $\gamma \to 0$. We note that for $\mathbf{x} = (x_1, x_2 \ldots x_N) \in \mathbf{R}^N$ and if we let J be the set of indices for which $x_j = max_k x_k$, then

$$\frac{c_0^j x_j^p}{\sum_{k=1}^N c_0^k x_k^p} = \begin{cases} 0 & if\ j \notin J \\ c_0^j \ / \sum_{k\in J} c_0^k & if\ j \in J \end{cases}$$

as $p \to \infty$. By letting $x_j = \tilde{\Lambda}_t^j$ and $p = 1/\gamma$, we find that the limit of c_t^j as $\gamma \to 0$ is either zero or $\frac{\delta_t c_0^j}{\sum_{k\in J} c_0^k}$, where J is now a random set which denotes the set of indices for which $\tilde{\Lambda}_t^j = max_k \tilde{\Lambda}_t^k$. Therefore, when $\gamma \to 0$, the locally dominant beliefs at time t are selected such that the output is consumed among the investors with these beliefs. This, of course, is in accordance with their initial consumptions.

3.3.7 Models with heterogeneous risk aversions

In this section, we consider investors with different coefficients of relative risk aversions $\gamma_j, j = 1, 2, \ldots N$. In this case, the defining equation for the state-price density becomes

$$\sum_{j=1}^N \left(\frac{\upsilon_j \zeta_t e^{\rho_j t}}{\Lambda_t^j}\right)^{-1/\gamma_j} = \sum_{j=1}^N c_0^j (\zeta_t^{-1}\tilde{\Lambda}_t^j)^{1/\gamma_j} = \delta_t. \tag{3.52}$$

This is a transcendental equation, which can be solved by numerical methods. However, by working in the limit, we can still obtain an analytical expression for the solution. We normalise the above equation by dividing through by $\delta_0 = \sum_{k=1}^N c_0^k$ and replace $c_0^j / \sum_{k=1}^N c_0^k$ by the "initial density of consumption" function $c_0(j)$ on the interval $(0, \infty)$. The interval $(0, \infty)$ is used to index the different investors with positive coefficients of relative risk aversion $\gamma(j), j \in (0, \infty)$. In the limit, the expression $\sum_{j=1}^{N} c_0^j (\zeta_t^{-1}\tilde{\Lambda}_t^j)^{1/\gamma_j}$ in equation (3.52) becomes

$$\int_0^\infty c_0(j)\tilde{\Lambda}_t(j)^{1/\gamma(j)}\zeta_t^{-1/\gamma(j)}dj. \tag{3.53}$$

If we take $\gamma(j)^{-1} = j$ (which amounts to indexing the continuum of investors by the inverse of their respective coefficients of relative risk aversion), then equation (3.53) can be written as

$$\int_0^\infty c_0(j)\tilde{\Lambda}_t(j)^j\zeta_t^{-j}dj = \int_0^\infty c_0(j)\tilde{\Lambda}_t(j)^j e^{-j\ln\zeta_t}dj.$$

By letting $s = \ln\zeta_t$ and $f(j) = c_0(j)(\tilde{\Lambda}_t(j))^j = c_0(j)e^{-j\rho(j)t+j\ln\Lambda_t(j)}$, we arrive at the equation

$$F(s) = \int_0^\infty f(j)e^{-js}dj = \int_0^\infty c_0(j)e^{-j\rho(j)t+j\ln\Lambda_t(j)}e^{-js}dj. \tag{3.54}$$

The function $F(s)$ is called the Laplace transform of the random function $f(j) = c_0(j)e^{-j\rho(j)t}\Lambda_t(j)^j$. Therefore, the modelling problem reduces to choosing a suitable function $c_0(j)$, compute the Laplace transform $F(s)$ and then find its inverse. This problem simplifies when agents have beliefs, which can be modelled by geometric Brownian motion: $\Lambda_t(j) = e^{-\alpha(j)^2t/2+\alpha(j)W_t}$. The Laplace transform becomes

$$F(s) = \int_0^\infty c_0(j)e^{-(j\rho(j)+0.5j\alpha(j)^2)t+j\alpha(j)w}e^{-js}dj,$$

where w is a realisation of W_t. However, this is not possible as no choice of the functions $\alpha(j)$ is amenable to an easy computation of $F(s)$. The problem becomes solvable if we drop the diverse beliefs assumption: that is, we assume that $\Lambda_t(j) = 1$ for all j, and also let $\rho(j) = \rho$ (constant). In this case, we are considering a continuum of agents who have access to the same information and have same beliefs and the same discount factor. The agents differ, however, in their risk aversion. Therefore, $f(j) = c_0(j)e^{-j\rho t}$ and

$$F(s) = \int_0^\infty c_0(j)e^{-j\rho t}e^{-js}dj$$

$$= C(s + \rho t),$$

where $C(s)$ is the Laplace transform of $c_0(j)$.

3.3.8 A PDE for the generating function

In this subsection, we will go over to log-consumption coordinates and introduce the function $g : \mathbb{R}^N \to \mathbb{R}_{>0}$ defined by

$$g(x) \equiv G(e^{x_1}, \dots e^{x_N}), \quad x = (x_1, \dots x_N) \tag{3.55}$$

so that

$$g(x) = \int_0^\infty \mathbb{E}^0(f_\gamma(x + at + bW_t))dt \tag{3.56}$$

with $f_\gamma(x) = (\sum_{j=1}^N e^{x_j})^\gamma$. Here, $at = (a_1 t, \dots a_N t)$ and $bW_t = (b_1 W_t, \dots b_N W_t)$; a_j and b_j are as given in equation (3.39). The following theorem gives the partial differential equation satisfied by the function $g(x)$.

Theorem 3.3.9 The function $g(x)$ satisfies

$$\frac{1}{2}(\sum_{j=1}^N b_j \partial_{x_j})^2 g + \sum_{j=1}^N a_j \partial_{x_j} g = -(e^{x_1} + \cdots e^{x_N})^\gamma, \tag{3.57}$$

and can be characterised as the unique solution of this PDE satisfying

$$\mathbb{E}^0(g(x + aT + bW_T)) \to 0 \quad as \quad T \quad \to \infty. \tag{3.58}$$

Proof. We first note that

$$g(x + at + bW_t) = \int_t^{\infty} \mathbb{E}^0(f_\gamma(x + au + bW_u) \mid \mathcal{F}_t)du,$$

which follows from the fact that $u = t + (u - t)$ and $W_u = W_t + (W_u - W_t)$, using the basic properties of Brownian motion (independence of increments and stationarity). Hence, for a small interval $[t, t + h] = [t, t + dt]$,

$$\mathbb{E}^0(g(x + a(t + h) + bW_{t+h}) \mid \mathcal{F}_t) - \mathbb{E}^0(g(x + at + bW_t) \mid \mathcal{F}_t)$$

$$= \mathbb{E}^0(\int_{t+h}^{\infty} \mathbb{E}^0(f_\gamma(x+au+bW_u) \mid \mathcal{F}_{t+h}) \mid \mathcal{F}_t)du - \int_t^{\infty} \mathbb{E}^0(f_\gamma(x+au+bW_u) \mid \mathcal{F}_t)du$$

$$= -\int_t^{t+h} \mathbb{E}^0(f_\gamma(x + au + bW_u) \mid \mathcal{F}_t)du$$

from which it follows that

$$\mathbb{E}^0(dg(x + at + bW_t) \mid \mathcal{F}_t) = -f_\gamma(x + at + bW_t)dt.$$

Further, $g(x)$ is well-defined and is easily seen to be a C^2 (and even a C^∞) function. By Itô's lemma (see [42], [7]),

$$dg(x + at + bW_t) = \left(\sum_j a_j \partial_{x_j} g + \frac{1}{2}(\sum_j b_j \partial_{x_j})^2 g \right) dt + (\sum_j b_j \partial_{x_j} g) dW_t$$

$$= \mathcal{L}(g)dt + (\sum_j b_j \partial_{x_j} g)dW_t,$$

with the derivatives of g evaluated in $x + at + bW_t$, and where $\mathcal{L}(g)$ is the left-hand side of equation (3.57). On taking expectations,

$$\mathbb{E}^0(dg(x + at + bW_t) \mid \mathcal{F}_t) = \mathcal{L}(g)(x + at + bW_t)dt,$$

which proves the first part of the theorem.

To prove that g is the unique solution of the PDE satisfying (3.58), observe that since g is a solution, by Itô's lemma (see [42], [7]),

$$g(x + aT + bW_T) - g(x) = \int_0^T dg(x + at + bW_t)$$

$$= \int_0^T \mathcal{L}(g)(x + at + bW_t)dt + \int_0^T \sum_{j=1}^N b_j \partial_{x_j} g(x + at + bW_t)dW_t$$

$$= -\int_0^T f_\gamma(x + at + bW_t)dt + \int_0^T \sum_{j=1}^N b_j \partial_{x_j} g(x + at + bW_t)dW_t.$$

Taking expectations,

$$g(x) = \int_0^T \mathbb{E}^0(f_\gamma(x + at + bW_t))dt + \mathbb{E}^0(g(x + aT + bW_T))$$

and the result follows by letting $T \to \infty$. ■

3.3.9 An SDE for ζ_t

If we introduce

$$Y_t = \sum_{j=1}^N c_0^j \left(\widetilde{\Lambda}_t\right)^{1/\gamma}, \tag{3.59}$$

then

$$\zeta_t = \delta_t^{-\gamma} Y_t^\gamma. \tag{3.60}$$

We first derive an SDE for Y_t in terms of the optimal consumption plans c_t^j, $j = 1, \ldots, N$.

Lemma 3.3.10 Let $\varphi_t^j = c_t^j/\delta_t$ be the fraction of the output consumed by agent j. Then

$$\frac{dY_t}{Y_t} = \sum_j (-\gamma^{-1}\rho_j + \frac{1}{2}\gamma^{-1}(\gamma^{-1}-1)\alpha_j^2)\varphi_t^j dt + \gamma^{-1}(\sum_j \alpha_j \varphi_t^j) dW_t. \quad (3.61)$$

Proof. By Itô's lemma (see [42], [7]),

$$d(\widetilde{\Lambda}_t)^{1/\gamma} = d\left(e^{-(\gamma^{-1}\rho_j + \frac{1}{2}\gamma^{-1}\alpha_j^2)t + \gamma^{-1}\alpha_j W_t}\right)$$

$$= (-\gamma^{-1}\rho_j + \frac{1}{2}\gamma^{-1}(\gamma^{-1}-1)\alpha_j^2)(\widetilde{\Lambda}_t^j)^{1/\gamma} dt + \gamma^{-1}\alpha_j (\widetilde{\Lambda}_t^j)^{1/\gamma} dW_t\,.$$

If we multiply by c_0^j, divide by Y_t and observe that

$$\frac{c_0^j(\widetilde{\Lambda}_t^j)^{1/\gamma}}{Y_t} = \frac{c_0^j(\widetilde{\Lambda}_t)^{1/\gamma}}{\sum_k c_0^k(\widetilde{\Lambda}_t^k)^{1/\gamma}} = \frac{c_t^j}{\sum_k c_t^k} = \frac{c_t^j}{\delta_t} = \varphi_t^j,$$

the lemma follows after summing over j. ■

Theorem 3.3.11 We have

$$d\zeta_t = -r_t\zeta_t dt + \Sigma_t \zeta_t dW_t, \quad (3.62)$$

where

$$r_t = \sum_j \left(\rho_j + \gamma\kappa_j - \frac{1}{2}(\gamma^{-1}-1)\alpha_j^2\right)\varphi_t^j + \frac{1}{2}(\gamma^{-1}-1)\left(\sum_j \alpha_j\varphi_t^j\right)^2 - \frac{1}{2}\gamma(\gamma+1)\eta^2 + \gamma\eta\left(\sum_j \alpha_j\varphi_t^j\right), \quad (3.63)$$

with $\kappa_j = \kappa + \alpha_j\eta$ the drift according to agent j, and

$$\Sigma_t := \sum_j \alpha_j\varphi_t^j - \gamma\eta. \quad (3.64)$$

Proof. Let us momentarily write

$$A_t = \sum_j (-\gamma^{-1}\rho_j + \frac{1}{2}\gamma^{-1}(\gamma^{-1} - 1)\alpha_j^2)\varphi_t^j, \ B_t := \gamma^{-1}(\sum_j \alpha_j \varphi_t^j) \tag{3.65}$$

for the drift and the volatility in the above expression for dY_t/Y_t. Then by Itô's lemma (see [42], [7]),

$$dY_t^\gamma = \gamma Y_t^{\gamma-1} dY_t + \frac{1}{2}\gamma(\gamma - 1)Y_t^{\gamma-2}(dY_t)^2$$

$$= (\gamma A_t + \frac{1}{2}\gamma(\gamma - 1)B_t^2)Y_t^\gamma dt + \gamma B_t Y_t^\gamma dW_t,$$

and, by a similar calculation,

$$d\delta_t^{-\gamma} = -\gamma\delta_t^{-\gamma-1} d\delta_t + \frac{1}{2}(-\gamma)(-\gamma - 1)\delta_t^{-\gamma-2}(d\delta_t)^2$$

$$= d(\delta_t)^{-\gamma} = (-\gamma\kappa + \frac{1}{2}\gamma(\gamma + 1)\eta^2)\delta_t^{-\gamma} dt - \gamma\eta\delta_t^{-\gamma} dW_t.$$

Since $\zeta_t = \delta_t^{-\gamma} Y_t^\gamma$, we then find that

$$d\zeta_t = \delta_t^{-\gamma} dY_t^\gamma + Y_t^\gamma d\delta_t^{-\gamma} + dY_t^\gamma d\delta_t^{-\gamma}$$

$$= \zeta_t \left((-\gamma\kappa + \frac{1}{2}\gamma(\gamma + 1)\eta^2 + \gamma A_t - \gamma^2\eta B_t + \frac{1}{2}\gamma(\gamma - 1)B_t^2)dt + \gamma(B_t - \eta)dW_t \right),$$

and the theorem follows by substituting the above expressions for A_t and B_t and rearranging. ■

Remark 3.3.12 As the notation suggests, r_t is indeed the risk-free short rate. The price at time t of an asset paying off one unit of consumption good at $t + dt$ is

$$\frac{E_t^0(\zeta_{t+dt})}{\zeta_t} = 1 + \frac{E_t^0(d\zeta_t)}{\zeta_t},$$

where $\zeta_{t+dt} = \zeta_t + d\zeta_t$. By the definition of the short rate, this is equal to $1/(1 + r_t dt) = 1 - rdt$, so $E^0(d\zeta_t) = -r\zeta_t dt$.

The expression for the short rate is quite complicated (note that it makes a difference whether $\gamma < 1$ or $\gamma > 1$) except for when $\gamma = 1$, which is the case of log utility, when

$$r_t = \sum_j (\rho_j + \kappa_j + \gamma\eta\alpha_j)\varphi_t^j - \eta^2.$$

There is, in all cases, a close relationship between the short rate and certain weighted sums of the optimal consumption plans, with the weights being related to the beliefs through the α_j (and also the subjective discount rates ρ_j). Therefore, it is important to inquire about the dynamics of the c_t^j or equivalently, the $\varphi_t^j = c_t^j/\delta_t$.

3.3.10 Consumption dynamics

We recall that

$$\varphi_t^j = \frac{c_0^j(\widetilde{\Lambda}_t^j)^{1/\gamma}}{\sum_k c_0^k(\widetilde{\Lambda}_t^k)^{1/\gamma}}. \tag{3.66}$$

Since

$$(\widetilde{\Lambda}_t^j)^{1/\gamma} = e^{p_j t + q_j W_t}, \tag{3.67}$$

with

$$p_j = -\gamma^{-1}(\rho_j + \frac{1}{2}\alpha_j^2), \quad q_j := \gamma^{-1}\alpha_j, \tag{3.68}$$

we see that $\varphi_t^j = f_j(W_t, t)$, where

$$f_j(w,t) = \frac{c_0^j e^{p_j t + q_j w}}{\sum_k c_0^k e^{p_k t + q_k w}}. \tag{3.69}$$

We will apply Itô's lemma (see [42], [7]) to compute the derivatives of f_j. We first note that

$$\frac{\partial f_j}{\partial w} = \frac{\sum_k (q_j - q_k) c_0^j c_0^k e^{p_j t + q_j w} e^{p_k t + q_k w}}{\left(\sum_i c_0^i e^{p_i t + q_i w}\right)^2},$$

so that

$$\frac{\partial f_j}{\partial w} = \sum_k (q_j - q_k) f_j f_k.$$

Similarly,

$$\frac{\partial f_j}{\partial t} = \sum_k (p_j - p_k) f_j f_k.$$

By differentiating once more and using the formula we derived above, we have

$$\begin{aligned}
\frac{\partial^2 f_j}{\partial w^2} &= \sum_k (q_j - q_k) \frac{\partial}{\partial w}(f_j f_k) = \sum_k (q_j - q_k) \left(f_k \frac{\partial f_j}{\partial w} + f_j \frac{\partial f_k}{\partial w} \right) \\
&= \sum_{k,l} \left((q_j - q_k)(q_j - q_l) + (q_j - q_k)(q_k - q_l) \right) f_j f_k f_l \\
&= \left(\sum_{k,l} (q_j^2 - 2 q_j q_l + 2 q_k q_l - q_k^2) f_k f_l \right) f_j.
\end{aligned}$$

As a further preparation, note that $\sum_k f_k = 1$, so that

$$\partial_w f_j = \left(q_j - (\sum_k q_k f_k) \right) f_j,$$

and similarly for $\partial_t f_j$ with q_j and q_k replaced by p_j and p_k, respectively. The second derivative is

$$\partial_w^2 f_j = \left(q_j^2 - 2 q_j (\sum_l q_l f_l) + 2 (\sum_k q_k f_k)^2 - (\sum_k q_k^2 f_k) \right) f_j.$$

Insert these expressions into the Itô's lemma (see [42], [7])

$$d\varphi_t^j = (\partial_t f_j(W_t,t) + \frac{1}{2}\partial_w^2 f_j(W_t,t))dt + \partial_w f_j(W_t,t)dW_t,$$

where $\varphi_t^j = f_j(W_t,t)$. We substitute the expressions in equation (3.68) for p_j and q_j and, after doing some algebra, we arrive at the following theorem.

Theorem 3.3.13 (*Consumption dynamics*) Let

$$A_{1,t} := \sum_k \alpha_k \varphi_t^k, \quad A_{2,t} := \sum_k \alpha_k^2 \varphi_t^k, \tag{3.70}$$

and

$$\Delta_t := \sum_k \rho_k \varphi_t^k. \tag{3.71}$$

Then

$$\frac{d\varphi_t^j}{\varphi_t^j} = \Big(-\gamma^{-1}(\rho_j - \Delta_t + \frac{1}{2}(\alpha_j^2 - A_{2,t})) + \frac{1}{2}\gamma^{-2}(\alpha_j^2 - 2\alpha_j A_{1,t} + 2A_{1,t}^2 - A_{2,t})\Big)dt$$

$$+ \gamma^{-1}\Big(\alpha_j - A_{1,t}\Big)dW_t. \tag{3.72}$$

Remark 3.3.14 We make some remarks on the interpretation of these quite complicated-looking dynamics. First of all, we note that this is a coupled system of SDEs driven by a single Brownian motion. The α_j's represent (or more precisely, parametrise) the different beliefs, and $A_{1,t}$ is a weighted average of these belief parameters, with the weights φ_t^k being the fraction of the total output δ_t consumed by the different agents. Similarly, $A_{2,t}$ is the weighted average of the α_k^2, while Δ_t is the weighted average of the different discount rates ρ_k. Terms like

$$\rho_j - \Delta_t\ ,\ \alpha_j - A_{1,t}\ ,\ \alpha_j^2 - A_{2,t}\ ,$$

which occur in the coefficients are simply the difference of agent j's discount and belief parameters with their respective averages. We note in this respect that the term multiplying the $\frac{\gamma^{-2}}{2}$ in the drift can also be written as

$$(\alpha_j - A_{1,t})^2 + A_{1,t}^2 - A_{2,t},$$

where $A_{2,t} - A_{1,t} = \sum_k \alpha_k^2 \varphi_t^k - (\sum_k \alpha_k \varphi_t^k)^2$. This can be viewed as the variance of $k \to \alpha_k$, interpreted as a random variable on $\{1, \ldots, N\}$, with respect to the probability measure which assigns a probability of φ_t^k to k.

Below, we will only need the diffusion coefficient, $\gamma^{-1}(\alpha_j - A_{1,t})$. Note that the quadratic variation of $d\varphi_t^j / \varphi_t^j$,

$$v_t^j dt := \gamma^{-2}(\alpha_j - A_{1,t})^2 dt,$$

can be interpreted as the instantaneous variance of j's consumption. The individual consumptions are, in principle at least, observable and a natural question would be to what extent these would allow us to determine the individual belief parameters α_j and the common risk-aversion parameter γ. For example, knowing the instantaneous variances at some point in time, what can we deduce about the α_j and γ?

3.3.11 Stock price dynamics

We finally derive an SDE for the stock price S_t from the SDE for ζ_t. To keep the calculations transparent, we introduce some (not entirely standard) terminology.

Terminology: If X_t is an Itô process with $dX_t = A_t dt + B_t dW_t$ (with respect to P^0), we will call A_t the drift and B_t the diffusion of X_t, so drift$(X_t) = E_t^0(dX_t)$ and diffusion$(X_t)^2 = E_t^0((dX_t)^2)$. We note that if X_t and Y_t are Itô processes, then it follows from the product rule

$$d(X_tY_t) = X_tdY_t + Y_tdX_t + (dX_t)(dY_t)$$

that

$$\mathrm{diff}(X_tY_t) = X_t\,\mathrm{diff}(Y_t) + Y_t\,\mathrm{diff}(X_t), \tag{3.73}$$

and

$$\mathrm{drift}(X_tY_t) = X_t\,\mathrm{drift}(Y_t) + Y_t\,\mathrm{drift}(X_t) + \mathrm{diff}(X_t)\,\mathrm{diff}(Y_t). \tag{3.74}$$

Lemma 3.3.15 The drift of (ζ_tS_t) equals

$$\mathrm{drift}(\zeta_tS_t) = E_t^0(d(\zeta_tS_t)) = -\delta_t\zeta_tdt. \tag{3.75}$$

Proof. From the definition of S_t,

$$\zeta_tS_t = \int_t^\infty E_t^0(\delta_u\zeta_u)du,$$

and the tower property of conditional expectations, we get that

$$E_t^0(\zeta_{t+h}S_{t+h} - \zeta_tS_t) = \int_{t+h}^\infty E_t^0(E_{t+h}^0(\delta_u\zeta_u))du - \int_t^\infty E_t^0(\delta_u\zeta_u)du$$

$$= -\int_t^{t+h} E_t^0(\delta_u\zeta_u)du,$$

from which the result follows on taking $h = dt$ infinitesimal. ■

Writing $S_t = \zeta_t^{-1}\cdot\zeta_tS_t$, we see that to compute dS_t, it now suffices to compute the diffusion coefficient of ζ_tS_t, since the drift and diffusion of ζ_t^{-1} can be computed from those of ζ_t. Similarly, writing $\zeta_tS_t = (\zeta_t\delta_t)(\delta_t^{-1}S_t)$, we see that it suffices to compute the diffusion coefficient of $\delta_t^{-1}S_t$, which we will do using Theorem 3.3.3 in combination with the SDE for the φ_t^j. This will introduce the wealth w_t^j into the equations.

Lemma 3.3.16 If we put $\psi_t^j = w_t^j / S_t$, the fraction of total wealth retained by agent j, then

$$\frac{d(\delta_t^{-1} S_t)}{\delta_t^{-1} S_t} = (\cdots)dt + \left((\sum_j \alpha_j \psi_t^j) - A_{1,t} \right) dW_t. \tag{3.76}$$

Proof. By Theorem 3.3.3, $\delta_t^{-1} S_t = G(\varphi_t^1, \ldots, \varphi_t^N)$, and therefore, by Itô's lemma (see [42], [7]), the SDE for φ_t^j and by writing $\varphi_t = (\varphi_t^1, \ldots, \varphi_t^N)$,

$$d(\delta_t^{-1} S_t) = (\cdots)dt + \sum_j \frac{\partial G}{\partial c_j}(\varphi_t) d\varphi_t^j$$

$$= (\cdots)dt + \sum_j \varphi_t^j \frac{\partial G}{\partial c_j}(\varphi_t) \cdot \gamma^{-1}(\alpha_j - A_{1,t}) dW_t.$$

$$= (\cdots)dt + \sum_j (\alpha_j - A_{1,t}) \frac{w_t^j}{\delta_t} dW_t,$$

where we used equation (3.44) and equation (3.72). Writing $w_t^j = \psi_t^j S_t$ and noting that $\sum A_{1,t} \psi_t^j = A_{1,t}$ since $\sum_j \psi_t^j = 1$, the lemma follows. ∎

Remark 3.3.17 One can, of course, also compute the drift in this way, but this would introduce terms such as

$$\varphi_t^j \varphi_t^k \frac{\partial^2 G}{\partial c_j \partial c_k}(\varphi_t),$$

whose economic interpretation is not clear. The argument we outlined above circumvents this problem.

We can now carry out our program. The diffusion coefficient of $\zeta_t S_t = \delta_t \zeta_t \cdot \delta_t^{-1} S_t$ equals

$$\mathrm{diff}(\zeta_t S_t) = \mathrm{diff}(\delta_t \zeta_t) \cdot \delta_t^{-1} S_t + \delta_t \zeta_t \cdot \mathrm{diff}(\delta_t^{-1} S_t)$$

$$= (\eta + \Sigma_t)\delta_t\zeta_t \cdot \delta_t^{-1}S_t + \delta_t\zeta_t \cdot (\sum_j \alpha_j\psi_t^j - A_{1,t})\delta_t^{-1}S_t$$
$$= (\eta + \Sigma_t + \sum_j \alpha_j\psi_t^j - A_{1,t})\zeta_t S_t.$$

Now $\Sigma_t = A_{1,t} - \gamma\eta$, by equation (3.64) and equation (3.70), and combining this with lemma 3.3.15, we arrive at the following intermediary result, which may be of independent interest.

Lemma 3.3.18

$$d(\zeta_t S_t) = -\delta_t\zeta_t dt + \left((1-\gamma)\eta + \sum_j \alpha_j\psi_t^j\right)\zeta_t S_t\, dW_t. \tag{3.77}$$

It is now routine to compute the SDE for S_t.

Theorem 3.3.19 (*Stock price dynamics*) We have

$$dS_t = (\nu_t S_t - \delta_t)dt + \sigma_t S_t dWt, \tag{3.78}$$

where

$$\sigma_t = \sum_j \alpha_j \left(\frac{w_t^j}{S_t} - \frac{c_t^j}{\delta_t}\right) + \eta, \tag{3.79}$$

and

$$\nu_t = r_t - \sigma_t\Sigma_t, \tag{3.80}$$

where we recall that Σ_t is the volatility of the state-price process ζ_t and r_t, the risk-free short rate.

Remark 3.3.20 It is interesting to note that the volatility σ_t does not explicitly depend on the coefficient of relative risk aversion γ, although it does implicitly, since w_t^j and c_t^j do. Similarly, the drift, written in this form, does not explicitly contain γ, although both r_t and Σ_t do. The formulas (3.78) - (3.80) present the stock price dynamics in a form which is universal for the class of CRRA utility functions, which suggests that another derivation may be possible.

One can substitute the explicit formulas (3.63) and (3.64) for r_t and Σ_t into formula (3.80) to obtain an explicit formula for ν_t in terms of γ, κ , and η. We can also use the expressions for $A_{1,t}$, $A_{2,t}$ and Δ_t introduced in Theorem 3.3.13 and the Ψ_t defined in the proof below, but this does not seem to lead to new insights, and it may be preferable to leave it in the form (3.80).

Proof of Theorem 3.3.19. By Itô's lemma (see [42], [7]), $d\zeta_t^{-1} = -\zeta_t^{-2} d\zeta_t + \zeta_t^{-3}(d\zeta_t)^2 = \zeta_t^{-1}((r_t + \Sigma_t^2)dt - \Sigma_t dW_t)$. It then follows from the preceding lemma that $S_t = \zeta_t^{-1} \cdot \zeta_t S_t$ has diffusion coefficient

$$\left((1-\gamma)\eta + \sum_j \alpha_j \psi_t^j - \Sigma_t\right) S_t$$

$$= \left(\eta + \sum_j \alpha_j \psi_t^j - A_{1,t}\right) S_t dW_t$$

$$= \left(\eta + \sum_j \alpha_j (\psi_t^j - \varphi_t^j)\right) S_t dW_t = \sigma_t S_t dW_t,$$

and drift coefficient

$$-\delta_t + \left(r_t + \Sigma_t^2 - \left((1-\gamma)\eta + \sum_j \alpha_j \psi_t^j\right) \Sigma_t\right) S_t.$$

We then note that, writing momentarily $\Psi_t := \sum_j \alpha_j \psi_t^j$,

$$\Sigma_t^2 - ((1-\gamma)\eta + \Psi_t)\,\Sigma_t = \Sigma_t\,(\Sigma_t - (1-\gamma)\eta - \Psi_t) = \Sigma_t(A_{1,t} - \Psi_t - \eta) = -\sigma_t \Sigma_t.$$

The drift coefficient then becomes

$$-\delta_t + r_t S_t - \sigma_t \Sigma_t S_t = \nu_t S_t - \delta_t$$

after using equation (3.80). ■

Conclusion

This work provides a new structure for the pricing and hedging of contingent claims in incomplete markets by extending the Theory of No Strictly Acceptable Opportunities to general probability spaces in a static setting and to finite probability spaces in a dynamic setting. We also used the equilibrium theory to study the implications for asset prices when agents have heterogeneity of beliefs, discount factors, and coefficients of relative risk aversion.

In the first chapter, we considered a general probability space on which a countable number of valuation test measures are defined. We assumed these valuation test measures are absolutely continuous with respect to the reference measure. We obtained an equivalence between the condition of no strictly acceptable opportunities and the existence of a representative state pricing function. The proof of this theorem made use of results from functional analysis, especially the Dieudonné's Separation Theorem (see [19]). We also showed that the concept of acceptable completeness extends, rather straightforwardly, with minimal additional mathematical complexity. We concluded by giving methods for generating valuation measures from utility functions.

The second chapter provides a framework for the dynamic extension (in discrete time) of the Theory of No Strictly Acceptable Opportunities (see [12]).

Each agent is assumed to have a probability measure generated by pasting together single-period measures. We obtained an equivalence between the condition of no strictly acceptable opportunities in the multi-period case and the condition of no strictly acceptable opportunities in each of the single-period models. This result is significant because it guarantees the existence of a representative state pricing function, which can be used as a pricing measure. In terms of pricing contingent claims, a key challenge is that the existence of a representative state pricing function is not sufficient for the existence of the condition of no strictly acceptable opportunities in the multi-period case. This observation contradicts the result in Carr *et al.* [12]. Therefore, more exploratory work needs to be done to obtain an equivalence between the condition of no strictly acceptable opportunities and the existence of a representative state pricing function.

In the third chapter, we discussed the homogeneity of agents' beliefs and its implications for the nature of the equilibrium, asset price, bond price, and the short rate. We reviewed the paper by Brown and Rogers [11], which shows that agents' beliefs can be modelled as probability measures, where the likelihood processes are important inputs in obtaining the optimality condition, and in the computation of the asset price. Thereafter, we assumed that agents have a power utility function with a homogeneous coefficient of relative risk aversion, but heterogeneous discount factors. We showed that each agent's optimal consumption is a function of their initial consumption, the state price density, and their discounted belief. Further, we are able to show that the asset price is a function of a certain generating function, which depends on the optimal consumption of the agents. The wealth of each agent is expressed as a function of the derivative of the generating function, where the arguments are the proportion of the total output consumed by the agents. We also obtained expressions for the bond price and the short rate. The discussion was extended to models with heterogeneous risk aversion, where we showed that if we relax the assumptions of heterogeneity of beliefs and heterogeneity of discount factors, the Laplace transform of the function of discounted beliefs takes on a simple form.

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Appendix to Section 1.2

A.1 Dieudonné's Separation Theorem Standard versions for the existence of a separating hyperplane for two disjoint convex sets A and B of a Banach space X (or, more generally, a locally convex topological vector space) suppose that either one of these sets is open, or that one is closed and the other compact. This is called the geometric form of the Hahn-Banach Theorem. Dieudonné [19] and Klee [36] independently proved this theorem where the compactness condition is replaced by that of being locally compact and closed. This necessitates a further condition on the so-called *recession cones* of A and B.

Definition A.1.1 Let $A \subset X$ be a convex set. Then the *recession cone* is the set of all $x \in X$ such that, for all $\lambda > 0$ and all $a \in A$, $a + \lambda x \in A$.

Equivalently, it is the set of all x such that for all $a \in A$, the positive half-line starting from a with direction x is entirely contained in A. Yet another way of stating the definition is that

$$C_A = \bigcap_{a \in A} \bigcap_{\lambda > 0} \lambda(A - a):$$

indeed, $x \in \bigcap_{\lambda>0} \lambda(A - a)$ iff for all $\lambda > 0$, there exists an $a' \in A$ such that $x = \lambda(a' - a)$. Therefore, $a + \lambda^{-1}x = a' \in A$.

One can show that if A is closed, then $x \in C_A$ iff $a+\lambda x \in A$ for *some* $a \in A$ and all $\lambda > 0$; equivalently, for any $a \in A$, $C_A = \bigcap_{\lambda>0} \lambda(A-a)$ where the left-hand side is independent of $a \in A$. This is the definition given by Dieudonné (see [19]). One can easily check that C_A is a closed convex cone. The point 0 is always in the recession cone of any non-empty convex set A, and if bounded, its recession cone reduces to $C_A = \{0\}$. This is because A cannot contain vectors of arbitrarily large norm, and $||a+\lambda x|| \geq |\lambda|\,||x|| - ||a|| \to \infty$ as $\lambda \to \infty$ if $x \neq 0$.

Theorem A.1.2 (Dieudonné, [19]) *Suppose that A and B are two disjoint closed convex subsets of a Banach space X with A locally compact such that $C_A \cap C_B = \{0\}$. Then there exists a continuous linear functional $f \in X^*$ and a $c > 0$ such that for all $a \in A, b \in B$,*

$$f(a) < c + f(b).$$

Dieudonné proved this theorem for general locally convex vector spaces (see [19]). His proof uses filters and their limits, which may make it less accessible to the present-day reader. The underlying idea is quite elegant, though, and we give a simplified proof for the case of a Banach space, using converging sequences instead of filters. The next lemma can be seen as a partial converse to the observation above that $C_A = \{0\}$ if A is bounded.

Lemma A.1.3 *Suppose that $A \subset X$ is closed and convex and contains a sequence of points $a_k \in A$, $k \in \mathbb{N}^*$, whose norm $||a_k|| \to \infty$. If a is any limit point of the bounded sequence $a_k/||a_k||$, then $a \in C_A$.*

Proof. Assume, without essential loss of generality, that $0 \in A$. If a is a limit point, then we can suppose after passing to a subsequence if necessary, that $a_k/||a_k|| \to a$. Hence, if $\lambda > 0$,

$$\lambda a = \lim_{k\to\infty} \frac{\lambda}{||a_k||} a_k,$$

and if $||a_k|| > \lambda$, then $\frac{\lambda}{||a_k||}a_k$ lies on the line segment $\{ra_k : 0 \leq r \leq 1\}$ connecting 0 and a_k. Since A is convex, this segment is contained in A, showing that $\frac{\lambda}{||a_k||}a_k \in A$; λa is the limit of a sequence of points of A, which implies that $\lambda a \in A$ since A is closed. ■

Proof of Theorem A.1.2. We first prove that $B - A$ is closed. Let $c \in \overline{B - A}$, $c = \lim_{k\to\infty}(b_k - a_k)$ with $a_k \in A$ and $b_k \in B$. We have to show that $c \in B - A$.

If the sequence a_k has a subsequence a_{k_ν}, which is bounded in norm

$$||a_{k_\nu}|| \leq R,$$

say, then since $A \cap \overline{B}(0, R)$ is compact by local compactness of A, this subsequence has itself a converging subsequence. After passing to this subsequence, we may therefore suppose that $a_k \to a$ for some $a \in A$. But $b_k = c + a_k \to c + a$, and $c + a \in B$ since B is closed. Hence $c = c + a - a \in B - A$ and we are done.

Suppose that no such norm-bounded subsequence exists, which implies that $||a_k|| \to \infty$. The sequence of points $\frac{a_k}{||a_k||}$ is in $A \cap \overline{B(0,1)}$, which is compact since A is locally compact, by hypothesis. Hence we may assume, after passing to a subsequence if necessary, that $\frac{a_k}{||a_k||} \to a$ with $||a|| = 1$. By Lemma A.1.3, it follows that $a \in C_A$. We also claim that

$$\frac{b_k}{||b_k||} \to a. \tag{3.81}$$

By Lemma A.1.3 again, this implies that $a \in C_B$, so that then $a \in C_A \cap C_B \neq \{0\}$, contradicting the hypothesis.

To prove (3.81), we write

$$\frac{b_k}{||b_k||} = \frac{a_k}{||b_k||} + \frac{b_k - a_k}{||b_k||} = \frac{||a_k||}{||b_k||} \cdot \frac{a_k}{||a_k||} + \frac{b_k - a_k}{||b_k||}. \tag{3.82}$$

Since

$$||a_k|| - ||b_k - a_k|| \le ||b_k|| = ||a_k + (b_k - a_k)|| \le ||a_k|| + ||b_k - a_k||,$$

and $||b_k - a_k|| \to ||c||$, it follows that $||b_k|| \to \infty$. Further, after dividing by $||a_k||$, we have that $||b_k||/||a_k|| \to 1$. Hence (3.82) implies that $b_k/||b_k|| \to a$. This completes the proof that $B - A$ is closed.

Hyperplane separation now follows by a standard argument: first note that $A \cap B = \emptyset$ is equivalent to $0 \notin B - A$. Since $B - A$ is closed and $\{0\}$ is compact, there exists an $f \in X^*$ and a $c > 0$ such that

$$f(0) = 0 < c < f(b - a), \quad \forall a \in A, \forall b \in B,$$

or $f(a) < c + f(a) < f(b)$, from which it follows that

$$f(a) < c + \sup_{a' \in A} f(a') \le f(b), \quad \forall a \in A, \forall b \in B,$$

which proves the theorem with $\gamma := c + \sup_{a' \in A} f(a')$. ■

A.2 A note on Theorem 1.2.5. If $\mathcal{C} \subset \mathbb{R}^p$ is convex and $0 \notin \mathcal{C}$, then there exists a $v \in \mathbb{R}^p$ such that $v \cdot x \ge 0$ for all $x \in \mathcal{C}$. For a proof, see [45, 24].

To show that there always exists at least one $x \in \mathcal{C}$ such that $v \cdot x > 0$, we can use induction on the dimension p. If $p = 1$, then we can, without loss of generality, assume that $\mathcal{C} \subset (0, \infty)$ and we can take $v = 1$. Since there is a non-zero element in $\mathcal{C}$, the claim is true.

Suppose now that we have proven the theorem for convex subsets in $\mathbb{R}^{p-1}$, let $\mathcal{C} \subset \mathbb{R}^p$ be a convex subset not containing 0, and let $v \in \mathbb{R}^p$ be such that $v \cdot x \ge 0$ for all $x \in \mathcal{C}$. If there is no $x \in \mathcal{C}$ for which $v \cdot x > 0$, then clearly $v \cdot x = 0$ for all $x \in \mathcal{C}$. This means that $\mathcal{C}$ is contained in the hyperplane $v^\perp := \{x \in \mathbb{R}^p : v \cdot x = 0\}$, which is an $\mathbb{R}^{p-1}$ space. By induction, there exists a vector w in this hyperplane such that $w \cdot x \geq 0$ for all $x \in \mathcal{C}$ and strictly positive for at least one such x.