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Supercards, Sunshines and Caterpillar Graphs

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Abstract

The vertex-deleted subgraph $G - v$, obtained from the graph $G$ by deleting the vertex $v$ and all edges incident to $v$, is called a card of $G$. The deck of $G$ is the multiset of its unlabelled cards. The number of common cards $b(G, H)$ of $G$ and $H$ is the cardinality of the multiset intersection of the decks of $G$ and $H$. A supercard $G^+$ of $G$ and $H$ is a graph whose deck contains at least one card isomorphic to $G$ and at least one card isomorphic to $H$. We show how maximum sets of common cards of $G$ and $H$ correspond to certain sets of permutations of the vertices of a supercard, which we call maximum saturating sets. We apply the theory of supercards and maximum saturating sets to the case when $G$ is a sunshine graph and $H$ is a caterpillar graph. We show that, for large enough $n$, there exists some maximum saturating set that contains at least $b(G, H) - 2$ automorphisms of $G^+$, and that this subset is always isomorphic to either a cyclic or dihedral group. We prove that $b(G, H) \leq \frac{2(n+1)}{5}$ for large enough $n$, and that there exists a unique family of pairs of graphs that attain this bound. We further show that, in this case, the corresponding maximum saturating set is isomorphic to the dihedral group.

Keywords: Graph reconstruction, reconstruction numbers, vertex-deleted subgraphs, common cards, supercards, maximum saturating sets, graph automorphisms, sunshine graph, caterpillar graph

1 Introduction

In this paper all graphs are finite, undirected and contain no loops or multiple edges. Any graph-theoretic terminology and notation not explicitly explained below can be found in Bondy and Murty’s text [5]. For more information on the action of a permutation group on the vertices of a graph, we refer the reader to the book by Lauri and Scapellato [16].

Let $G$ be a graph of order $n$ and let $u, v \in V(G)$. We denote the group of all permutations of $V(G)$ by $S_{V(G)}$ and the identity permutation of $S_{V(G)}$ by $1_{V(G)}$. A transposition of $V(G)$ is a permutation that swaps two vertices in $V(G)$ and leaves the rest unchanged.

The neighbourhood of $v$ in $G$ is the set $N_G(v)$ consisting of all vertices of $G$ adjacent to $v$. The cardinality of this set is the degree of $v$ in $G$, i.e., $d_G(v) = |N_G(v)|$. A leaf of $G$ is a vertex of degree 1, and an isolated vertex of $G$ is a vertex of degree 0. We denote the number

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of vertices of degree \( k \) in \( G \) by \( d_k(G) \), so \( \sum_i d_i(G) = n \). A non-trivial component of a graph is one of order at least two.

Suppose that \( H \) is another graph and that \( \gamma \) is a bijection from \( V(G) \) to \( V(H) \). For any \( Z \subseteq V(G) \), we write the image of \( Z \) under \( \gamma \) as \( \gamma(Z) \). When \( \gamma \) is, moreover, an isomorphism from \( G \) to \( H \), i.e., \( xy \) is an edge of \( G \) if and only if \( \gamma(x)\gamma(y) \) is an edge of \( H \), we write \( \gamma(G) = H \). We write \( G \cong H \) to indicate that \( G \) and \( H \) are isomorphic. The group of all automorphisms of \( G \), i.e., isomorphisms from \( G \) to itself is denoted by \( \text{Aut}(G) \). We note that any transposition of \( V(G) \) that swaps a pair of leaves adjacent to the same vertex is in \( \text{Aut}(G) \).

Now let \( Z \subseteq V(G) \). The \( Z \)-deleted subgraph \( G - Z \) is obtained from \( G \) by deleting all the vertices of \( Z \) together with all edges of \( G \) incident to a vertex in \( Z \). So \( d_{G-Z}(v) = d_G(v) - |N_G(v) \cap Z| \), for all \( v \in V(G - Z) \). When \( Z = \{v\} \) or \( Z = \{u, v\} \), we write \( G - Z \) as \( G - v \) or \( G - u - v \), respectively. The vertex-deleted subgraph \( G - v \) is also known as a card of \( G \), and the multi-set of all \( n \) unlabelled cards of \( G \) is called the deck of \( G \), which we denote by \( \mathcal{D}(G) \). If \( Z \) is the set of leaves and isolated vertices of \( G \) then \( G - Z \) is called the skeleton of \( G \), denoted by \( \text{skeletal}(G) \).

Clearly, if \( G \cong H \) then \( \mathcal{D}(G) = \mathcal{D}(H) \). The Reconstruction Conjecture, first proposed by Kelly and Ulam in 1941 [13, 14, 22], asserts that, when \( n > 2 \), the converse also holds, i.e., \( G \) is isomorphic to \( H \) if and only if \( G \) has the same collection of \( n \) unlabelled cards as \( H \). However, despite the efforts of many graph theorists, the status of the sufficiency of the condition remains unresolved. Surveys on the reconstruction problem can be found in [3] [4] [16].

Since the conjecture remains unresolved, attention has focused on related reconstruction problems. One such area is proving that certain classes of graphs are reconstructible (i.e., that the conjecture is true when \( G \) and \( H \) belong to that class of graphs), or even recognisable (i.e., that membership of the class can be determined from the deck). Many classes of graphs have been shown to be reconstructible, including trees by Kelly [14] and also Bondy [2], disconnected graphs by Greenwell and Hemminger [11], and also Manvel [18], unicyclic graphs by Manvel [17], and maximal planar graphs by Lauri [15].

Another area of research has been to consider how many cards are required to reconstruct a graph - either its existential (ally) or universal (adversary) reconstruction number (see [6] [20]) - or even just to recognise that it is a member of a particular class. An equivalent approach to finding universal reconstruction numbers is to consider the maximum number of common cards of two graphs. A common card of \( G \) and \( H \) is any card in the multiset intersection \( \mathcal{D}(G) \cap \mathcal{D}(H) \), and the number of common cards of \( G \) and \( H \), denoted by \( b(G, H) \), is the cardinality of this multiset intersection. The Reconstruction Conjecture can then be reformulated as follows: if \( G \) and \( H \) are not isomorphic then \( b(G, H) < n \) when \( n > 2 \).

Until a few years ago, there were no known families of pairs of non-isomorphic graphs that had \( b(G, H) > \frac{n}{2} + \frac{1}{8}(3 + \sqrt{8n + 9}) \). However, Bowler, Brown and Fenner [6] showed that there are, in fact, several infinite families of pairs of non-isomorphic graphs \( G \) and \( H \) with \( b(G, H) = 2 \left\lfloor \frac{2n - 1}{3} \right\rfloor \). Moreover, they conjectured that \( b(G, H) \) is bounded above by \( \frac{2n - 1}{3} \) for large enough \( n \). Results for small graphs, i.e., for \( n \leq 11 \), have been provided by Baldwin [1], McMullen [19] and Rivshin [21].

In a subsequent paper [7], Bowler, Brown, Fenner and Myrvold showed that if \( G \) is disconnected and \( H \) is connected then \( b(G, H) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \), i.e., the connectedness of a graph can be recognised from any \( \left\lfloor \frac{n}{2} \right\rfloor + 2 \) of its cards. They also characterised all pairs of graphs that attain this bound (most of these infinite families can also be found in [6]).
A similar recognition question is to ask how many cards are required to recognize whether a graph is a tree. Since $H$ is a tree in some of the families in \cite{7}, it follows that at least $\left\lfloor \frac{n}{2} \right\rfloor + 2$ cards may be required. To fully answer the question, it is necessary to determine how many cards are required to distinguish a non-tree $G$ from a tree $H$ when it is known that $G$ is connected. It is easy to show that, in this case, $b(G, H) \leq 2$ when $G$ contains more than one cycle. Furthermore, if $G - v \cong H - t$, for some $t \in V(H)$, then $v$ must lie on every cycle of $G$. We can therefore restrict our attention to graphs where $G$ is unicyclic and the length of its cycle is reasonably large. For the maximum value of $b(G, H)$ to be attained, it has been conjectured that $G$ must be a sunshine graph (a connected graph for which skel($G$) is a cycle) and $H$ must be a caterpillar (a connected graph for which skel($H$) is a path); see \cite{6,11}. Support for restricting our attention to sunshine graphs and caterpillars is the observation that $b(G, H) \leq 6$ when either $G$ is a sunshine graph and $H$ is a non-caterpillar tree, or when $G$ is a non-sunshine unicyclic graph and $H$ is a caterpillar.

In \cite{9}, Brown gave an intricate proof that, for large $n$, the number of common cards between a sunshine graph and a caterpillar of order $n$ is at most $2 \left\lfloor \frac{n+1}{2} \right\rfloor$ and, moreover, that this bound is only attained by a unique family of pairs of graphs for which $n \equiv 4 \pmod{5}$. In this paper, we prove this result using supercards, a new approach to the study of the maximum number of common cards that we introduced in \cite{8}.

A supercard of $G$ and $H$ is a graph $G^+$, the deck of which contains at least one card isomorphic to $G$ and at least one card isomorphic to $H$. In \cite{8}, we showed the existence of certain subsets of $S_{V(G^+)}$ of cardinality $b(G, H)$, the elements of which are in one-to-one correspondence with the elements of $D(G) \cap D(H)$. We called these subsets maximum saturating sets. We further showed that, for many families of pairs of graphs with a large number of common cards, there exist maximum saturating sets containing a large number of automorphisms of $G^+$.

We use supercards to investigate the case when $G$ is a sunshine graph and $H$ is a caterpillar. We show that, when $b(G, H) \geq 2 \left\lfloor \sqrt{2n} + 1 \right\rfloor + 4$, we may construct supercards $G^+$ of $G$ and $H$, and suitable maximum saturating sets for which almost all of the elements are automorphisms of $G^+$. We further show that, in each case, the subset of elements that are automorphisms forms a group isomorphic to either a cyclic or dihedral group. We present several families of sunshine-caterpillar pairs that have such supercards and a large number of common cards. Finally, we show that $b(G, H) \leq \frac{2n+1}{3}$ when $n \geq 62$, and we exhibit the unique family of pairs of graphs that attain this bound. In this case, the maximum saturating set is a subgroup of $\text{Aut}(G^+)$ isomorphic to a dihedral group.

## 2 Supercards and Maximum Saturating Sets

We now recall the main definitions and results in the theory of supercards. Detailed explanations and proofs of the results can be found in \cite{8}.

**Lemma 2.1** Let $G$ and $H$ be graphs, and let $\gamma$ be a bijection from $V(G)$ to $V(H)$. Suppose that there is some vertex $v$ of $G$ such that $\gamma(G - v) = H - \gamma(v)$ and $\gamma(N_G(v)) = N_H(\gamma(v))$. Then $\gamma(G) = H$. \hfill $\square$

**Corollary 2.2** Let $G$ be a graph and let $v \in V(G)$. Suppose that $\gamma \in S_{V(G)}$. Then $\gamma \in \text{Aut}(G)$ if and only if \[ \gamma(G - v) = G - \gamma(v) \quad \text{and} \quad \gamma(N_G(v)) = N_G(\gamma(v)). \] \hfill $\square$
Definition 2.3 A supercard of $G$ is any graph of order $n + 1$ whose deck contains a card isomorphic to $G$.  

Definition 2.4 A common supercard of $G$ and $H$ is any graph that is a supercard of both $G$ and $H$, i.e., a graph whose deck contains some card $\hat{G}$ isomorphic to $G$ and another card $\hat{H}$ isomorphic to $H$. For brevity, we refer to such graphs as supercards of $G$ and $H$.  

Lemma 2.5 There exists a graph $G^+$ that is a supercard of $G$ and $H$ if and only if $b(G, H) \geq 1$. \hfill $\Box$  

It is easy to verify that, if $v \in V(G)$, $t \in V(H)$ and $\gamma$ is an isomorphism such that $\gamma(G - v) = H - t$, then the graph $G^*$ defined by

\[
\begin{align*}
V(G^*) &= V(G) \cup \{w\}, \\
E(G^*) &= E(G) \cup \{xw \mid x \in V(G - v) \text{ and } \gamma(x)t \in E(H)\},
\end{align*}
\]

for some $w \notin V(G) \cup V(H)$, is a supercard of $G$ and $H$. In this case, $\hat{G} = G^+ - w = G$ and $\hat{H} = G^+ - v \cong H$ (see Lemma 3.3 in [8]).

For the rest of this section we assume that $b(G, H) \geq 1$. We then let $G^+$ be a supercard of $G$ and $H$, and let $v$ and $w$ be vertices of $G^+$ such that $\hat{G} = G^+ - w \cong G$ and $\hat{H} = G^+ - v \cong H$.  

Definition 2.6 The set of active permutations of $G^+$ with respect to $v$ and $w$, denoted by $B_{vw}(G^+)$, is the subset of $S_{V(G^+)}$ defined by

\[
B_{vw}(G^+) = \{\lambda \in S_{V(G^+)} \mid \lambda((G^+ - w) - \lambda^{-1}(v)) = (G^+ - v) - \lambda(w)\} \quad (2)
\]

We note that $1_{V(G^+)} \in B_{vw}(G^+)$, and that if $\lambda \in B_{vw}(G^+)$ then $\lambda(w) \neq v$, since $G$ and $H$ are not isomorphic.  

Definition 2.7 $B_{vw}^G(G^+)$ is the subset of $B_{vw}(G^+)$ defined by

\[
B_{vw}^G(G^+) = \{\lambda \in B_{vw}(G^+) \mid \lambda(\hat{G}) = G^+ - \lambda(w)\}. \quad (3)
\]

Definition 2.8 A maximum saturating set of $B_{vw}(G^+)$ is a subset $X \subseteq B_{vw}(G^+)$ that satisfies the following three properties:

(a) $1_{V(G^+)} \in X$;  
(b) if $\lambda$ and $\pi$ are distinct permutations in $X$ then $\lambda^{-1}(v) \neq \pi^{-1}(v)$ and $\lambda(w) \neq \pi(w)$;  
(c) there is no $\sigma$ in $B_{vw}(G^+) \setminus X$ such that $X \cup \{\sigma\}$ satisfies (b).

We note that, for any pair of distinct permutations $\lambda$ and $\pi$ in $X$, (b) guarantees that $G^+ - \lambda^{-1}(v) \neq G^+ - \pi^{-1}(v)$ and $G^+ - \lambda(w) \neq G^+ - \pi(w)$, although either pair of graphs could be isomorphic.

Although condition (c) only ensures that $X$ is maximal with respect to (a) and (b), it follows from Theorem 2.11 below that all maximum saturating sets have the same cardinality. This implies that such sets are in fact of maximum cardinality with respect to (a) and (b), and justifies our terminology in Definition 2.8.
Definition 2.9 Let \( X \) be a maximum saturating set of \( B_{vw}(G^+) \). Then \( X_G = X \cap B_{vw}^G(G^+) \).
\[ \square \]

Definition 2.10 A \( G^+ \)-optimum saturating set of \( B_{vw}(G^+) \) is a maximum saturating set \( X \) of \( B_{vw}(G^+) \) for which \( |X_G| \) takes its maximum possible value. We define \( \chi(G^+) = |X_G| \) for any \( G^+ \)-optimum saturating set \( X \).
\[ \square \]

Theorem 2.11 Let \( Y \subseteq B_{vw}(G^+) \) satisfy properties (a) and (b) of Definition 2.8.
(a) If \( Y \) is not a maximum saturating set of \( B_{vw}(G^+) \) then \( |Y| < b(G, H) \).
(b) If \( |Y| < b(G, H) \) then there is a maximum saturating set \( X \) such that \( Y \subseteq X \) (so \( Y \) is not a maximum saturating set).
(c) \( Y \) is a maximum saturating set of \( B_{vw}(G^+) \) if and only if \( |Y| = b(G, H) \).
\[ \square \]

We make frequent use of the fact that every maximum saturating set of \( B_{vw}(G^+) \) has cardinality \( b(G, H) \) without explicitly quoting this theorem.

3 Sunshine graphs and caterpillars

We recall that \( \text{ske}(G) \), the skeleton of the graph \( G \), is the graph \( G - X \), where \( X \) is the set of leaves and isolated vertices of \( G \). A sunshine graph is a connected graph whose skeleton is a cycle and a caterpillar is a connected graph whose skeleton is a path. Clearly, all caterpillars are trees and all sunshine graphs are unicyclic. We shall use supercards to investigate the number of common cards between pairs of such graphs.

We denote the diameter, i.e., the length of a longest path, of a connected graph \( G \) by \( \text{diam}(G) \). If \( G \) consists of a tree \( T \) plus a collection of isolated vertices, then we define \( \text{diam}(G) = \text{diam}(T) \). A leaf at the end of any longest path in a graph is called a peripheral leaf. For any vertex \( v \) of \( G \), we denote the number of vertices in \( N_G(v) \) of degree 2 by \( \tau_G(v) \). A d-leaf of \( G \) is a leaf \( w \) (in a component of order at least three) for which \( \tau_G(w) = 0 \), i.e., the degree of its neighbour is at least 3.

We are interested in pairs of sunshine graphs \( U \) and caterpillars \( T \) with a large number of common cards relative to their order \( n \). We therefore assume, for the rest of this paper, that all the pairs \( U \) and \( T \) that we consider have \( b(U, T) \geq 5 \). By inspection, it is easy to show that, for such pairs, the unique cycle of \( U \) is of length at least 6, and that \( U \) has at least one leaf.

We use the following conventions for any sunshine graph \( S \) with skeleton \( x_0x_1 \ldots x_{c-1}x_0 \); for any integer \( k \), we interpret \( x_k \) to be the vertex \( x_i \), where \( i \) is the unique integer such that \( 0 \leq i \leq c - 1 \) and \( k \equiv i \pmod{c} \); in addition, for \( 0 \leq b < a \leq c - 1 \), we abbreviate the path \( x_ax_{a+1} \ldots x_{c-1}x_0x_1 \ldots x_b \) on \( \text{ske}(S) \) to \( x_ax_{a+1} \ldots x_b \).

We make frequent use of the following easy result concerning the cards of a sunshine graph.

Lemma 3.1 Let \( S \) be a sunshine graph with skeleton \( x_0x_1 \ldots x_{c-1}x_0 \) and let \( x_i \) be in \( V(\text{ske}(S)) \). Then \( S - x_i \) consists of a caterpillar \( Q \) of diameter \( c - \tau_S(x_i) \), together with \( d_S(x_i) - 2 \) isolated vertices. In addition:
(a) \( \text{ske}(Q) \) is \( x_{i+1}x_{i+2} \ldots x_{i-1} \) if and only if \( d_S(x_{i+1}) \geq 3 \) and \( d_S(x_{i-1}) \geq 3 \), i.e., \( \tau_S(x_i) = 0 \);
(b) \( \text{ske}(Q) \) is \( x_{i+1}x_{i+2} \ldots x_{i-2} \) if and only if \( d_S(x_{i+1}) \geq 3 \) and \( d_S(x_{i-1}) = 2 \), i.e., \( \tau_S(x_i) = 1 \);
that is adjacent to a peripheral leaf. A similar observation holds for $x_i$ and (d). Similarly, cases (b) and (d).

Moreover, $x_{i+1}$ is a peripheral leaf of $Q$ adjacent to $x_{i+2}$ when $d_S(x_{i+1}) = 2$, i.e., in cases (c) and (d). Similarly, $x_{i-1}$ is a peripheral leaf of $Q$ adjacent to $x_{i-2}$ when $d_S(x_{i-1}) = 2$, i.e., in cases (b) and (d).

Proof Since $x_{i+1}$ and $x_{i-1}$ are the only non-leaves in $N_S(x_i)$ and $x_i$ is on the unique cycle of $S$, clearly $S - x_i$ consists of a tree $Q$ together with $d_S(x_i) - 2$ isolated vertices. Moreover, since $\text{skel}(S)$ is a cycle, $\text{skel}(Q)$ is a path, so $Q$ is a caterpillar. It is easy to see that $x_{i+1}$ is a peripheral leaf of $Q$ when $d_S(x_{i+1}) = 2$; otherwise $x_{i+1}$ is one of the two vertices of $\text{skel}(Q)$ that is adjacent to a peripheral leaf. A similar observation holds for $x_{i-1}$. Cases (a) to (d) then follow immediately. Finally, it follows from (a) to (d) that $\text{diam}(Q) = c - \tau_S(x_i)$ as $\text{diam}(Q) = |V(\text{skel}(Q))| + 1$.

We also make the following easy observation about the possible cards of a caterpillar.

Lemma 3.2 Let $T$ be a caterpillar with skeleton $y_1y_2\ldots y_p$.

(a) $T - y_i$ contains precisely one non-trivial component if and only if $i \in \{1, p\}$.

(b) If $t$ is a leaf of $T$ that is not a $d$-leaf then $t$ is a peripheral leaf, and is adjacent to either $y_1$ or $y_p$. Moreover, $t$ is the only leaf that is adjacent to $y_1$ or $y_p$, respectively.

(c) There exist at most two leaves of $T$ that are not $d$-leaves.

Proof The results follow immediately by considering the structure of $T$. \hfill \Box

These two results yield the following important lemma.

Lemma 3.3 Let $U$ be a sunshine graph and $T$ be a caterpillar. Then there exists a supercard of $U$ and $T$ that is a sunshine graph.

Proof Since $U$ is a sunshine graph, clearly no card of $U$, and hence no common card of $U$ and $T$, can contain more than one non-trivial component. So, since $b(U, T) \geq 5$, it follows from Lemma 3.2 that there exists a vertex $v$ of $U$, a $d$-leaf $t$ of $T$, and an isomorphism $\gamma$ such that $\gamma(U - v) = T - t$.

Let $s$ be the unique vertex of $T$ adjacent to $t$, and let $U^*$ be the supercard of $U$ and $T$ constructed as in equation [11]. Now, since $t$ is a leaf of $T$, clearly $\gamma^{-1}(s)$ is the only vertex of $U^*$ adjacent to $w$. Moreover, since $t$ is a $d$-leaf of $T$, it is easy to see that $d_U(\gamma^{-1}(s)) \geq 2$, so $\gamma^{-1}(s)$ is on the unique cycle of $U$. Thus $U^*$ must be a sunshine graph. \hfill \Box

Let $U$ be a sunshine graph and $T$ a caterpillar. By Lemma 3.3 there exists some supercard $U^+$ of $U$ and $T$ that is a sunshine graph. So, for any such supercard $U^+$, let $w$ and $v$ be vertices of $U^+$ such that $\bar{U} = U^+ - w \cong U$ and $\bar{T} = U^+ - v \cong T$. Clearly, $w$ is a leaf of $U^+$ as $U$ is a sunshine graph. In addition, since $T$ is a tree, $v$ must be on the cycle of $U^+$ and $d_{U^+}(v) = 2$. We therefore label the skeleton of $U^+$ as $x_0x_1\ldots x_{c-1}x_0$ (so the cycle is of length $c \geq 6$), where $x_0$ is adjacent to $w$, $x_1$ is $v$, for some $\nu$, $1 \leq \nu \leq c - 1$, and $d_{U^+}(x_{\nu-1}) \geq d_{U^+}(x_{\nu+1})$. We further arbitrarily label all the leaves of $U^+$, so that each distinct leaf adjacent to $x_i$ is labelled $x_i^j$ for some unique $j$, $1 \leq j \leq d_{U^+}(x_i) - 2$, where $w$ is labelled $x_0^0$. Our supercard $U^+$ thus satisfies

$$V(U^+) = V(U) \cup \{w\} \quad \text{and} \quad E(U^+) = E(U) \cup \{x_0w\}, \quad (4)$$
assuming the above restrictions on $\text{skel}(U^+)$. 

For the rest of this section, we assume that $U^+$ is the supercard of $U$ and $T$ specified in (4). Clearly, any supercard of $U$ and $T$ is also a supercard of any pair of graphs isomorphic to $U$ and $T$, respectively. So, for ease of notation, we shall write $U = U^+ - w$ instead of $U = U^+ - w$, and $T = U^+ - x_\nu$, instead of $T = U^+ - x_\nu$. We note that $\text{skel}(U) = \text{skel}(U^+)$. Applying Lemma 3.1 to $U^+$ and $x_\nu$ yields the following result.

**Lemma 3.4** We have the following possibilities for the skeleton of $T$.

(a) If $\tau_{U^+}(x_\nu) = 0$ then $\text{skel}(T)$ is $x_{\nu+1}x_{\nu+2}\ldots x_{\nu-1}$.
(b) If $\tau_{U^+}(x_\nu) = 1$ then $\text{skel}(T)$ is $x_{\nu+2}x_{\nu+3}\ldots x_{\nu-1}$.
(c) If $\tau_{U^+}(x_\nu) = 2$ then $\text{skel}(T)$ is $x_{\nu+2}x_{\nu+3}\ldots x_{\nu-2}$.

It follows that $x_{\nu+1}$ is a peripheral leaf of $T$ adjacent to $x_{\nu+2}$ when $\tau_{U^+}(x_\nu) \geq 1$, and $x_{\nu-1}$ is a peripheral leaf of $T$ adjacent to $x_{\nu-2}$ when $\tau_{U^+}(x_\nu) = 2$.

**Proof** This follows immediately by Lemma 3.1 with $S = U^+$ and $i = \nu$, noting that case (b) of that lemma cannot occur as $d_{U^+}(x_{\nu-1}) \geq d_{U^+}(x_{\nu+1})$. \hfill \Box

We recall that $B_{vw}(U^+)$ is the set of active permutations of $U^+$ with respect to $v$ and $w$, i.e.,

$$B_{vw}(U^+) = B_{x_{\nu}x_{\nu}^{-1}}(U^+) = \{ \lambda \in S_{V(U^+)} \mid \lambda(U - \lambda^{-1}(x_\nu)) = T - \lambda(w) \}. \quad (5)$$

**Lemma 3.5** Let $\lambda \in B_{vw}(U^+)$. Then exactly one of the following holds:

(a) $\lambda(w)$ is a d-leaf of $T$, in which case $\text{skel}(T - \lambda(w)) = \text{skel}(T)$ and $\text{diam}(T - \lambda(w)) = \text{diam}(T)$;
(b) $\lambda(w)$ is a peripheral leaf of $T$ that is not a d-leaf, in which case $\text{diam}(T - \lambda(w))) = \text{diam}(T) - 1$;
(c) $\lambda(w)$ is adjacent to a peripheral leaf of $T$, in which case $\text{diam}(T - \lambda(w)) \leq \text{diam}(T) - 1$.

**Proof** By Lemmas 3.1 and 3.2(a), $\lambda(w)$ is either a leaf or adjacent to a peripheral leaf. Cases (a), (b) and (c) then follow easily by considering the structure of $T$. \hfill \Box

**Lemma 3.6** Let $\lambda \in B_{vw}(U^+)$. Then $\lambda^{-1}(x_\nu) \in V(\text{skel}(U))$, i.e., $\lambda^{-1}(x_\nu)$ is $x_\mu$ for some $\mu$, $0 \leq \mu \leq c - 1$, and $d_U(x_\mu) = d_T(\lambda(w)) + 1$. In addition, $U - x_\mu$, and therefore also $T - \lambda(w)$, consists of a caterpillar of diameter $c - \tau_T(x_\mu)$, together with $d_U(x_\mu) - 2$, equivalently $d_T(\lambda(w)) - 1$, isolated vertices.

**Proof** Since $T$ is a tree, clearly $\lambda^{-1}(x_\nu) \in V(\text{skel}(U))$, so $\lambda^{-1}(x_\nu)$ is $x_\mu$, for some $\mu$. In addition, $d_U(x_\mu) = d_T(\lambda(w)) + 1$ as $|E(U)| = |E(T)| + 1$. Finally, by Lemma 3.1 $U - x_\mu$ consists of a caterpillar of diameter $c - \tau_T(x_\mu)$, together with $d_U(x_\mu) - 2$ isolated vertices. \hfill \Box

From Lemma 3.6 and Theorem 2.11 it immediately follows that $b(U, T) \leq c$.

For brevity, we frequently use Lemma 3.6 in the rest of this paper without explicit reference. Moreover, given any $\lambda \in B_{vw}(U^+)$, unless otherwise stated, we let $x_\mu = \lambda^{-1}(x_\nu)$. So, by Definition 2.6

$$\lambda(U - x_\mu) = \lambda(U^+ - w - x_\mu) = (U^+ - x_\nu) - \lambda(w) = T - \lambda(w). \quad (6)$$
respectively. On applying Lemma 3.1 with $S = U$ and $i = \mu$, we have the following result.

**Corollary 3.7** Let $\text{skel}(U - x_\mu)$ be as in (7). Then $x_a \in \{x_{\mu+1}, x_{\mu+2}\}$ and $x_b \in \{x_{\mu-1}, x_{\mu-2}\}$. □

Since $c \geq 6$, it immediately follows that $a \neq b$.

**Lemma 3.8** Let $\lambda \in B_{vw}(U^+)$. Then $b - a \equiv s - r \pmod{c}$. Moreover, either

(a) $\lambda(x_i) = x_{(r-a)+i}$ for all $x_i \in V(\text{skel}(U - x_\mu))$, or
(b) $\lambda(x_i) = x_{(s+a)-i}$ for all $x_i \in V(\text{skel}(U - x_\mu))$.

We note that $\lambda(x_a) = x_s$ in (a) and $\lambda(x_b) = x_r$ in (b).

**Proof** $\lambda$ maps the skeleton of $U - x_\mu$ onto the skeleton of $T - \lambda(w)$. So $b - a \equiv s - r \pmod{c}$, and either $\lambda(x_a) = x_r$ and $\lambda(x_b) = x_s$, or $\lambda(x_a) = x_s$ and $\lambda(x_b) = x_r$. It is then easy to see that either (a) or (b) must hold. □

**Lemma 3.9** Let $\lambda \in B_{vw}(U^+)$. Then

$$c - \tau_{U^+}(x_\nu) = \text{diam}(T) \geq \text{diam}(T - \lambda(w)) = \text{diam}(U - x_\mu) = c - \tau_U(x_\mu) \geq c - 2. \quad (8)$$

**Proof** This follows easily by applying Lemma 3.1 to $U^+$ and $x_\nu$, and then $U$ and $x_\mu$. □

**Lemma 3.10** Let $\lambda \in B_{vw}(U^+)$ be such that $\lambda(w)$ is a $d$-leaf of $T$. Suppose that $\lambda(x_{\mu+2}) \notin \{x_{\nu-2}, x_{\nu+2}\}$. Then $\tau_U(x_\mu) = \tau_{U^+}(x_\nu) = 1$. Moreover, $\text{skel}(U - x_\mu)$ is either

(a) $x_{\mu+1}x_{\mu+2} \ldots x_{\mu-2}$, in which case $\lambda(x_i) = x_{(\nu-\mu+1)+i}$ for all $x_i \in V(\text{skel}(U - x_\mu))$, or
(b) $x_{\mu-2}x_{\mu-3} \ldots x_{\mu-1}$, in which case $\lambda(x_i) = x_{(\nu+\mu+1)-i}$ for all $x_i \in V(\text{skel}(U - x_\mu))$.

**Proof** $\text{skel}(T - \lambda(w)) = \text{skel}(T)$ by Lemma 3.3(a). So it follows from Corollary 3.4(a) that the possible skeletons of $T - \lambda(w)$ are determined by $\tau_{U^+}(x_\nu)$. In addition, $\tau_U(x_\mu) = \tau_{U^+}(x_\nu)$ by (8). It then follows from Lemma 3.1 that the possible skeletons of $U - x_\mu$ are also determined by $\tau_{U^+}(x_\nu)$. Using Lemma 3.8, it is now straightforward to determine all the possibilities for $\lambda(x_{\mu+2})$, for each of the three values of $\tau_{U^+}(x_\nu)$.

It is easy to see that $\lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}$ when $\tau_{U^+}(x_\nu) \neq 1$. Since this excluded by assumption, it follows that $\tau_U(x_\nu) = \tau_{U^+}(x_\nu) = 1$. In this case, $x_r$ is $x_{\nu+2}$ and $x_s$ is $x_{\nu-1}$ by Corollary 3.4(b). So, since $\lambda(x_{\mu+2}) \notin \{x_{\nu-2}, x_{\nu+2}\}$, it is straightforward to show that either (i) $x_a$ is $x_{\mu+1}$ and $x_b$ is $x_{\mu-2}$ and Lemma 3.8(a) holds, or (ii) $x_a$ is $x_{\mu+2}$, $x_b$ is $x_{\mu-1}$ and Lemma 3.8(b) holds. In case (i), we have $r - a \equiv \nu - \mu + 1 \pmod{c}$, and in case (ii), we have $s + a \equiv \nu + \mu + 1 \pmod{c}$. Cases (a) and (b) of the lemma then immediately follow from Lemma 3.8. □

**Definition 3.11** We define $\tilde{B}_{vw}(U^+)$ to be the subset of $B_{vw}(U^+)$ containing those permutations $\lambda$ such that $\lambda(w)$ is a leaf of $U^+$ and a $d$-leaf of $T$. We further define $\tilde{X} = X \cap \tilde{B}_{vw}(U^+)$ for any maximum saturating set $X$ of $B_{vw}(U^+)$. □
Lemma 3.12 \ Let $\lambda \in B_{uv}(U^+) \setminus \tilde{B}_{uv}(U^+)$. \\
(a) If $\tau_{U^+}(x_\nu) = 0$ then $\lambda(w)$ is not a $d$-leaf of $T$. Moreover, \\
(i) if $\lambda(w)$ is a leaf of $T$ then either $\lambda(w) = x_{\nu+1}^1$ and $d_{U^+}(x_{\nu+1}) = 3$, or $\lambda(w) = x_{\nu-1}^1$ and $d_{U^+}(x_{\nu-1}) = 3$; \\
(ii) if $\lambda(w)$ is a cut-vertex of $T$ then $\lambda(w) \in \{x_{\nu+1}, x_{\nu-1}\}$. \\
(b) If $\tau_{U^+}(x_\nu) = 1$ then \\
(i) if $\lambda(w)$ is a $d$-leaf of $T$ then $\lambda(w) = x_{\nu+1}$ and $d_{U^+}(x_{\nu+1}) \geq 3$; \\
(ii) if $\lambda(w)$ is a leaf but not a $d$-leaf of $T$ then either $\lambda(w) = x_{\nu+1}$ and $d_{U^+}(x_{\nu+1}) = 2$, or $\lambda(w) = x_{\nu-1}^1$ and $d_{U^+}(x_{\nu-1}) = 3$; \\
(iii) if $\lambda(w)$ is a cut-vertex of $T$ then $\lambda(w) \in \{x_{\nu+1}, x_{\nu-1}\}$. \\
(c) If $\tau_{U^+}(x_\nu) = 2$ then $\lambda(w)$ is a $d$-leaf of $T$ and $\lambda(w) \in \{x_{\nu+1}, x_{\nu-1}\}$. \\

Proof \ Since $\lambda \not\in \tilde{B}_{uv}(U^+)$, either $\lambda(w)$ is a $d$-leaf of $T$ that is not a leaf of $U^+$, or it is not a $d$-leaf of $T$, in which case (b) or (c) of Lemma 3.5 must hold. \\
(a) Suppose that $\tau_{U^+}(x_\nu) = 0$. Then $\text{skel}(T)$ is given by Corollary 3.4(a), so every leaf of $T$ is a leaf of $U^+$. It immediately follows that $\lambda(w)$ cannot be a $d$-leaf of $T$. Now, if case (b) of Lemma 3.5 holds then it is easy to see that either $\lambda(w) = x_{\nu+1}^1$ and $d_{U^+}(x_{\nu+1}) = 3$, or $\lambda(w) = x_{\nu-1}^1$ and $d_{U^+}(x_{\nu-1}) = 3$. On the other hand, if case (c) holds then $\lambda(w)$ is either $x_{\nu+1}$ or $x_{\nu-1}$. \\
(b) Suppose that $\tau_{U^+}(x_\nu) = 1$. Then $\text{skel}(T)$ is given by Corollary 3.4(b), so the only possible $d$-leaf of $T$ that is not a leaf of $U^+$ is $x_{\nu+1}$; in this case, clearly $d_{U^+}(x_{\nu+1}) \geq 3$. Now, if case (b) of Lemma 3.5 holds then it is easy to see that either $\lambda(w) = x_{\nu+1}$ and $d_{U^+}(x_{\nu+1}) = 2$, or $\lambda(w) = x_{\nu-1}^1$ and $d_{U^+}(x_{\nu-1}) = 3$. On the other hand, if case (c) holds then $\lambda(w)$ is either $x_{\nu+1}$ or $x_{\nu-1}$. \\
(c) Suppose that $\tau_{U^+}(x_\nu) = 2$. Then $\text{skel}(T)$ is given by Corollary 3.4(c), so the only possible $d$-leaves of $T$ that are not leaves of $U^+$ are $x_{\nu+1}$ and $x_{\nu-1}$. Since equality holds throughout, clearly $\text{diam}(T - \lambda(w)) = \text{diam}(T)$ and, therefore, neither case (b) nor case (c) of Lemma 3.5 holds.

Corollary 3.13 \ Let $X$ be a maximum saturating set of $B_{uv}(U^+)$. \\
(a) If $\tau_{U^+}(x_\nu) = 0$ then $|X \setminus \tilde{X}| \leq 4$. \\
(b) If $\tau_{U^+}(x_\nu) = 1$ then $|X \setminus \tilde{X}| \leq 4$. \\
(c) If $\tau_{U^+}(x_\nu) = 2$ then $|X \setminus \tilde{X}| \leq 2$. \\

Proof \ This follows immediately from Lemma 3.12 and part (b) of Definition 2.8. \\

We recall from Definition 2.10 that a $U^+$-optimum saturating set $X$ of $B_{uv}(U^+)$ is a maximum saturating set of $B_{uv}(U^+)$ such that $|X_U| = \chi(U^+)$. \\

Corollary 3.14 \ If there exists a $U^+$-optimum saturating set $X$ of $B_{uv}(U^+)$ such that $\tilde{X} \subseteq X_U$ then $b(U, T) \leq \chi(U^+) - 4$. \\

We now consider the permutations in $B_{uv}^U(U^+)$, i.e., those permutations $\lambda$ in $B_{uv}(U^+)$ such that $\lambda(U) = U^+ - \lambda(w)$. \\

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Lemma 3.15 Let $\lambda \in B_{vw}^U(U^+).$ Then $\lambda(w)$ is a leaf of $U^+$ adjacent to a vertex of degree $d_{U^+}(x_0), d_U(x_0) = 2,$ and either

(a) $\lambda(x_i) = x_{(\nu - \mu)+i}$ for all $x_i$ (a rotation), or
(b) $\lambda(x_i) = x_{(\nu + \mu)-i}$ for all $x_i$ (a reflection).

Proof Since $\lambda(U) = U^+ - \lambda(w),$ clearly $\lambda(w)$ must be a leaf of $U^+$ adjacent to a vertex of degree $d_{U^+}(x_0).$ Thus $d_U(x_0) = 2$ by Lemma 3.6. Now $\lambda$ must preserve the cycle structure of $U.$ So, since $\lambda(x_\mu) = x_\nu,$ it follows that either $\lambda(x_{\mu+i}) = x_{\nu+i}$ and $\lambda(x_{\mu-i}) = x_{\nu-i}$ for all $i,$ or $\lambda(x_{\mu+i}) = x_{\nu-i}$ and $\lambda(x_{\mu-i}) = x_{\nu+i}$ for all $i.$ Cases (a) and (b) then follow immediately. \qed

The following lemma gives a methodology for replacing permutations in $B_{vw}(U^+)$ by “equivalent” permutations in $B_{vw}^U(U^+).$

Lemma 3.16 Let $\lambda \in B_{vw}(U^+).$ Suppose that $\lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}.$ Then there exists $\tilde{\lambda} \in B_{vw}^U(U^+)$ such that $\tilde{\lambda}^{-1}(x_\mu) = \lambda^{-1}(x_\mu) = x_\mu$ and $\tilde{\lambda}(w) = \lambda(w).$

Proof By Corollary 3.7, $x_a \in \{x_{\mu+1}, x_{\mu+2}\}$ and $x_b \in \{x_{\mu-1}, x_{\mu+1}\};$ so $x_{\mu+2} \in \{x_a, x_{a+1}\}.$ Since $\lambda(w)$ is a $d$-leaf of $T,$ it follows from Lemma 3.5(a) that $\text{skel}(T - \lambda(w)) = \text{skel}(T).$ So $x_r \in \{x_{\mu+1}, x_{\mu+2}\}$ and $x_s \in \{x_{\mu-1}, x_{\mu-2}\}$ by Corollary 3.3. We now assume that $\lambda(x_{\mu+2}) = x_{\nu+2}$ and prove the result in this case. The case when $\lambda(x_{\mu+2}) = x_{\nu-2}$ can be proved in a similar manner.

Suppose that Lemma 3.8(b) holds. Then $\{\lambda(x_a), \lambda(x_{a+1})\} \subseteq \{x_s, x_{s-1}\} \subseteq \{x_{\nu-1}, x_{\nu-2}, x_{\nu-3}\}.$ Since $\lambda(x_{\mu+2}) = x_{\nu+2}$ and $x_{\mu+2} \in \{x_a, x_{a+1}\},$ this implies that $x_{\nu+2} \in \{x_{\nu-1}, x_{\nu-2}, x_{\nu-3}\}.$ This is impossible as $c \geq 6.$ Therefore Lemma 3.8(a) must hold. Hence $r = a \equiv s = b \equiv \nu - \mu$ (mod $c$), and $\lambda(x_i) = x_{(\nu-\mu)+i}$ for all $x_i$ in $V(\text{skel}(U - x_\mu)).$

Let $\theta$ be the transposition of $V(T - \lambda(w))$ that swaps $\lambda(x_{\mu+1})$ and $x_{\nu+1}.$ We show that $\theta \in \text{Aut}(T - \lambda(w)).$ If $\lambda(x_{\mu+1}) = x_{\nu+1}$ then $\theta$ is $1_{V(T - \lambda(w))},$ so there is nothing to prove. Suppose therefore that $\lambda(x_{\mu+1}) \neq x_{\nu+1}.$ It is then easy to show that $x_{\mu+1} \notin V(\text{skel}(U - x_\mu)).$ Thus $x_a$ is $x_{\mu+2}$ and $x_{\mu+1}$ is a leaf of $U - x_a$ adjacent to $x_a.$ Hence, correspondingly, $\lambda(x_{\mu+1})$ is a leaf of $T - \lambda(w)$ adjacent to $x_r.$ Furthermore, $x_r$ is $x_{\nu+2},$ so $x_{\nu+1}$ is also a leaf of $T - \lambda(w)$ adjacent to $x_r.$ Thus $\theta$ swaps a pair of leaves adjacent to $x_{\nu+2},$ so $\theta \in \text{Aut}(T - \lambda(w))$ in this case also.

By considering the vertices $x_b$ and $x_s,$ it is easy to show that the transposition $\theta'$ of $V(T - \lambda(w))$ that swaps $\lambda(x_{\mu-1})$ and $x_{\nu-1}$ is also in $\text{Aut}(T - \lambda(w)).$ Let us define $\tilde{\lambda} \in S_{V(U^+)}$ by $\tilde{\lambda}(x_\mu) = x_\nu, \tilde{\lambda}(\nu) = \lambda(\nu)$ and $\tilde{\lambda}(\nu) = \theta' \theta(\nu)$ for all other vertices $u$ of $U^+.$ Then

$$\tilde{\lambda}(U - x_\mu) = \theta' \theta(\lambda(U - x_\mu)) = \theta' \theta(T - \lambda(w)) = T - \lambda(w),$$

so $\tilde{\lambda} \in B_{vw}(U^+).$ Moreover, since $\tilde{\lambda}(x_{\mu+1}) = x_{\nu+1}$ and $\tilde{\lambda}(x_{\mu-1}) = x_{\nu-1},$ it follows that $\tilde{\lambda}(\text{skel}(U - x_\mu)) = \{x_{\nu-1}, x_{\nu+1}\}$ as $d_U(x_\mu) = 2.$ Hence $\tilde{\lambda}(U) = U^+ - \lambda(w)$ by Lemma 2.1, i.e., $\tilde{\lambda} \in B_{vw}^U(U^+).$ \qed

Lemma 3.17 Let $X$ be a maximum saturating set of $B_{vw}(U^+)$ and let $L$ be the subset of $X$ defined by $L = \{\lambda \in \tilde{X} | \lambda(x_{\mu+2}) \in \{x_{\nu-2}, x_{\nu+2}\}\}.$ Then

(a) $|L| \leq \chi(U^+);$
(b) if $X$ is a $U^+$-optimum saturating set of $B_{vw}(U^+)$ then $X_U \cap \tilde{X} = L.$
Proof For each $\lambda \in L$, we let $\tilde{\lambda}$ be the permutation in $B_{vw}^U(U^+)$ as defined in Lemma 3.16. We then define $\hat{L}$ to be the set of all such $\tilde{\lambda}$. Since $X$ is a maximum saturating set of $B_{vw}(U^+)$, clearly $X \setminus L$ and $\hat{L}$ are disjoint, and the set $\hat{X}$ defined by $\hat{X} = (X \setminus L) \cup \hat{L}$ is also a maximum saturating set. Moreover $\hat{X} = (X_U \setminus L) \cup \hat{L}$. Therefore $|L| = |\hat{L}| \leq |\hat{X}| \leq \chi(U^+)$. 

Now suppose that $X$ is $U^+$-optimum. If $\lambda \in X_U \cap \hat{X}$ then $\lambda \in L$ by Lemma 3.15. Thus $X_U \cap \hat{X} \subseteq L$. So suppose that $L \not\subseteq X_U$. Then $|\hat{X}| = |X_U \setminus L| + |\hat{L}| > |X_U|$. This is impossible since $X$ is $U^+$-optimum. So $L \subseteq X_U$, and therefore $X_U \cap \hat{X} = L$. \hfill $\square$

Corollary 3.18 Let $X$ be a $U^+$-optimum saturating set of $B_{vw}(U^+)$. If there exists $\lambda \in \hat{X} \setminus X_U$ then $\tau_{U^+}(x_{\nu}) = 1$ and $\{\lambda(x_{\mu+2}), \lambda(x_{\mu-2})\} = \{x_{\nu+3}, x_{\nu-1}\}$.

Proof This follows easily from Lemma 3.17(b) and Lemma 3.10. \hfill $\square$

It follows from this result that $\hat{X} \subseteq X_U$ when $\tau_{U^+}(x_{\nu}) \neq 1$.

If there exists a $U^+$-optimum saturating set $X$ of $B_{vw}(U^+)$ such that $\hat{X} \subseteq X_U$ then $b(U, T) \leq \chi(U^+) + 4$ by Corollary 3.14. When there is no such set, we must construct another supercard of $U$ and $T$.

Let $\sigma \in B_{vw}(U^+)$, and let $x_\xi = \sigma^{-1}(x_\nu)$, so $\sigma(U - x_\xi) = T - \sigma(w)$. Let $u$ be the unique vertex of $U$ adjacent to $\sigma(w)$. Since $\sigma(w)$ is a $d$-leaf of $T$, clearly $d_{T - \sigma(w)}(u) = 2$, and thus $d_{\sigma^{-1}(u)}(x_\eta)$ for some $\eta$, $0 \leq \eta \leq c - 1$. We note that $x_\eta$ cannot be $x_\xi$.

We now define a new sunshine graph $U^+_{\sigma}$ constructed from $U^+$. First we delete from $U^+$ the edge $x_0w$ and add an additional edge $x_\eta w$. We then relabel the skeleton of $U^+$ as $z_0z_1 \ldots z_{c-1}z_0$, where $x_\eta$ is relabelled as $z_0$, $x_\xi$ as $z_\xi$, and the other vertices on the cycle in the natural way, reversing the labelling around the cycle if necessary in order to ensure that $d_{U^+_{\sigma}}(z_{\xi-1}) \geq d_{U^+_{\sigma}}(z_{\xi+1})$. Finally, as before, we relabel all the leaves of $U^+$ so that each distinct leaf adjacent to $z_i$ is labelled $z_i^j$ for some unique $j$, $1 \leq j \leq d_{U^+_{\sigma}}(z_i) - 2$, where $w$ is labelled $z_0^1$. We note this labelling is analogous to the original labelling of $U^+$.

Since $|V(U^+)| = |V(U_{\sigma}^+)|$, we may define a bijection $\theta$ from $V(U^+)$ to $V(U_{\sigma}^+)$ that encapsulates the relabelling described above, i.e., $\theta(x_i) = z_{\eta-i}$ if the order of the labels around the cycle did not need reversing, and $\theta(x_i) = z_{\eta+i}$ if it did. We also specify that $\theta(w) = w$, and that $\theta$ maps the remaining leaves of $U^+$ so that those adjacent to $x_i$ map to leaves adjacent to $\theta(x_i)$ for each $x_i$. We note that $\theta(x_\eta) = z_0$ and $\theta(x_\xi) = z_\xi$.

Clearly, $d_{U_{\sigma}^+}(z_\xi) = 2$, as $d_{U^+}(x_\xi) = 2$ and $x_\eta$ is not $x_\xi$. Let $U_\sigma = U_{\sigma}^+ - w$ and $T_\sigma = U_{\sigma}^+ - z_\xi$. Then $U_{\sigma}^+$ is a supercard of $U_\sigma$ and $T_\sigma$ that satisfies $V(U_{\sigma}^+) = V(U_\sigma) \cup \{w\}$ and $E(U_{\sigma}^+) = E(U_\sigma) \cup \{z_0w\}$. (9)

For ease of notation, we write $B_{vw}(U_{\sigma}^+)$ for the set of active permutations of $U_{\sigma}^+$ with respect to $z_\xi$ and $w$, i.e.,

$$B_{vw}(U_{\sigma}^+) = \{\pi \in S_{V(U_{\sigma}^+)} \mid \pi(U_\sigma - \lambda^{-1}(z_\xi)) = T_\sigma - \pi(w)\}.$$ 

Clearly, $U_{\sigma}^+$ is a sunshine graph with skeleton $z_0z_1 \ldots z_{c-1}z_0$, labelled analogously to the labelling of $U^+$, and $U_\sigma$ is a sunshine graph. Furthermore, since $d_{U_{\sigma}^+}(z_\xi) = 2$, it follows from Lemma 3.17 that $T_\sigma$ is a caterpillar. Hence $U_{\sigma}^+$ is a supercard of the sunshine graph $U_\sigma$ and the
caterpillar \( T_\sigma \). We may therefore use results corresponding to Lemma 3.4 to Corollary 3.18 by substituting \( U^+_\sigma \), \( U_\sigma \), \( T_\sigma \) and \( B_{vw}(U^+_\sigma) \) for \( U^+ \), \( U \), \( T \) and \( B_{vw}(U^+) \), respectively.

We now show that \( U^+_\sigma \) is also a supercard of \( U \) and \( T \), and thence relate the maximum saturating sets of \( B_{vw}(U^+) \) and \( B_{vw}(U^+_\sigma) \). In each of the following four lemmas, we define \( U^+_\sigma \) to be the supercard in equation (2) for the given permutation \( \sigma \), satisfying \( \sigma(U - x_\xi) = T - \sigma(w) \), where \( x_\xi = \sigma^{-1}(x_\nu) \). As above, \( \theta \) denotes the corresponding map from \( V(U^+) \) to \( V(U^+_\sigma) \).

**Lemma 3.19** Let \( \sigma \in \tilde{B}_{vw}(U^+_\sigma) \). Then \( \theta^{-1}(U_\sigma) = U \), \( \sigma\theta^{-1}(T_\sigma) = T \), so \( U^+_\sigma \) is a supercard of \( U \) and \( T \).

**Proof** Since \( \theta(w) = w \), the restriction of \( \theta \) to \( U \) is clearly a relabelling of the vertices of \( U \) that preserves neighbourhoods. So \( \theta(U) = U_\sigma \). It now follows that

\[
\sigma\theta^{-1}(U^+_\sigma - w - z_\zeta) = \sigma(U - \theta^{-1}(z_\zeta)) = \sigma(U - x_\xi) = T - \sigma(w).
\]

So, since \( \sigma\theta^{-1}(N(U^+_\sigma - z_\zeta)(w)) = \{\sigma\theta^{-1}(z_0)\} = \{\sigma(x_\eta)\} = N_T(\sigma(w)) \), it follows from Lemma 2.1 that \( \sigma\theta^{-1}(U^+_\sigma - z_\zeta) = T \).

\( \square \)

For any \( \lambda \in B_{vw}(U^+) \), we define \( \lambda_\sigma \in S_{V(U^+_\sigma)} \) by \( \lambda_\sigma = \theta\sigma^{-1}\lambda\theta^{-1} \).

**Lemma 3.20** Let \( \sigma \in \tilde{B}_{vw}(U^+_\sigma) \) and let \( \lambda \in B_{vw}(U^+) \). Then \( \lambda_\sigma^{-1}(z_\zeta) = \theta(x_\mu) \) and \( \lambda_\sigma \in B_{vw}(U^+_\sigma) \).

**Proof** \( \lambda_\sigma^{-1}(z_\zeta) = \theta^{-1}\sigma\theta^{-1}(z_\zeta) = \theta(x_\mu) \). In addition, since \( \theta^{-1}(U_\sigma) = U \), \( \sigma\theta^{-1}(T_\sigma) = T \) and \( \lambda(U - x_\mu) = T - \lambda(w) \), we have

\[
\lambda_\sigma(U_\sigma - \lambda_\sigma^{-1}(z_\zeta)) = \lambda_\sigma(U_\sigma - \theta(x_\mu)) = \theta\sigma^{-1}\lambda(U - x_\mu) = \theta\sigma^{-1}(T - \lambda(w)) = T_\sigma - \theta\sigma^{-1}\lambda\theta^{-1}(w),
\]

as \( \theta(w) = w \). Hence \( \lambda_\sigma(U_\sigma - \lambda_\sigma^{-1}(z_\zeta)) = T_\sigma - \lambda_\sigma(w) \), so \( \lambda_\sigma \in B_{vw}(U^+_\sigma) \).

\( \square \)

**Lemma 3.21** Let \( X \) be a maximum saturating set of \( B_{vw}(U^+) \) and suppose that \( \sigma \in \tilde{X} \). Then the set \( X_\sigma \) defined by \( X_\sigma = \{\lambda_\sigma \mid \lambda \in X\} \) is a maximum saturating set of \( B_{vw}(U^+_\sigma) \).

**Proof** By Lemma 3.20, \( X_\sigma \subseteq B_{vw}(U^+_\sigma) \). Moreover, since \( X \) is a maximum saturating set of \( B_{vw}(U^+) \) that contains \( \sigma \), it is straightforward to show that \( X_\sigma \) satisfies conditions (a) and (b) of Definition 2.8. So, since \( |X_\sigma| = |X| \), it follows from Theorem 2.11(c) that \( X_\sigma \) is a maximum saturating set of \( B_{vw}(U^+_\sigma) \).

\( \square \)

For the final lemma in this section, we make use of the fact that if \( \theta(x_i) = z_j \) then \( \theta(x_{i+k}) \in \{z_{j-k}, z_{j+k}\} \), and \( \theta^{-1}(z_{j+k}) \in \{x_{i-k}, x_{i+k}\} \) for all \( k \).

**Lemma 3.22** Let \( X \) be a \( U^+ \)-optimum saturating set of \( B_{vw}(U^+) \). Suppose there exists some \( \sigma \in \tilde{X} \setminus X_U \). Then \( |\tilde{X}| \leq 2\max(\chi(U^+), \chi(U^+_\sigma)) + 1 \).

**Proof** Let \( X_\sigma = \{\pi_\sigma \mid \pi \in X\} \), and let \( P \) be the subset of \( X_\sigma \) defined by \( P = \{\pi_\sigma \mid \pi \in \tilde{X} \setminus X_U\} \). We show that \( |P| \leq \chi(U^+_\sigma) + 1 \). As \( |P| = |\tilde{X} \setminus X_U| \), it will then follow that

\[
|\tilde{X}| \leq \chi(U^+) + \chi(U^+_\sigma) + 1 \leq 2\max(\chi(U^+), \chi(U^+_\sigma)) + 1.
\]

We note that, since \( X_\sigma \) is a maximum saturating set of \( B_{vw}(U^+_\sigma) \) by Lemma 3.21, we may define \( \tilde{X}_\sigma = X_\sigma \cap \tilde{B}_{vw}(U^+_\sigma) \) as in Definition 3.11.
It follows from the definition of $P$ that, given any $\pi_\sigma \in P$, there exists a corresponding $\pi \in \hat{X} \setminus X_U$. We next show that if $\lambda_\sigma \in P \cap \hat{X}_\sigma$ then $\lambda_\sigma(z_{\zeta-2}) \in \{z_{\zeta-2}, z_{\zeta+2}\}$, where $z_\alpha = \lambda_\sigma^{-1}(z_\alpha)$. So $P \cap \hat{X}_\sigma \subseteq L_\sigma$, where $L_\sigma$ is the subset of $X_\sigma$ that corresponds to the subset $L$ of $X$ in Lemma 3.17. Using Lemma 3.17(a) for $X_\sigma$, it will immediately follow that $|P \cap \hat{X}_\sigma| \leq \chi(U_\sigma^+)$. Let $\lambda_\sigma \in P \cap \hat{X}_\sigma$. On using Corollary 3.18 first for $\lambda$ and then for $\sigma$, we see that $$\{\lambda(x_{\mu-2}), \lambda(x_{\mu+2})\} = \{\sigma(x_{\xi-2}), \sigma(x_{\xi+2})\}. \quad \therefore$$ Thus $$\{\sigma^{-1}\lambda(x_{\mu-2}), \sigma^{-1}\lambda(x_{\mu+2})\} = \{x_{\xi-2}, x_{\xi+2}\}.$$ So, since $z_\alpha = \theta(x_\mu)$ by Lemma 3.20, it follows that $\theta^{-1}(z_{\alpha+2}) \in \{x_{\mu-2}, x_{\mu+2}\}$, and hence $\alpha^{-1}\theta^{-1}(z_{\alpha+2}) \in \{x_{\xi-2}, x_{\xi+2}\}$. Therefore $\lambda_\sigma(z_{\alpha+2}) \in \{\theta(x_{\xi-2}), \theta(x_{\xi+2})\} = \{z_{\zeta-2}, z_{\zeta+2}\}$ as required.

It remains to be shown that $|P \setminus \hat{X}_\sigma| \leq 1$. So suppose that there exists $\lambda_\sigma \in P \setminus \hat{X}_\sigma$. Now $\lambda(w)$ is a d-leaf of $T$ as $\lambda \in \hat{X}$. So, since $\theta^{-1}(T) = T_\sigma$ by Lemma 3.19 and $\lambda_\sigma = \theta^{-1}\lambda_\sigma^{-1}$, it follows that $\lambda_\sigma(w)$ must be a d-leaf of $T_\sigma$. As $\lambda_\sigma \notin \hat{X}_\sigma$, this implies that $\lambda_\sigma(w)$ is a leaf of $T_\sigma$ but not a leaf of $U_\sigma^+$. Now we show that $z_{\zeta+1}$ is the unique leaf of $T_\sigma$ that is not a leaf of $U_\sigma^+$. Since $\pi_\sigma(w)$ must be distinct for each $\pi_\sigma \in X_\sigma$, this will imply that $P \setminus \hat{X}_\sigma = \{\lambda_\sigma\}$, and therefore $|P \setminus \hat{X}_\sigma| \leq 1$.

Now $\text{diam}(T) = \text{diam}(T_\sigma)$ as $T \cong T_\sigma$. So, by applying Lemma 3.1 to $U_\sigma^+$ and $x_\nu$, and then to $U_\sigma^+$ and $z_\zeta$, it is easy to see that $\tau_{U_\sigma^+}(x_{\nu}) = \tau_{U_\sigma^+}(z_\zeta)$. Now, since $X$ is $U_\sigma^+$-optimum and $\sigma \in \hat{X} \setminus X_U$, it follows from Corollary 3.18 that $\tau_{U_\sigma^+}(x_{\nu}) = 1$, and thus $\tau_{U_\sigma^+}(z_\zeta) = 1$. On applying Lemma 3.4 to $U_\sigma^+$, $T_\sigma$ and $z_\zeta$, it then follows that $z_{\zeta+1}$ is the only leaf of $T_\sigma$ that is not a leaf of $U_\sigma^+$. This completes the proof. \hfill $\Box$

**Theorem 3.23** Let $U_\sigma^+$ be a supercard of $U$ and $T$ that is a sunshine graph that has the largest possible value of $\chi(U_\sigma^+)$ over all supercards of $U$ and $T$ that are sunshine graphs. Then $b(U, T) \leq 2\chi(U_\sigma^+) + 5$.

**Proof** Let $X$ be a $U_\sigma^+$-optimum saturating set of $B_{vw}(U_\sigma^+)$. Now, if $\hat{X} \subseteq X_U$ then the result holds immediately by Corollary 3.14. So suppose that there exists $\sigma \in \hat{X} \setminus X_U$, and let $U_\sigma^+$ be the supercard of $U$ and $T$ as defined in equation (9). Then $|\hat{X}| \leq 2\max(\chi(U_\sigma^+), \chi(U_\sigma^+)) + 1$ by Lemma 3.22. So, since $U_\sigma^+$ is a sunshine graph, $b(U, T) \leq 2\chi(U_\sigma^+) + 5$ by Corollary 3.14. \hfill $\Box$

**4 The set $B_{vw}^U(U_\sigma^+)$**

Let $U$ be a sunshine graph and $T$ be a caterpillar. For the rest of this paper, we shall assume that $U_\sigma^+$ is a supercard of $U$ and $T$ that satisfies the conditions of Theorem 3.23. In light of the bound in this theorem, we now concentrate on the set $B_{vw}^U(U_\sigma^+)$. For ease of notation, we write $B_U$ instead of $B_{vw}^U(U_\sigma^+)$. Let $\lambda \in B_U$. By Lemma 3.15, $\lambda(w)$ is a leaf of $U_\sigma^+$ adjacent to a vertex of degree $d_{U_\sigma^+}(x_0)$. Moreover, $\lambda$ is either a rotation of the cycle $\text{ske}(U)$ and $\lambda(x_1) = x_{\nu+\mu+4}$ for each $x_1$, or $\lambda$ is a reflection of the cycle $\text{ske}(U)$ and $\lambda(x_1) = x_{\nu+\mu-1}$ for each $x_1$. We may therefore partition $B_U$ into the rotations $\text{Rot}(B_U)$, and the reflections $\text{Ref}(B_U)$.

We make frequent use of the following well-known results concerning the rotations and reflections of a cycle.

**Lemma 4.1** Let $\lambda, \pi \in B_U$, and let $x_\alpha = \lambda(x_0)$ and $x_\beta = \pi(x_0)$. The following results holds for all $x_1$. 13
(a) If $\lambda \in \text{Rot}(B_U)$ then $\lambda(x_i) = x_{\alpha+i}$ and $\lambda^{-1}(x_i) = x_{i-\alpha}$.
(b) If $\lambda \in \text{Ref}(B_U)$ then $\lambda(x_i) = \lambda^{-1}(x_i) = x_{\alpha-i}$.
(c) If $\lambda, \pi \in \text{Rot}(B_U)$ then $\lambda \pi(x_i) = \pi \lambda(x_i) = x_{(\alpha+\beta)+i}$.
(d) If $\lambda, \pi \in \text{Ref}(B_U)$ then $\pi \lambda(x_i) = x_{(\beta-\alpha)+i}$ and $\lambda \pi(x_i) = x_{(\alpha-\beta)+i}$.
(e) If $\lambda \in \text{Rot}(B_U)$ and $\pi \in \text{Ref}(B_U)$ then $\lambda \pi(x_i) = x_{(\beta-\alpha)-i}$, and $\pi \lambda(x_i) = x_{(\beta-\alpha)+i}$.

Corollary 4.2 Let $\lambda \in B_U$. Suppose there exists $\sigma \in B_U$ such that $\sigma(x_i) = \lambda^{-1}(x_i)$ for all $x_i$. Then either $\lambda$ and $\sigma$ are both in $\text{Rot}(B_U)$, or they are both in $\text{Ref}(B_U)$.

Corollary 4.3 Let $\lambda, \pi$ be in $B_U$. Suppose there exists $\sigma \in B_U$ such that $\sigma(x_i) = \lambda \pi(x_i)$ for all $x_i$. Then $\sigma \in \text{Rot}(B_U)$ if and only if either $\lambda$ and $\pi$ are both in $\text{Rot}(B_U)$, or they are both in $\text{Ref}(B_U)$; otherwise $\sigma \in \text{Ref}(B_U)$.

We call any $\lambda \in \text{Rot}(B_U)$ such that $\lambda(x_0) = x_0$, a trivial rotation (equivalently, $\lambda(x_i) = x_i$ for all $x_i$ by Lemma 4.1(a)). Clearly, $\lambda^{-1}(x_\nu) = x_\nu$ for every trivial rotation $\lambda$ in $B_U$. It then immediately follows from Definition 2.8 that $1_{V(U^+)}$ is the only trivial rotation in any maximum saturating set $X$ of $B_{vw}(U^+)$.

Definition 4.4 Let $X$ be a maximum saturating set of $B_{vw}(U^+)$. We define $X_{\text{Rot}} = X \cap \text{Rot}(B_U)$, $X_{\text{Ref}} = X \cap \text{Ref}(B_U)$ and $X_{\text{Aut}} = X \cap \text{Aut}(U^+)$.

Lemma 4.5 Let $X$ be a maximum saturating set of $B_{vw}(U^+)$, and let $\lambda$ and $\pi$ be distinct permutations in $X$. If $\lambda$ and $\pi$ are both in $X_{\text{Rot}}$ then $\lambda(x_i) \neq \pi(x_i)$ for all $x_i$. Similarly, if $\lambda$ and $\pi$ are both in $X_{\text{Ref}}$ then $\lambda(x_i) \neq \pi(x_i)$ for all $x_i$.

Proof Let $x_\alpha = \lambda(x_0)$ and $x_\beta = \pi(x_0)$. Suppose that there exists $x_k$ such that $\lambda(x_k) = \pi(x_k)$. Now, if both $\lambda$ and $\pi$ are in $\text{Rot}(B_U)$ then it follows from Lemma 4.1(a) that $\alpha = \beta$, so $\lambda^{-1}(x_\nu) = \pi^{-1}(x_\nu)$. Similarly, if both $\lambda$ and $\pi$ are in $\text{Ref}(B_U)$ then it follows from Lemma 4.1(b) that $\lambda^{-1}(x_\nu) = \pi^{-1}(x_\nu)$. Either contradicts property (b) of Definition 2.8.

Corollary 4.6 Let $X$ be a maximum saturating set of $B_{vw}(U^+)$, and let $Z = \{\lambda(x_0) \mid \lambda \in B_U\}$. Then $|X_U| \leq 2|Z|$.

Proof For each $z \in Z$, it follows from Lemma 4.5 that there is at most one $\lambda \in X_{\text{Rot}}$ and at most one $\pi \in X_{\text{Ref}}$ such that $\lambda(x_0) = \pi(x_0) = z$.

We now make some further observations about $B_U$.

Lemma 4.7 Let $\theta \in B_U$ and let $u \in V(U^+)$. Then

(a) $d_{U^+}(u) = d_{U^+}(\theta(u)) + 1$ if and only if $u$ is $x_0$ and $\theta(w)$ is not adjacent to $\theta(x_0)$;
(b) $d_{U^+}(u) = d_{U^+}(\theta(u)) - 1$ if and only if $u$ is not $x_0$ and $\theta(w)$ is adjacent to $\theta(u)$;
(c) $d_{U^+}(u) = d_{U^+}(\theta(u))$ otherwise.

Proof Since $\theta(U) = U^+ - \theta(w)$, it follows that $d_{U^+}(u) = d_{U^+ \theta(w)}(\theta(u))$ for any $u \in V(U)$. It is now easy to show that one of (a), (b) and (c) must hold as $w$ is adjacent to $x_0$ in $U^+$.  

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Corollary 4.8 Let $\theta \in B_U$ be such that $\theta(\pi) = x_0$. If $x_i$ is not $x_0$ and $\theta(x_i) \neq x_0$ then $d_{U^+}(x_i) = d_{U^+}(\theta(x_i))$. \hfill $\blacksquare$

Corollary 4.9 Let $\theta \in B_U$.

(a) $d_{U^+}(x_0) = d_{U^+}(\theta(x_0)) + 1$ if and only if $\theta \not\in \text{Aut}(U^+)$;
(b) $d_{U^+}(x_0) = d_{U^+}(\theta(x_0))$ if and only if $\theta \in \text{Aut}(U^+)$.

Proof By Corollary 2.2, then follow by putting $u = x_0$ in Lemma 4.7 when $u$ is $x_0$. Part (c) is immediate from part (b). \hfill $\blacksquare$

It follows from Corollary 4.9(c) that any trivial rotation in $B_U$ is an automorphism of $U^+$.

Lemma 4.10 Let $\lambda \in \text{Aut}(U^+)$ and $\pi \in B_U \setminus \text{Aut}(U^+)$. Then $\lambda \pi \in B_U \setminus \text{Aut}(U^+)$ and $d_{U^+}(\lambda \pi(x_0)) = d_{U^+}(x_0) - 1$.

Proof Since $\pi \in B_U$ and $\lambda \in \text{Aut}(U^+)$, clearly $\lambda \pi(U) = \lambda(U^+ - \pi(w)) = U^+ - \lambda \pi(w)$, so $\lambda \pi \in B_U$. Now, if $\lambda \pi \in \text{Aut}(U^*)$ then $\lambda = \lambda^{-1} \lambda \pi \in \text{Aut}(U^+)$ as $\text{Aut}(U^+)$ is a group. Since this is impossible, $\lambda \pi \in B_U \setminus \text{Aut}(U^+)$. So $d_{U^+}(\lambda \pi(x_0)) = d_{U^+}(x_0) - 1$ by Corollary 4.9(a). \hfill $\blacksquare$

Lemma 4.11 Let $\lambda \in B_U \setminus \text{Aut}(U^+)$ be such that $\lambda^2(x_0) = x_0$. Then $\lambda(w)$ is adjacent to $x_0$.

Proof $d_{U^+}(x_0) = d_{U^+}(\lambda(x_0)) + 1$ by Lemma 4.9(a). So, since $\lambda^2(x_0) = x_0$, clearly $d_{U^+}(\lambda(x_0)) = d_{U^+}(\lambda^2(x_0)) - 1$. Therefore $\lambda(w)$ is adjacent to $\lambda^2(x_0)$, i.e. $x_0$, by Lemma 4.7(b). \hfill $\blacksquare$

Corollary 4.12 Let $\lambda \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$. Then $\lambda(w)$ is adjacent to $x_0$.

Proof As $\lambda \in \text{Ref}(B_U)$, we have $\lambda^2(x_0) = x_0$ by Lemma 4.1(b). \hfill $\blacksquare$

We next show that if any non-trivial rotation is an automorphism of $U^+$ then $B_U = \text{Aut}(U^+)$. 

Lemma 4.13 Suppose there exists $\lambda \in \text{Rot}(B_U) \cap \text{Aut}(U^+)$ such that $\lambda(x_0) \neq x_0$. Then $B_U = \text{Aut}(U^+)$. 

Proof Let us assume, to the contrary, that there exists $\pi \in B_U \setminus \text{Aut}(U^+)$. Suppose first that $\pi \in \text{Rot}(B_U)$. Then, by Lemma 4.1(c) and Lemma 4.10 $d_{U^+}(\pi \lambda(x_0)) = d_{U^+}(\lambda \pi(x_0)) = d_{U^+}(x_0) - 1$. Thus $d_{U^+}(\lambda(x_0)) = d_{U^+}(\pi \lambda(x_0)) + 1$ as $\lambda \in \text{Aut}(U^+)$. However by using Lemma 4.7(a) with $\theta = \pi$ and $u = \lambda(x_0)$ yields $\lambda(x_0) = x_0$, which is impossible.

Suppose, on the other hand, that $\pi \in \text{Ref}(B_U)$. Then, by Lemma 4.11 and Corollary 4.3 $\lambda \pi \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$. So $\lambda \pi(w)$ is adjacent to $x_0$ by Corollary 4.12. By Lemma 4.1(b), $\lambda \pi(x_0) = \lambda(x_0)$. So, since $\pi(x_0) \neq x_0$ by Corollary 4.3(c) and $\lambda(x_0) \neq x_0$ by assumption, it follows from Corollary 4.3 with $\theta = \lambda \pi$ and $x_i = \pi(x_0)$ that $d_{U^+}(\pi(x_0)) = d_{U^+}(\lambda \pi(x_0)) = d_{U^+}(\lambda(x_0))$. However, Corollary 4.3(a) and (b) applied to $\pi$ and $\lambda$ respectively, implies that $d_{U^+}(\pi(x_0)) \neq d_{U^+}(\lambda(x_0))$, contradicting the previous statement. Hence $B_U = \text{Aut}(U^+)$. \hfill $\blacksquare$

Corollary 4.14 Let $X$ be a $U^+$-optimum saturating set of $B_{vw}(U^+)$. 

(a) If $B_U \neq \text{Aut}(U^+)$ then $X_{\text{Aut}} \subseteq X_{\text{Ref}} \cup \{1_{V(U^+)}\}$. 

(b)
Lemma 4.7 that

\( \Delta B \)

then (a) 1

Proof Now, since Aut(\( \Lambda \))

Moreover, since \( \Lambda \)

we may assume that both \( \Lambda \) and \( \lambda \) are reflections as 1\((U^+)\) is the only trivial rotation in \( X \).

Now, by Corollary 4.9(c), we have that \( \lambda \pi(x_0) \neq x_0 \) by Lemma 4.1(d). Therefore \( B_U = \text{Aut}(U^+) \) by Lemma 4.13.

(c) \(|X_{\text{Aut}}| = \chi(U^+)\) as \( X \) is \( U^+ \)-optimum and \( B_U = \text{Aut}(U^+) \). The result now follows from Theorem 3.23 and part (b).

We now consider those rotations that are not automorphisms of \( U^+ \).

Lemma 4.15 Suppose there exists \( \lambda, \pi \in \text{Rot}(B_U) \setminus \text{Aut}(U^+) \) such that \( \lambda(x_0) \neq \pi(x_0) \). Then both \( \lambda(w) \) and \( \pi(w) \) are adjacent to \( x_0 \).

Proof Let \( x_\alpha = \lambda(x_0) \) and \( x_\beta = \pi(x_0) \), so \( \lambda(x_i) = x_{\alpha+i} \) and \( \pi(x_i) = x_{\beta+i} \) by Lemma 4.1(a). Now, by Corollary 4.9(c), \( \alpha \neq 0 \) as \( \lambda \not\in \text{Aut}(U^+) \). So, by Lemma 4.7, \( d_{U^+}(x_\alpha) \leq d_{U^+}(\pi(x_\alpha)) = d_{U^+}(x_{\beta-\alpha}) \).

Suppose that \( \lambda(w) \) is not adjacent to \( x_0 \), i.e. \( \lambda(x_\alpha) \). Then, since \( \alpha \neq 0 \), it follows from Lemma 4.7 that \( d_{U^+}(x_\alpha) = d_{U^+}(\lambda(x_\alpha)) = d_{U^+}(x_0) \). So \( d_{U^+}(x_{\beta-\alpha}) \geq d_{U^+}(x_0) \). Now, by Corollary 4.9(a), we have that \( d_{U^+}(x_0) = d_{U^+}(\pi(x_0)) + 1 \) as \( \pi \not\in \text{Aut}(U^+) \). Thus

\[
\begin{align*}
d_{U^+}(x_{\beta-\alpha}) &\geq d_{U^+}(x_0) > d_{U^+}(\pi(x_0)) = d_{U^+}(x_\beta) = d_{U^+}(\lambda(x_{\beta-\alpha})).
\end{align*}
\]

Hence, \( x_{\beta-\alpha} \) must be \( x_0 \) by Lemma 4.7(a), and therefore, \( \beta = \alpha \), which is impossible. So \( \lambda(w) \) must be adjacent to \( x_0 \). By symmetry, clearly \( \pi(w) \) is also adjacent to \( x_0 \).

Corollary 4.16 Let \( X \) be a maximum saturating set of \( B_{\text{ws}}(U^+) \). If \(|X_{\text{Rot}} \setminus X_{\text{Aut}}| \geq 2 \) then \( \lambda(w) \) is adjacent to \( x_0 \) for all \( \lambda \in X_{\text{Rot}} \setminus X_{\text{Aut}} \).

Proof Suppose that there exist distinct rotations \( \lambda, \pi \in X_{\text{Rot}} \setminus X_{\text{Aut}} \). Then \( \lambda(x_0) \neq \pi(x_0) \) by Lemma 4.5 so the result follows immediately from Lemma 4.15.

The following lemma shows that there exists a correspondence between those permutations \( \lambda \) for which \( \lambda(w) \) is adjacent to \( x_0 \), and a subset of \( \text{Aut}(U) \).

Lemma 4.17 (a) For each \( \lambda \in B_U \) for which the leaf \( \lambda(w) \) of \( U^+ \) is adjacent to \( x_0 \), there exists \( \lambda^* \in \text{Aut}(U) \) such that \( \lambda^*(x_i) = \lambda(x_i) \) for all \( x_i \).

(b) For each \( \lambda^* \in \text{Aut}(U) \), there exists \( \lambda \in B_U \) such that \( \lambda(w) = w \) and \( \lambda(x_i) = \lambda^*(x_i) \) for all \( x_i \).

Proof (a) Let \( \lambda \in B_U \) be such that \( \lambda(w) \) is adjacent to \( x_0 \). Let \( \phi \) be the transposition that swaps the leaves \( w \) and \( \lambda(w) \), and let \( \lambda^* = \phi \lambda \). Then \( \lambda^*(U) = \phi \lambda(U) = \phi(U^+ - \lambda(w)) = U \), as \( \phi \in \text{Aut}(U^+) \).

(b) Let \( \lambda^* \in \text{Aut}(U) \). Then the permutation \( \lambda \) defined by \( \lambda(w) = w \) and \( \lambda(u) = \lambda^*(u) \) for all \( u \in V(U) \) clearly has the required properties.
Lemma 4.18 Suppose there exists $\lambda, \pi \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$ such that $\lambda(x_0) \neq \pi(x_0)$. Then there exists $\sigma \in \text{Rot}(B_U) \setminus \text{Aut}(U^+)$ such that $\sigma(w) = w$ and $\sigma(x_i) = \lambda(x_i)$ for all $x_i$.

Proof Since $\lambda(w)$ and $\pi(w)$ are both adjacent to $x_0$ by Corollary 4.12 it follows from Lemma 4.17(a) that there exist $\lambda^*$ and $\pi^*$ in $\text{Aut}(U)$ such that $\lambda^*(x_i) = \lambda(x_i)$ and $\pi^*(x_i) = \pi(x_i)$ for all $x_i$. Since $\text{Aut}(U)$ is a group, $\lambda^* \pi^* \in \text{Aut}(U)$. So, by Lemma 4.17(b), there exists $\sigma \in B_U$ such that $\sigma(w) = w$ and $\sigma(x_i) = \lambda(x_i)$ for all $x_i$.

Now, by Corollary 4.13, $\sigma \in \text{Rot}(B_U)$. Moreover, since $\lambda(x_0) \neq \pi(x_0)$, it follows from Lemma 4.1(d) that $\sigma(x_0) \neq x_0$. As $B_U \neq \text{Aut}(U^+)$, it immediately follows from Lemma 4.13 that $\sigma \notin \text{Aut}(U^+)$. \qed

Lemma 4.19 Suppose there exists $\lambda, \pi \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$ such that $\lambda(x_0) \neq \pi(x_0)$. Then $\psi(w)$ is adjacent to $x_0$ for all $\psi \in B_U$.

Proof Let $\sigma$ be the rotation from Lemma 4.18 and let $\psi \in B_U$. If $\psi(x_0) = x_0$ then $\psi \in \text{Aut}(U^+)$ by Corollary 4.19(c), so $\psi(w)$ is adjacent to $x_0$. We therefore assume that $\psi(x_0) \neq x_0$. We first show that $\psi \notin \text{Aut}(U^+)$. Suppose that $\psi \in \text{Aut}(U^+)$. Since $\psi(x_0) \neq x_0$ and $B_U \neq \text{Aut}(U^+)$, it follows from Lemma 4.13 that $\psi$ must be a reflection. By Lemma 4.10 and Corollary 4.13 we see that $\psi \sigma \in \text{Ref}(B_U) \setminus \text{Aut}(U^+)$, $\psi \sigma$ is adjacent to $x_0$ by Corollary 4.12. However, $\psi \sigma(w) = \psi(w)$ as $\sigma(w) = w$, so $\psi \sigma(w)$ must be adjacent to $\psi(x_0)$ as $\psi \in \text{Aut}(U^+)$. Since $\psi(x_0) \neq x_0$ this is impossible. This contradiction shows that $\psi \notin \text{Aut}(U^+)$. It remains to be shown that $\psi(w)$ is adjacent to $x_0$. If $\psi \in \text{Ref}(B_U)$ then the result follows from Corollary 4.12. So suppose that $\psi \in \text{Rot}(B_U) \setminus \text{Aut}(U^+)$. Since the result follows from Lemma 4.15 when $\sigma(x_0) \neq \psi(x_0)$, we may assume that $\sigma(x_0) = \psi(x_0)$, thus $\psi^{-1}(x_0) = \sigma^{-1}(x_0)$ by Lemma 4.11(a). Hence $\sigma^{-1}(x_0) \neq x_0$. As $\sigma(w) = w$, it now follows from Lemma 4.17(b) with $\theta = \sigma$ and $u = \sigma^{-1}(x_0)$, that $d_{U^+}(\sigma^{-1}(x_0)) = d_{U^+}(\sigma \sigma^{-1}(x_0)) - 1$. So $d_{U^+}(\sigma^{-1}(x_0)) = d_{U^+}(x_0) - 1$, and thus, $d_{U^+}(\psi^{-1}(x_0)) = d_{U^+}(x_0) - 1$. The result now follows by again applying Lemma 4.17(b) to $\psi$ and $\psi^{-1}(x_0)$.

Corollary 4.20 Let $X$ be a maximum saturating set of $B_{vw}(U^+)$. If $|X_{\text{Ref}} \setminus X_{\text{Aut}}| \geq 2$ then $\lambda(w)$ is adjacent to $x_0$ for all $\lambda \in B_U$.

Proof Suppose that $\lambda$ and $\pi$ are two distinct reflections in $X_{\text{Ref}} \setminus X_{\text{Aut}}$. Then $\lambda(x_0) \neq \pi(x_0)$ by Lemma 4.5 so the result follows immediately from Lemma 4.19. \qed

We note that Corollary 4.20 is a stronger result than the analogous result for rotations in Corollary 4.16 since the conclusions of Corollary 4.20 applies to all permutations in $B_U$.

Lemma 4.21 Suppose that $B_U \neq \text{Aut}(U^+)$ and $\chi(U^+) \geq 5$. Let $X$ be a $U^+$-optimum saturating set of $B_{vw}(U^+)$. Then

(a) for each $\lambda \in X_U$, there exists a distinct leaf of $U^+$, namely $\lambda(w)$, adjacent to $x_0$;

(b) $d_{U^+}(x_0) \geq \chi(U^+) + 2$;

(c) for each $\lambda \in X_U$, there exists $\lambda^* \in \text{Aut}(U)$ such that $\lambda^*(x_0) = \lambda(x_0)$.

Proof Part (b) follows immediately from part (a). In addition, part (c) follows from part (a) by Lemma 4.17(a). Now, by property (b) of Definition 2.8 and Lemma 3.15, $\lambda(w)$ is a distinct
leaf of $U^+$ for each distinct $\lambda \in X_U$. To complete the proof, we now show that each such $\lambda(w)$ is adjacent to $x_0$.

By Corollary 4.20, there is nothing to prove when $|X_{\text{Ref}} \setminus X_{\text{Aut}}| \geq 2$; so we assume that $|X_{\text{Ref}} \setminus X_{\text{Aut}}| \leq 1$. Furthermore, since $B_U \neq \text{Aut}(U^+)$, it follows from Corollary 4.14 that $X_{\text{Aut}} \subseteq X_{\text{Ref}} \cup \{1_U^+\}$ and $|X_{\text{Aut}}| \leq 2$. Since $|X_U| \geq 5$, this implies that $|X_{\text{Rot}} \setminus X_{\text{Aut}}| \geq 2$. It now follows from Corollaries 4.16 and 4.12 that $\lambda(w)$ is adjacent to $x_0$ for each $\lambda \in X_U \setminus X_{\text{Aut}}$. As $1_{V(U)}(w)$ is clearly also adjacent to $x_0$, this concludes the proof in the case that $X_{\text{Ref}} \cap X_{\text{Aut}} = \emptyset$.

Suppose then there exists $\theta \in X_{\text{Ref}} \cap X_{\text{Aut}}$, and let $\phi$ and $\eta$ be permutations in $X_{\text{Rot}} \setminus X_{\text{Aut}}$. By Lemma 4.10 and Corollary 4.3, both $\theta \phi$ and $\theta \eta$ are in $\text{Ref}(B_U) \setminus \text{Aut}(U^+)$. Furthermore, $\theta \phi(x_0) \neq \theta \eta(x_0)$ as $\phi(x_0) \neq \eta(x_0)$ by Corollary 4.5. By applying Lemma 4.19 to $\theta \phi$ and $\theta \eta$, this implies that $\psi(w)$ is adjacent to $x_0$ for all $\psi \in B_U$.

\begin{theorem}
Suppose that $B_U \neq \text{Aut}(U^+)$ and $n \geq 12$. Then $b(U, T) \leq 2 \left\lceil \sqrt{2n + 1} \right\rceil + 3$.
\end{theorem}

\begin{proof}
We recall that $d_1(U)$ and $d_2(U)$ are the number of vertices of $U$ of degrees 1 and 2, respectively. Let $d_q(U) = n - d_1(U) - d_2(U)$, i.e., the number of vertices of $U$ of degree 3 or more.

Now, if $\chi(U^+) \leq 4$ then $b(U, T) \leq 13$ by Theorem 3.23, thus the bound holds. We may therefore assume that $\chi(U^+) \geq 5$, so the conclusions of Lemma 4.21 hold. It follows from part(b) of that lemma that $d_U(x_0) = d_{U^+}(x_0) - 1 \geq \chi(U^+) + 1 \geq 6$.

Let $X$ be a $U^+$-optimum saturating of $U^+$, and let $Z = \{\lambda(x_0) \mid \lambda \in X_U\}$. By Lemma 4.21(c), there exists a subset $\Lambda^*$ of $\text{Aut}(U)$ such that $\{\lambda(x_0) \mid \lambda^* \in \Lambda^*\} = Z$. This implies that $U$ contains at least $|Z|$ vertices of degree $d_U(x_0)$, and hence $d_q(U) \geq |Z|$. In addition, since each vertex $x_i$ in $Z$ is adjacent to precisely $d_U(x_i) - 2$ leaves, it follows that $d_1(U) \geq |Z|(d_U(x_0) - 2)$. Therefore $d_1(U) \geq |Z|(\chi(U^+) - 1)$.

Now, by property (b) of Definition 2.8 and Lemma 3.15, for each distinct $\lambda$ in $X_U$ there exists a distinct vertex of $U$ of degree 2, namely $\lambda^{-1}(x_0)$. So $d_2(U) \geq |X_U| = \chi(U^+)$. Since each vertex in $Z$ has degree at least 6 in $U$, it therefore follows that

\[
n = |V(U)| = d_1(U) + d_2(U) + d_q(U) \geq |Z|(\chi(U^+) - 1) + \chi(U^+) + |Z| = (|Z| + 1)\chi(U^+).
\]

So, since $\chi(U^+) = |X_U| \leq 2|Z|$ by Corollary 4.6, we have $2n \geq \chi(U^+)(\chi(U^+) + 2)$. Solving for $\chi(U^+)$, yields $\chi(U^+) \leq \sqrt{2n + 1} - 1$. Therefore $b(U, T) \leq 2 \lceil \sqrt{2n + 1} - 1 \rceil + 5$ by Theorem 3.23, yielding the bound.

\end{proof}

5 The case when $B_{uv}^U(U^+) = \text{Aut}(U^+)$

For the whole of this section, we assume that there exists some non-trivial rotation in $\text{Aut}(U^+)$. So $B_{uv}^U(U^+) = \text{Aut}(U^+)$ by Lemma 4.13 and thus $X_U = X_{\text{Aut}} = X_{\text{Rot}} \cup X_{\text{Ref}}$, for any maximum saturating set $X$. We note that, if $B_{uv}^U(U^+) = \text{Aut}(U^+)$ but there does not exist such a rotation, then $b(U, T) \leq 9$ by Corollary 4.14(c). This motivates our assumption.

For clarity, we write $\text{Rot}(\text{Aut}(U^+))$ and $\text{Ref}(\text{Aut}(U^+))$ instead of $\text{Rot}(B_U)$ and $\text{Ref}(B_U)$, respectively. Since $\text{Aut}(U^+)$ is a group, we can simplify Corollaries 4.2 and 4.3 as follows.

\begin{corollary}
Let $\lambda \in \text{Aut}(U^+)$. Then either $\lambda$ and $\lambda^{-1}$ are both in $\text{Rot}(\text{Aut}(U^+))$, or they are both in $\text{Ref}(\text{Aut}(U^+))$.
\end{corollary}
Corollary 5.2 Let $\lambda, \pi \in \text{Aut}(U^+)$. Then $\lambda\pi \in \text{Rot}(\text{Aut}(U^+))$ if and only if either $\lambda$ and $\pi$ are both in $\text{Rot}(\text{Aut}(U^+))$, or they are both in $\text{Ref}(\text{Aut}(U^+))$; otherwise $\lambda\pi \in \text{Ref}(\text{Aut}(U^+))$.

We recall that $x_i^j$ is the $j^{th}$ distinct leaf adjacent to $x_i$. The following results two are easy to prove.

Lemma 5.3 There exists $\lambda \in \text{Rot}(\text{Aut}(U^+))$, where $\lambda(x_i) = x_{\alpha+i}$ and $\lambda(x_i^j) = x_{\alpha+i}^j$ for all $i$ and $j$, if and only if $d_{U^+}(x_i) = d_{U^+}(x_{\alpha+i})$ for all $i$.

Lemma 5.4 There exists $\lambda \in \text{Ref}(\text{Aut}(U^+))$, where $\pi(x_i) = x_{\beta-i}$ and $\lambda(x_i^j) = x_{\beta-i}^j$ for all $i$ and $j$, if and only if $d_{U^+}(x_i) = d_{U^+}(x_{\beta-i})$ for all $i$.

If $Z \subseteq \text{Aut}(U^+)$, we define $Z(u) = \{\lambda(u) \mid \lambda \in Z\}$. We note that if $Z$ is a subgroup of $\text{Aut}(U^+)$ then $Z(u)$ is the orbit of $u$ under the group action of $Z$ on $V(U^+)$.

Lemma 5.5 Let $A = \{\beta \mid \beta \in \text{Rot}(\text{Aut}(U^+))(x_0)\}$, and let $\delta$ be the smallest positive element in $A$. Then

(a) $\delta$ divides every element of $A$;

(b) $2 \leq \delta \leq \frac{c}{2}$;

(c) there exists $\phi \in \text{Rot}(\text{Aut}(U^+))$ such that $\phi^\delta = 1_{U^+}$, $\phi(x_i) = x_{\delta+i}$ and $\phi(x_i^j) = x_{\delta+i}^j$ for all $i, j$.

Proof By assumption, there exists some non-trivial rotation in $\text{Aut}(U^+)$; so $0 < \delta < c$. Let $\sigma \in \text{Rot}(\text{Aut}(U^+))$ be such that $\sigma(x_0) = x_\delta$.

(a) Suppose that there exists $\beta \in A$ such that $\delta$ does not divide $\beta$, and let $\pi(x_0) = x_\beta$. Let $d$ be the highest common factor of $\delta$ and $\beta$. By the Euclidean algorithm, there exist integers $a$ and $b$ such that $bd + a\beta = d$. Now, on using Lemma 4.1 and Corollaries 5.1 and 5.2 repeatedly, it is easy to see that $\sigma^d \pi^a \in \text{Rot}(\text{Aut}(U^+))$ and $\sigma^b \pi^a(x_i) = x_{b\delta+a\beta+i} = x_{d+i}$ for all $x_i$. Therefore $d_{U^+}(x_i) = d_{U^+}(x_{d+i})$ for all $x_i$, as $\sigma^d \pi^a$ is in $\text{Aut}(U^+)$. It now follows from Lemma 5.3 that there exists $\lambda \in \text{Rot}(\text{Aut}(U^+))$ such that $\lambda(x_0) = x_d$, and thus $d \in A$. Since $0 < d < \delta$, this contradicts the minimality of $\delta$.

(b) $\delta$ divides $c$ as $c \in A$, thus $\delta \leq \frac{c}{2}$. So suppose that $\delta = 1$. Then, on using Corollary 5.2 and Lemma 4.1(c) repeatedly, we see that $\sigma^\alpha(x_0) = x_\nu$. This is impossible as $d_{U^+}(x_0) \geq 3$ and $d_{U^+}(x_\nu) = 2$. Therefore $2 \leq \delta \leq \frac{c}{2}$.

(c) $d_{U^+}(x_i) = d_{U^+}(x_{\delta+i})$ for all $x_i$, as $\sigma \in \text{Rot}(\text{Aut}(U^+))$. Hence by Lemma 5.3, there exists $\phi \in \text{Rot}(\text{Aut}(U^+))$ such that $\phi(x_i) = x_{\delta+i}$ and $\phi(x_i^j) = x_{\delta+i}^j$ for all $i, j$. As $\delta$ divides $c$, it is easy to see that $\phi^\delta(x_i) = x_i$ and $\phi^\delta(x_i^j) = x_i^j$ for all $i, j$. So $\phi^\delta = 1_{U^+}$.

By Corollary 5.2, the composition of two rotations is also a rotation. We may therefore make the following definition.

Definition 5.6 Let $\delta$ be the positive integer and $\phi$ be the rotation from Lemma 5.3. We define $\Phi$ to be the cyclic subgroup of $\text{Aut}(U^+)$ of order $\frac{c}{\delta}$ generated by $\phi$, i.e., $\Phi = \{\phi^j \mid 0 \leq j < \frac{c}{\delta}\}$.

\[\square\]
For the rest of this section we assume that on that δ, φ and Φ are as in Definition 5.6. Since the cycle length \( c = \delta|\Phi| \), and the number of leaves of \( U^+ \) is \( d_1(U^+) \), we see that

\[ n + 1 = \delta|\Phi| + d_1(U^+). \] (10)

For any \( \pi \in \text{Ref}(\text{Aut}(U^+)) \), we denote the right and left cosets of \( \Phi \) with respect to \( \pi \) by \( \Phi\pi \) and \( \pi\Phi \), respectively. It follows from Corollary 5.2 that \( \Phi\pi \subseteq \text{Ref}(\text{Aut}(U^+)) \) and \( \pi\Phi \subseteq \text{Ref}(\text{Aut}(U^+)) \).

The orbit \( \Phi(u) \) of a vertex \( u \) of \( U^+ \) under \( \Phi \) is the set \( \{ \phi^j(u) \mid 0 \leq j < \frac{c}{\delta} \} \). It follows from a well-known result from Group Theory that, for any two vertices \( u \) and \( t \) of \( U^+ \), either \( \Phi(u) = \Phi(t) \) or \( \Phi(u) \cap \Phi(t) = \emptyset \). Thus \( t \) is in \( \Phi(u) \) if and only if \( \Phi(u) = \Phi(t) \). Now, for every vertex \( u \) of \( U^+ \), clearly \( 1_{V(U^+)} \) is the only element of \( \Phi \) that fixes \( u \). It therefore follows from the Orbit-Stabiliser and Lagrange Theorems [12] that \( |\Phi(u)| = |\Phi| = |\Phi\pi| = \frac{c}{\delta} \).

**Lemma 5.7** Let \( x_i \in V(\text{skel}(U^+)) \). Then

(a) \( \text{Rot}(\text{Aut}(U^+))(x_i) = \Phi(x_i) \);
(b) if \( \lambda \in \text{Rot}(\text{Aut}(U^+)) \) then \( \lambda^{-1}(x_i) \in \Phi(x_i) \);
(c) if \( \pi \in \text{Ref}(\text{Aut}(U^+)) \) then \( \Phi\pi(x_i) = \pi\Phi(x_i) \);
(d) if \( \text{Ref}(\text{Aut}(U^+))(x_i) \cap \Phi(x_i) \neq \emptyset \) then \( \text{Ref}(\text{Aut}(U^+))(x_i) \subseteq \Phi(x_i) \).

**Proof** (a) Clearly \( \Phi(x_i) \subseteq \text{Rot}(\text{Aut}(U^+))(x_i) \). So let \( \lambda \in \text{Rot}(\text{Aut}(U^+)) \) and let \( x_\alpha = \lambda(x_0) \). Now, by Lemma 5.5, there exists some positive integer \( k \) such that \( \alpha = k\delta \). Moreover, on using Corollary 5.2 and Lemma 4.1, we see that \( \lambda(x_i) = x_{k\delta+i} = \phi^k(x_i) \). Therefore \( \lambda(x_i) \in \Phi(x_i) \), and thus \( \text{Rot}(\text{Aut}(U^+))(x_i) \subseteq \Phi(x_i) \).

(b) This follows immediately from part (a) as \( \lambda^{-1} \in \text{Rot}(\text{Aut}(U^+)) \) for all \( \lambda \in \text{Rot}(\text{Aut}(U^+)) \) by Corollary 5.1.

(c) Let \( x_\beta = \pi(x_0) \), and let \( 0 \leq j < \frac{c}{\delta} \). Then, on using Corollary 5.2 and Lemma 4.1 we see that \( \phi^j\pi(x_i) = x_{j\delta+i} = \pi\phi^{-j}(x_i) \). So \( \phi^j\pi(x_i) \in \pi\Phi(x_i) \) by part (b), and therefore \( \Phi\pi(x_i) \subseteq \pi\Phi(x_i) \). It similarly follows that \( \pi\Phi(x_i) \subseteq \Phi\pi(x_i) \).

(d) Suppose there exists \( \pi \in \text{Ref}(\text{Aut}(U^+)) \) such that \( \pi(x_i) \in \Phi(x_i) \). Then \( \Phi\pi(x_i) = \Phi(x_i) \), so \( \pi\Phi(x_i) = \Phi(x_i) \) by part (c). Now let \( \lambda \in \text{Ref}(\text{Aut}(U^+)) \). Then \( \pi\lambda(x_i) \in \Phi(x_i) \) by Corollary 5.2 and part (a). So \( \pi\lambda(x_i) \in \pi\Phi(x_i) \), and therefore \( \lambda(x_i) \in \Phi(x_i) \), yielding the result. \( \square \)

We now show that we can always find some maximum saturating set \( X \) such that \( X_{\text{Aut}} \) is isomorphic to a subgroup of \( \text{Aut}(U^+) \).

**Theorem 5.8** Suppose that \( \text{Aut}(U^+) \) contains a non-trivial rotation. We consider the following two (not necessarily disjoint) cases:

(i) \( \text{Ref}(\text{Aut}(U^+))(x_0) \subseteq \Phi(x_0) \);
(ii) \( d_{U^+}(x_0) = 3 \) and \( \text{Ref}(\text{Aut}(U^+))(x_0) \subseteq \Phi(x_0) \).

Then

(a) if either case (i) or case (ii) holds, there exists a \( U^+ \)-optimum saturating set \( X \) of \( B_{uv}(U^+) \) such that \( X_{\text{Aut}} = \Phi \cong C(\frac{c}{\delta}) \), the cyclic group of order \( \frac{c}{\delta} \);
(b) if neither case holds, there exists \( \pi \in \text{Ref}(\text{Aut}(U^+)) \) and a \( U^+ \)-optimum saturating set \( X \) of \( B_{uv}(U^+) \) such that \( X_{\text{Aut}} = \Phi \cup \pi\Phi \cong D(\frac{2c}{\delta}) \), the dihedral group of order \( \frac{2c}{\delta} \).
Lemma 5.9

Let \( X \) be the \( U^+ \)-optimum saturating set from Theorem 5.8.

\( \text{(a)} \) \( X_{\text{Aut}}(x_v) = \{ \lambda^{-1}(x_v) \mid \lambda \in X_{\text{Aut}} \} \) and \( |X_{\text{Aut}}(x_v)| = |X_{\text{Aut}}| \). In addition, \( d_{U^+}(x_i) = 2 \) and \( \tau_{U^+}(x_i) = \tau_{U^+}(x_v) \) for each vertex \( x_i \) in \( X_{\text{Aut}}(x_v) \).

\( \text{(b)} \) \( |X_{\text{Aut}}(w)| = |X_{\text{Aut}}| \), and each vertex in \( X_{\text{Aut}}(w) \) is a leaf adjacent to a vertex of degree \( d_{U^+}(x_0) \).

\( \text{(c)} \) If \( \lambda \in X \setminus X_{\text{Aut}} \) then \( \lambda^{-1}(x_v) \notin X_{\text{Aut}}(x_v) \) and \( \lambda(w) \notin X_{\text{Aut}}(w) \).

\( \square \)

Lemma 5.10

Let \( X \) be the \( U^+ \)-optimum saturating set from Theorem 5.8 and let \( B = \{ \lambda(w) \mid \lambda \in X \setminus X_{\text{Aut}} \text{ and } d_{U^+}(\lambda(w)) = 1 \} \). Let \( A \) be any subset of \( B \) such that \( \Phi(a_1) \cap \Phi(a_2) = \emptyset \) for all distinct \( a_1 \) and \( a_2 \) in \( A \). Then
Proof. Suppose there exists $a_1 \in A$. By Lemma 5.9(c), $a_1 \notin X_{\text{Aut}}(w)$. So $a_1 \notin \Phi(w)$, and thus $\Phi(a_1) \cap \Phi(w) = \emptyset$. Similarly, $\Phi(a_1) \cap \Phi(w) = \emptyset$ when $X_{\text{Aut}} = \Phi \cup \Phi \pi$. Therefore $\Phi(a_1) \cap X_{\text{Aut}}(w) = \emptyset$. So, since $\Phi(a_1) \cap \Phi(a_2) = \emptyset$ for all $a_2 \in A \setminus \{a_1\}$, it follows that $d_1(U^+) \geq |A||\Phi| + |X_{\text{Aut}}(w)|$. Hence $d_1(U^+) \geq |A||\Phi| + |X_{\text{Aut}}|$, by Lemma 5.9(b). Together with equation (10), this yields $n + 1 \geq |X_{\text{Aut}}| + (|A| + \delta)|\Phi|$. Cases (a) and (b) immediately follow. Clearly, when $|A| = 0$, equality holds if and only if $d_1(U^+) = |X_{\text{Aut}}|$. \done

6 Upper bounds on $b(U, T)$

We now obtain upper bounds on $b(U, T)$ for the three possible values of $\tau_{U^+}(x_\nu)$, and the two possible cases for $X_{\text{Aut}}$ from Theorem 5.8. We further show that, for each of the six cases, the maximum value of $b(U, T)$ is attained if and only if the structure of $U^+$ is as described below.

(S0a) $n \equiv 2 \pmod{3}$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod{2}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod{2}$.

(S0b) $n \equiv 6 \pmod{7}$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod{4}$, $d_{U^+}(x_i) = 3$ when $i \equiv 2 \pmod{4}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod{3}$.

(S1a) $n \equiv 3 \pmod{4}$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod{3}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod{3}$.

(S1b) $n \equiv 4 \pmod{5}$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod{3}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 1 \pmod{3}$.

(S2a) $n \equiv 4 \pmod{5}$, $d_{U^+}(x_i) = 3$ when $i \equiv 0 \pmod{4}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, $\nu \equiv 2 \pmod{4}$.

(S2b) $n \equiv 6 \pmod{7}$, $d_{U^+}(x_i) = 4$ when $i \equiv 0 \pmod{5}$, and $d_{U^+}(x_i) = 2$ otherwise; in addition, either $\nu \equiv 2 \pmod{5}$ or $\nu \equiv 3 \pmod{5}$.

We note that, $\tau_{U^+}(x_\nu) = 0$ when $U^+$ has structure (S0a) or (S0b); $\tau_{U^+}(x_\nu) = 1$ when $U^+$ has structure (S1a) or (S1b); $\tau_{U^+}(x_\nu) = 2$ when $U^+$ has structure (S2a) or (S2b). We will see that for structures (S0a), (S1a) and (S2a), the corresponding $X_{\text{Aut}}$ is $\Phi$, and for structures (S0b), (S1b) and (S2b), the corresponding $X_{\text{Aut}}$ is $\Phi \cup \Phi \pi$.

Examples of the structures of the six possible supercards are shown in Figures 1, 2 and 3.

We first prove that if $U^+$ has any of these structures then $B_{\text{Aut}}^U(U^+) = \text{Aut}(U^+)$, and specify the value of $b(U, T)$ in each case.

Lemma 6.1 Suppose that $U^+$ has any of the six structures (S0a) through to (S2b). Then there exists a non-trivial rotation in $\text{Aut}(U^+)$ and $B_{\text{Aut}}^U(U^+) = \text{Aut}(U^+)$.

Proof. It is easy to see that if $U^+$ has any of the six structures, then there exists a non-trivial rotation in $\text{Aut}(U^+)$. So $B_{\text{Aut}}^U(U^+) = \text{Aut}(U^+)$ by Lemma 4.13 in each case. \done
Lemma 6.2 Let \( n \geq 14 \). Suppose that \( U^+ \) has any of the six structures (S0a) through to (S2b).

(a) If \( U^+ \) has structure (S0a) then \( b(U, T) = \frac{n+1}{3} + 2 \).
(b) If \( U^+ \) has structure (S0b) then \( b(U, T) = \frac{2(n+1)}{7} + 1 \).
(c) If \( U^+ \) has structure (S1a) then \( b(U, T) = \frac{n+1}{3} + 1 \).
(d) If \( U^+ \) has structure (S1b) then \( b(U, T) = \frac{2(n+1)}{5} \).
(e) If \( U^+ \) has structure (S2a) then \( b(U, T) = \frac{n+1}{3} \).
(f) If \( U^+ \) has structure (S2b) then \( b(U, T) = \frac{2(n+1)}{7} \).

Proof In each of the six cases, there exists a non-trivial rotation in \( \text{Aut}(U^+) \) by Lemma 6.1 so we therefore apply all the results from Section 5. In particular, we may define \( \Phi \) and \( \delta \) as in Definition 5.6

Suppose that \( U^+ \) has structure (S0a), (S1a) or (S2a). Then clearly, case (ii) of Theorem 5.8 holds. So there exists a \( U^+ \)-optimum saturating set \( X \) such that \( X_{\text{Aut}} = \Phi \).

Suppose instead that \( U^+ \) has structure (S0b), (S1b) or (S2b). Then, it is easy to see that there exists \( \lambda \in \text{Ref}(\text{Aut}(U^+)) \) such that \( \lambda(x_0) = x_0 \), and to verify that \( \lambda(x_\nu) \), i.e., \( x_{c-\nu} \), is not in \( \Phi(x_\nu) \). So case (i) of Theorem 5.8 cannot hold. Moreover, case (ii) of Theorem 5.8 cannot hold as \( d_{U^+}(x_0) \geq 4 \). Therefore, there exists \( \pi \in \text{Ref}(\text{Aut}(U^+)) \) and a \( U^+ \)-optimum saturating set \( X \) such that \( X_{\text{Aut}} = \Phi \cup \Phi\pi \).

(a) Since \( d_1(U^+) = |\Phi| \), it follows from equation (10) that \( |X_{\text{Aut}}| = |\Phi| = \frac{n+1}{3} + 2 \). It is easy to see that \( U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1} \cong T - x_{\nu-1} \), and to check using Lemma 3.5 that \( \lambda(w) \in \{x_{\nu-1}, x_{\nu+1}\} \) for any \( \lambda \in X \setminus X_{\text{Aut}} \). Therefore \( |X \setminus X_{\text{Aut}}| = 2 \), and \( b(U, T) = |X| = \frac{n+1}{3} + 2 \).

(b) Since \( d_1(U^+) = 3|\Phi| \), it follows from equation (10) that \( |X_{\text{Aut}}| = 2|\Phi| = \frac{2(n+1)}{7} + 1 \). It is easy to see that \( U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1} \), and to check using Lemma 3.5 that \( \lambda(w) = x_{\nu+1} \) for any \( \lambda \in X \setminus X_{\text{Aut}} \). Therefore \( |X \setminus X_{\text{Aut}}| = 1 \), and \( b(U, T) = |X| = \frac{2(n+1)}{7} + 1 \).

(c) Since \( d_1(U^+) = |\Phi| \), it follows from equation (10) that \( |X_{\text{Aut}}| = |\Phi| = \frac{n+1}{4} + 1 \). It is easy to see that \( U - x_2 \cong U - x_{c-2} \cong T - x_{\nu+1} \), and to check that to check using Lemma 3.5 that \( \lambda(w) = x_{\nu+1} \) for any \( \lambda \in X \setminus X_{\text{Aut}} \). Therefore \( |X \setminus X_{\text{Aut}}| = 1 \), and \( b(U, T) = |X| = \frac{n+1}{4} + 1 \).

For (d), (e), and (f), it is easy to check using Lemma 3.5 that \( \lambda(w) \in V(\text{skel}(U^+)) \) for any \( \lambda \in X \setminus X_{\text{Aut}} \). Furthermore, it is easy to see that \( U - x_i \not\cong T - x_j \), for all \( i, j \), and hence \( X = X_{\text{Aut}} \) in each case.

Figure 1: The supercard \( U^+ \) with structure (S0a) when \( n = 17 \) and structure (S0b) when \( n = 20 \).
(d) Since $d_1(U^+) = 2|\Phi|$, it follows from equation (10) that $|X_{\text{Aut}}| = 2|\Phi| = \frac{2(n+1)}{5}$. Therefore $b(U, T) = |X| = \frac{2(n+1)}{5}$.

(e) Since $d_1(U^+) = |\Phi|$, it follows from equation (10) that $|X_{\text{Aut}}| = |\Phi| = \frac{n+1}{5}$. Therefore $b(U, T) = |X| = \frac{n+1}{5}$.

(f) Since $d_1(U^+) = 2|\Phi|$, it follows from equation (10) that $|X_{\text{Aut}}| = 2|\Phi| = \frac{2(n+1)}{7}$. Therefore $b(U, T) = |X| = \frac{2(n+1)}{7}$.

For each of the following four lemmas, we assume that there exists a non-trivial rotation in $\text{Aut}(U^+)$, and we let $X$ be the $U^+$-optimum saturating set of $B_{vw}(U^+)$ from Theorem 5.8. We recall that $\tilde{X}$ is the subset of $X$ containing those permutations $\lambda$ such that $\lambda(w)$ is a leaf of $U^+$ and a $d$-leaf of $T$. We note that $X_U = X_{\text{Aut}}$ by Lemma 4.13.

Lemma 6.3 Let $n \geq 56$ and suppose that $\tau_{U^+}(x_\nu) = 0$.

(a) If $X_{\text{Aut}} = \Phi$ then $b(U, T) = |X| \leq \frac{n+1}{3} + 2$. Furthermore, equality holds if and only if $U^+$ has structure (S0a).

(b) If $X_{\text{Aut}} = \Phi \cup \Phi\pi$ then $b(U, T) = |X| \leq \frac{2(n+1)}{7} + 1$. Furthermore, equality holds if and only if $U^+$ has structure (S0b).

Proof It follows from Corollary 3.13(a) that $\tilde{X} \subseteq X_{\text{Aut}}$, so $X \setminus X_{\text{Aut}} \subseteq X \setminus \tilde{X}$. Therefore $|X \setminus X_{\text{Aut}}| \leq 4$ by Corollary 3.13(a).

(a) By Lemma 5.10(a), $|X_{\text{Aut}}| \leq \left\lfloor \frac{n+1}{3} \right\rfloor$. Simple calculations then show that the bound holds for $\delta \geq 3$ with strict inequality.
So suppose that $\delta = 2$. If $d_{U^+}(x_0) \geq 4$, then $d_1(U^+) \geq 2|\Phi|$, so $|X_{Aut}| = |\Phi| \leq \left\lceil \frac{\nu + 1}{6} \right\rceil$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 3$ then $U^+$ has structure $(S0a)$, so $b(U, T) = \frac{\nu + 1}{3} + 2$ by Lemma 6.2(a). Therefore, the bound holds in all cases, and is attained if and only if $U^+$ has structure $(S0a)$.

(b) By Lemma 5.9(a), $U^+$ contains $2|\Phi|$ vertices $x_i$ with $\tau_{U^+}(x_i) = 2$. As $\tau_{U^+}(x_\nu) = 0$, this implies that $\delta \geq 4$. In addition, it is easy to see by inspection, that $d_1(U^+) \geq 3|\Phi|$ when $\delta \geq 5$.

Suppose that $\delta \geq 5$. Then $|X_{Aut}| \leq 2 \left\lceil \frac{\nu + 1}{6} \right\rceil$ by (10). Simple calculations then show that the bound holds with strict inequality when $\delta \geq 6$ or $|X \setminus X_{Aut}| = 3$. So suppose that $\delta = 5$ and $|X \setminus X_{Aut}| = 4$. In this case, it is easy to see from Lemma 3.12(a) that $\{x_{\nu-1}, x_{\nu+1}\}$ is the set $B$ of leaves defined in Lemma 5.10. Moreover, since $\delta \geq 5$, we may clearly put $A = B$ in the lemma. Hence $|X_{Aut}| \leq 2 \left\lceil \frac{\nu + 1}{12} \right\rceil$ by Lemma 5.10(b), and again simple calculations show that the bound holds with strict inequality.

So suppose that $\delta = 4$. Then $d_{U^+}(x_{\nu-1}) > d_{U^+}(x_{\nu+1}) \geq 3$ and $d_{U^+}(x_{\nu+2}) = 2$. Now, if $d_{U^+}(x_{\nu+1}) \geq 4$, then $d_1(U^+) \geq 5|\Phi|$ and $|X_{Aut}| \leq 2 \left\lceil \frac{\nu + 1}{6} \right\rceil$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_{\nu-1}) = 4$ and $d_{U^+}(x_{\nu+1}) = 3$, then $U^+$ has structure $(S0b)$, so $b(U, T) = \frac{2(\nu + 1)}{3} + 1$ by Lemma 6.2(b). Hence the bound holds in all cases, and is attained if and only if $U^+$ has structure $(S0b)$.

\[ \square \]

Lemma 6.4 Let $n \geq 60$ and suppose that $\tau_{U^+}(x_\nu) = 2$.

(a) If $X_{Aut} = \Phi$ then $b(U, T) \leq \frac{\nu + 1}{5}$. Furthermore, equality holds in the bound if and only if $U^+$ has structure $(S2a)$.

(b) If $X_{Aut} = \Phi \cup \Phi_T$ then $b(U, T) \leq \frac{2(\nu + 1)}{7}$. Furthermore, equality holds in the bound if and only if $U^+$ has structure $(S2b)$.

Proof. It follows from Corollary 3.18 that $\tilde{X} \subseteq X_{Aut}$, so $X \setminus X_{Aut} \subseteq X \setminus \tilde{X}$. Therefore, $|X \setminus X_{Aut}| \leq 2$ by Corollary 3.13(c).

(a) By Lemma 5.10(a), $|X_{Aut}| \leq \left\lceil \frac{\nu + 1}{6} \right\rceil$. Simple calculations then show that the bound holds for $\delta \geq 5$ with strict inequality.

So suppose that $\delta \leq 4$. By Lemma 5.9(a), $U^+$ contains $|\Phi|$ vertices $x_i$ with $\tau_{U^+}(x_i) = 2$. This implies that $\delta = 4$, and that every cut-vertex of $U^+$ is in $\Phi(x_0)$. Now, if $d_{U^+}(x_0) \geq 4$ then $d_1(U^+) \geq 2|\Phi|$, so $|X_{Aut}| \leq \left\lceil \frac{\nu + 1}{6} \right\rceil$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 3$ then $U^+$ has structure $(S2a)$, so $b(U, T) = \frac{\nu + 5}{5}$ by Lemma 6.2(e). Therefore, the bound holds in all cases, and is attained if and only if $U^+$ has structure $(S2a)$.

(b) By Lemma 5.10(b), $|X_{Aut}| \leq 2 \left\lceil \frac{\nu + 1}{12} \right\rceil$. Simple calculations then show that the bound holds for $\delta \geq 6$ with strict inequality.

So suppose that $\delta \leq 5$. By Lemma 5.9(a), $U^+$ contains $2|\Phi|$ vertices $x_i$ with $\tau_{U^+}(x_i) = 2$. This implies that $\delta = 5$, and that every cut-vertex of $U^+$ is in $\Phi(x_0)$. Since $U^+$ contains at least $2|\Phi|$ leaves by Lemma 5.9(b), clearly $d_{U^+}(x_0) \geq 4$. Now, if $d_{U^+}(x_0) \geq 5$ then $d_1(U^+) \geq 3|\Phi|$, so $|X_{Aut}| \leq 2 \left\lceil \frac{\nu + 1}{8} \right\rceil$ by (10). Simple calculations then show that the bound holds with strict inequality in this case. On the other hand, if $d_{U^+}(x_0) = 4$ then $U^+$ has structure $(S2b)$, so $b(U, T) = \frac{2(\nu + 1)}{7}$ by Lemma 6.2(f). Therefore, the bound holds in all cases, and is attained if and only if $U^+$ has structure $(S2b)$.

\[ \square \]
Since $\bar{X}$ may not be contained in $X_{\text{Aut}}$ when $\tau_{U^+}(x_\nu) = 1$, we need an auxiliary result in this case.

**Lemma 6.5** Suppose that $\lambda \in \bar{X} \setminus X_{\text{Aut}}$ and that $\delta \leq \frac{c-3}{2}$. Then $\lambda(w)$ is adjacent to $x_{\nu+2}$ and $\lambda^{-1}(x_\nu) \in \{x_2, x_{c-2}\}$.

**Proof** Let $x_\mu = \lambda^{-1}(x_\nu)$. By Corollary 3.18 $\tau_{U^+}(x_\nu) = 1$ and $\{\lambda(x_{\mu+2}), \lambda(x_{\mu-2})\} = \{x_{\nu+3}, x_{\nu-1}\}$ as $\lambda \in \bar{X} \setminus X_{\text{Aut}}$. Moreover, by Lemma 3.10 either

(a) $\text{ske}(U - x_\mu) = x_{\nu+1}x_{\mu+2}...x_{\mu-2}$ and $\lambda(x_i) = x_{(\nu-i)}$ for all $x_i$ in $V(\text{ske}(U - x_\mu))$, or

(b) $\text{ske}(U - x_\mu) = x_{\mu+2}x_{\mu+3}...x_{\mu-1}$ and $\lambda(x_i) = x_{(\nu+i)}$ for all $x_i$ in $V(\text{ske}(U - x_\mu))$.

We recall that $d_{U^+}(x_i) = d_{U^+}(x_{i+\delta}) = d_{U^+}(x_{i-\delta})$ for all $x_i$, as $\phi(x_i) = x_{i+\delta}$.

Case (a): We first note that $\{x_{\mu+1+\delta}, x_{\mu+1+2\delta}, x_{\mu-2-\delta}\} \subseteq V(\text{ske}(U - x_\mu))$ as $\delta \leq \frac{c-3}{2}$. Suppose that $\lambda(w)$ is not adjacent to $x_{\nu+2}$. Then

$$d_{U^+}(x_{\mu+1}) > d_{U^+}(x_{\mu+1}) = d_{T-\lambda(w)}(\lambda(x_{\mu+1})) = d_{T-\lambda(w)}(x_{\nu+2}) = d_{U^+}(x_{\nu+2}).$$

So $d_{U^+}(x_{\mu+1+\delta}) > d_{U^+}(x_{\nu+2+\delta})$ and $d_{U^+}(x_{\mu+1+2\delta}) = d_{U^+}(x_{\nu+2+2\delta})$. However, since $\lambda(x_{\mu+1+\delta}) = x_{\nu+2+\delta}$ and $\lambda(x_{\mu+1+2\delta}) = x_{\nu+2+2\delta}$, we see that $d_{U^+}(x_{\mu+1+\delta}) = d_{T-\lambda(x_{\mu+1+\delta})}$ and $d_{U^+}(x_{\mu+1+2\delta}) = d_{T-\lambda(x_{\mu+1+2\delta})}$. Since $x_\mu$ is not adjacent to $x_{\mu+1+\delta}$ or $x_{\mu+1+2\delta}$, it follows that $w$ must be adjacent to both of these vertices. This is impossible since $w$ is a leaf. Therefore $\lambda(w)$ is adjacent to $x_{\nu+2}$.

Suppose now that $x_\mu$ is not $x_2$. Then $w$ is not adjacent to $x_{\nu-2}$, and thus

$$d_{U^+}(x_{\nu-1}) > d_{T-\lambda(x_{\nu-1})} = d_{U^+}(x_{\nu-2}) = d_{U^+}(x_{\nu-2}).$$

So $d_{U^+}(x_{\nu-1+\delta}) > d_{U^+}(x_{\nu-1+\delta})$. However, since $\lambda(x_{\nu-2+\delta}) = x_{\nu-1+\delta}$, it follows that $d_{U^+}(x_{\nu-1+\delta}) = d_{T-\lambda(x_{\nu-2+\delta})}$, which is impossible as neither $\lambda(w)$ nor $x_\nu$ are adjacent to $x_{\nu-1+\delta}$. Therefore $x_\mu$ is $x_2$.

Case (b) can be proved in a similar manner by replacing $x_{\mu+k}$ by $x_{\mu-k}$ for each $k$, and vice versa, and also substituting $x_{c-2}$ for $x_2$.

**Lemma 6.6** Let $n \geq 48$ and suppose that $\tau_{U^+}(x_{\nu}) = 1$.

(a) If $X_{\text{Aut}} = \Phi$ then $b(U, T) \leq \frac{n+4}{3} + 1$. Furthermore, equality holds if and only if $U^+$ has structure (S1a).

(b) If $X_{\text{Aut}} = \Phi \cup \Phi \pi$ then $b(U, T) \leq \frac{2(n+1)}{3}$. Furthermore, equality holds if and only if $U^+$ has structure (S1b).

**Proof** We first note that, $\delta \geq 3$, as $\tau_{U^+}(x_\nu) \geq 1$. When $\delta = 3$, it is easy to see that there exists a $\psi \in \text{Ref}(\text{Aut}(U^+))$ such that $\psi(x_\nu) = x_{\nu+1} \notin \Phi(x_\nu)$, and $\psi(x_0) \in \Phi(x_0)$. Thus $\text{Ref}(\text{Aut}(U^+))(x_\nu) \notin \Phi(x_\nu)$, whereas $\text{Ref}(\text{Aut}(U^+))(x_0) \subseteq \Phi(x_0)$ by Lemma 5.7(c).

Suppose that $\delta \geq \frac{c-2}{2}$, so $|\Phi| = \frac{c}{2} \leq 2$. Then, since $|X_{\text{Aut}}| = \chi(U^+)$, it follows from Theorem 3.23 that $b(U, T) \leq 9$ when $X_{\text{Aut}} = \Phi$, and $b(U, T) \leq 13$ when $X_{\text{Aut}} = \Phi \cup \Phi \pi$. Simple calculations show that both bounds hold with strict inequality.

We therefore assume that $\delta \leq \frac{c-2}{2}$. Now, if $\lambda \in \bar{X} \setminus X_{\text{Aut}}$ then $\lambda^{-1}(x_\nu) \in \{x_2, x_{c-2}\}$ by Lemma 6.3, hence $|\bar{X} \setminus X_{\text{Aut}}| \leq 2$ by Definition 2.3(b). Therefore, since $|X \setminus \bar{X}| \leq 4$ by Corollary 3.13(b), clearly $|X \setminus X_{\text{Aut}}| \leq 6$. 26
Let $B$ be the set of leaves defined in Lemma 5.10. Then, by Lemma 3.12(b) and Lemma 6.3, $B \subseteq \{x^{i}_{v-1}, x^{i}_{v+2}, x^{k}_{v+2}\}$, for some $j, k$. On the other hand, if $\lambda \in X \setminus X_{\text{Aut}}$ but $\lambda(w) \notin B$ then $\lambda(w) \in \{x_{v-1}, x_{v+1}, x_{v+2}\}$. It follows that if $|X \setminus X_{\text{Aut}}| \geq 4$, then there exists a subset $A$ of $B$ with $|A| \geq 1$ that satisfies the conditions of Lemma 5.10. In addition, if $\delta \geq 4$ and $|X \setminus X_{\text{Aut}}| \geq 5$, it is easy to see that there is some subset $A$ of $B$ with $|A| \geq 2$ that satisfies the conditions of Lemma 5.10.

(a) Suppose that $\delta \geq 4$. Then, by Lemma 5.10(a), $|X_{\text{Aut}}| \leq \left\lceil \frac{n+1}{3} \right\rceil$ when $|X \setminus X_{\text{Aut}}| \leq 3$, $|X_{\text{Aut}}| \leq \left\lceil \frac{n+4}{6} \right\rceil$ when $|X \setminus X_{\text{Aut}}| = 4$, and $|X_{\text{Aut}}| \leq \left\lceil \frac{n+1}{4} \right\rceil$ otherwise. Simple calculations then show that the bound holds with strict inequality.

(b) Suppose that $\delta \geq 4$. Then, by Lemma 5.10(b), $|X_{\text{Aut}}| \leq 2 \left\lceil \frac{n+1}{6} \right\rceil$ when $|X \setminus X_{\text{Aut}}| \leq 3$, $|X_{\text{Aut}}| \leq 2 \left\lceil \frac{n+4}{7} \right\rceil$ when $|X \setminus X_{\text{Aut}}| = 4$, and $|X_{\text{Aut}}| \leq 2 \left\lceil \frac{n+1}{8} \right\rceil$ otherwise. Simple calculations then show that the bound holds with strict inequality.

So, suppose that $\delta = 3$. Then, since $X_{\text{Aut}} = \Phi$ and $\text{Ref}(\text{Aut}(U^+))(x_{v}) \nsubseteq \Phi(x_{v})$, it follows from Theorem 5.8 that $d_{U^+}(x_{0}) = 3$. So $U^+$ has structure (S1a) and $b(U, T) = \frac{n+1}{4} + 1$ by Lemma 6.2(c). Therefore, the bound holds in all cases, and is attained if and only if $U^+$ has structure (S1a).

So, suppose that $\delta = 3$. Then, since $X_{\text{Aut}} = \Phi \cup \Phi_{\pi}$, $\text{Ref}(\text{Aut}(U^+))(x_{v}) \nsubseteq \Phi(x_{v})$ and $\text{Ref}(\text{Aut}(U^+))(x_{0}) \nsubseteq \Phi(x_{0})$, it follows from Theorem 5.8 that $d_{U^+}(x_{0}) = 4$. Now, by Lemma 5.9(a), $U^+$ contains $|X_{\text{Aut}}(x_{v})|$ vertices $x_{i}$ with $d_{U^+}(x_{i}) = 2$. As $d_{U^+}(x_{0}) = 4$, it then follows from Lemma 5.9(c) that if $\lambda \in X \setminus X_{\text{Aut}}$ then $\lambda^{-1}(x_{v})$ must be a cut-vertex of $U$. It is easy to see by inspection that there can be no such $\lambda$, and therefore $X = X_{\text{Aut}}$. The bound then holds by Lemma 5.10(b) with $|A| = 0$, with equality holding if and only if $d_{\lambda}(U^+) = |\Phi|$, i.e., when $d_{U^+}(x_{0}) = 4$. In this case, $U^+$ has structure (1b), and $b(U, T) = \frac{2(n+1)}{5}$ by Lemma 6.2(d). Therefore, the bound holds in all cases, and is attained if and only if $U^+$ has structure (S1b).

By combining the above results with Theorem 4.22 we finally obtain a bound on $b(U, T)$ which holds in all cases.

**Theorem 6.7** Let $U$ be a sunshine graph and $T$ be a caterpillar of order $n$, where $n \geq 60$. Suppose there exists a sunshine graph $U^+$ that is a supercard of $U$ and $T$ such that $\text{Aut}(U^+)$ contains a non-trivial rotation. Then $b(U, T) \leq \frac{2(n+1)}{5}$, with equality if and only if $U^+$ has structure (S1b), in which case $n \equiv 4 \pmod{5}$. Moreover, in all other cases, $b(U, T) \leq \frac{n+1}{3} + 2$.

**Proof** This follows immediately from Lemmas 6.1 to 6.6.

We note that, with more work, we can show that this bound holds for smaller values of $n$ (this is relatively straightforward for $n \geq 35$). However, since the proofs are slightly technical, in the interests of brevity, we have not included them in this paper.

**Theorem 6.8** Let $U$ be a sunshine graph and $T$ be a caterpillar, where $n \geq 62$. Then $b(U, T) \leq \frac{2(n+1)}{5}$, with equality if and only if there is a supercard of $U$ and $T$ that has structure (S1b), in which case $n \equiv 4 \pmod{5}$. Moreover, in all other cases, $b(U, T) \leq \frac{n+1}{3} + 2$ when $n \geq 74$.

**Proof** We may clearly assume that $b(U, T) \geq 10$. By Lemma 3.3, there exists a supercard $U^+$ of $U$ and $T$ that is a sunshine graph, and we may choose $U^+$ to have the largest possible value of $\chi(U^+)$ over all supercards of $U$ and $T$ that are sunshine graphs. If $B_{U}(U^+) = \text{Aut}(U^+)$
then $\text{Aut}(U^+)$ contains a non-trivial rotation by Corollary 4.14(c), so the results follow from Theorem 6.7. When $B_U(U^+) \neq \text{Aut}(U^+)$, they follow from Theorem 4.12 by straightforward calculation. □

References


