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Generalised Wendland functions for the sphere

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Abstract

In this paper, we compute the spherical Fourier expansion coefficients for the restriction of the generalised Wendland functions from d -dimensional Euclidean space to the $(d - 1)$ -dimensional unit sphere. We use results from the theory of special functions to show that they can be expressed in a closed form as a multiple of a certain ${}_3F_2$ hypergeometric function. We present tight asymptotic bounds on the decay rate of the spherical Fourier coefficients and, in the case where d is odd, we are able to provide the precise asymptotic rate of decay. Numerical evidence suggests that this precise asymptotic rate also holds when d is even and we pose this as an open problem. Finally, we observe a close connection between the asymptotic decay rate of the spherical Fourier coefficients and that of the corresponding Euclidean Fourier transform.

Keywords Positive definite kernels · Spherical basis functions · Compact support

Mathematics Subject Classification (2010) 33B10 · 33C45 · 42A16 · 42A82 · 42C10

1 Introduction

Positive definite functions are frequently used in scattered data fitting algorithms both in Euclidean space and on spheres: see [31]. The special case of the 2-sphere is of importance in geostatistics where positive definite functions are used as covariance functions of random fields on the surface of the earth, [8, 10, 15, 18] and [11]. The aim of this paper is to investigate the generalised Wendland functions, a parameterised family of compactly supported basis functions which, for a certain parameter range, are strictly positive definite on \mathbb{R}^d . In the opening section, we present the background to positive definite kernels defined on Euclidean space and on the unit

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sphere. We will pay particular attention to the restriction of Euclidean positive definite functions to the sphere and state an identity that connects the Fourier transform of the Euclidean function to the spherical Fourier coefficients of its restriction to the sphere. In Section 3, we introduce the family of generalised Wendland functions, setting out their known properties including their Euclidean Fourier transforms and the rates at which they decay. Then, in Section 4, we make use of the aforementioned connection identity to derive an expression for the spherical Fourier coefficients of the generalised Wendland functions restricted to the sphere. In the final section, we examine the asymptotic rate of decay of the spherical Fourier coefficients where we provide tight asymptotic bounds. In addition, when the functions are restricted to an even dimensional sphere, we provide the precise asymptotic rate of decay and present numerical results to suggest that the same decay holds for the restriction to odd-dimensional spheres too.

The class of generalised Wendland functions, as their name suggests, contains the original Wendland functions, which are popular in applications due to their simple polynomial form. Many researchers have employed the original Wendland functions on the sphere and, for these functions, the asymptotic behaviour of the spherical Fourier coefficients has been addressed. However, as far as the authors are aware, a precise formula for the coefficients has not previously been made available. A closed form of the coefficients is necessary for the use of recently proposed numerical methods such as the stable computation via Hilbert-Schmidt SVD ([12, Chapter 13]) and also for the spectral simulation of Gaussian random fields as described in [17].

Of the work that is related to ours, we mention [22, Proposition 3.1] which gives a precise asymptotic form for the spherical Fourier coefficients of the original Wendland functions; however, the constant multiplying the decay factor is not explicitly given. In addition, le Gia et al. [19, Section 6] have considered scaled versions of positive definite functions and show that if their Fourier transforms decay at a polynomial rate then, when restricted to the sphere, their corresponding Fourier coefficients decay at the expected analogous rate; however, precise asymptotic rates are not given.

2 Radial and zonal kernels

Definition 2.1 A kernel $\Phi : \Omega \times \Omega \rightarrow \mathbb{R}$ is said to be strictly positive definite on a domain Ω , if, for any $n \geq 2$ distinct locations $\mathbf{x}_1, \dots, \mathbf{x}_n \in \Omega$, the $n \times n$ matrix

$$\left(\Phi(\mathbf{x}_j, \mathbf{x}_k) \right)_{j,k=1}^n \quad (2.1)$$

is symmetric and positive definite.

For the class of radial kernels taking the form $\Phi(x, y) = \phi(\|x - y\|)$, we have the following characterisation theorem (see [31, Theorem 6.18]).

Theorem 2.2 A radial kernel $\Phi(\mathbf{x}, \mathbf{y}) = \phi(\|\mathbf{x} - \mathbf{y}\|)$ with $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $r \mapsto r^{d-1}\phi(r) \in L_1[0, \infty)$ is strictly positive definite in \mathbb{R}^d if and only if the d -dimensional Fourier transform

$$\widehat{\phi}(z) = z^{1-\frac{d}{2}} \int_0^\infty \phi(y) y^{\frac{d}{2}} J_{\frac{d}{2}-1}(yz) dy, \tag{2.2}$$

where $J_\nu(\cdot)$ denotes the Bessel function of the first kind with order ν , is non-negative and not identically equal to zero.

We let Φ_d denotes the class of continuous functions ϕ associated to the strictly positive definite radial kernel via $\Phi(\mathbf{x}, \mathbf{y}) = \phi(\|\mathbf{x} - \mathbf{y}\|)$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. If we assume that $\widehat{\phi}(z)$ (2.2) is positive for all $z \geq 0$, then we can appeal to the theory of radial basis functions (see [31]) to deduce the following result.

Theorem 2.3 Let $d \geq 1$ denote a fixed spatial dimension and Φ be a strictly positive definite radial kernel on \mathbb{R}^d with $\widehat{\phi}(z) > 0$ for all $z \geq 0$. Define

$$N_\phi := \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_\phi^2 = \int_{\mathbb{R}^d} \frac{|\widehat{f}(\boldsymbol{\omega})|^2}{\widehat{\phi}(\|\boldsymbol{\omega}\|)} d\boldsymbol{\omega} < \infty \right\}, \tag{2.3}$$

where $\|\cdot\|_\phi$ is a norm induced by the inner product

$$(f, g)_\phi := \int_{\mathbb{R}^d} \frac{\widehat{f}(\boldsymbol{\omega})\widehat{g}(\boldsymbol{\omega})}{\widehat{\phi}(\|\boldsymbol{\omega}\|)} d\boldsymbol{\omega}. \tag{2.4}$$

Then N_ϕ is a real Hilbert space with inner product $(\cdot, \cdot)_\phi$ and reproducing kernel Φ .

Remark 2.4 We note that if there exist positive constants $\kappa_1 < \kappa_2$ such that

$$\frac{\kappa_1}{(1+z^2)^\lambda} \leq \widehat{\phi}(z) \leq \frac{\kappa_2}{(1+z^2)^\lambda}, \quad z \geq 0, \tag{2.5}$$

where $\lambda > \frac{d}{2}$, then the space N_ϕ in (2.3) is a reproducing kernel Hilbert space which is isomorphic to the Sobolev space

$$H^\lambda(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d) : \|f\|_{H^\lambda(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1+\|\boldsymbol{\omega}\|^2)^\lambda d\boldsymbol{\omega} < \infty \right\}.$$

Furthermore, under the same assumption, it can be shown that N_ϕ and $H^\lambda(\mathbb{R}^d)$ are norm-equivalent (see [32]).

We now consider the spherical case. We let $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the $(d - 1)$ -dimensional unit sphere. Then, for any points $\boldsymbol{\xi}, \boldsymbol{\eta} \in S^{d-1}$, we write $\boldsymbol{\xi}^T \boldsymbol{\eta} = \cos(\theta)$ to denote their dot-product where T denotes the transpose of a vector and θ is the angular distance between the two points.

The zonal kernel $\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) = \psi(\boldsymbol{\xi}^T \boldsymbol{\eta})$ induced by any continuous function $\psi : [-1, 1] \rightarrow \mathbb{R}$, possesses a Fourier-type expansion in spherical harmonics

$$\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) = \psi(\boldsymbol{\xi}^T \boldsymbol{\eta}) = \sum_{m=0}^\infty \sum_{n=1}^{N_{m,d}} \widehat{\psi}_m \mathcal{Y}_{m,n}(\boldsymbol{\xi}) \mathcal{Y}_{m,n}(\boldsymbol{\eta}), \tag{2.6}$$

where $\{\mathcal{Y}_{m,n} : n = 1, \dots, N_{m,d}\}$ is a real orthonormal basis for the space of spherical harmonics of degree m and the collection $\{\mathcal{Y}_{m,n} : n = 1, \dots, N_{m,d}, m \geq 0\}$ forms a real orthonormal basis for $L_2(S^{d-1})$.

Using Schoenberg’s [30] pioneering work, it can be shown that if the expansion coefficients $\widehat{\psi}_m$ are positive for $m \geq 0$, then Ψ is a strictly positive definite kernel on S^{d-1} . This simple condition is sufficient for our purposes but the reader may consult [7] for a careful investigation of the necessary and sufficient conditions.

Spherical harmonics provide a Fourier analysis for the sphere. In particular, every $f \in L_2(S^{d-1})$ has an associated spherical Fourier expansion

$$f = \sum_{m=0}^{\infty} \sum_{n=1}^{N_{m,d}} \widehat{f}_{n,m} \mathcal{Y}_{m,n},$$

$$\text{where } \widehat{f}_{n,m} = (f, \mathcal{Y}_{m,n})_{L_2(S^{d-1})} = \int_{S^{d-1}} f(\xi) \mathcal{Y}_{m,n}(\xi) d\omega_{d-1}(\xi).$$

The following theorem is the spherical analogue of Theorem 2.3.

Theorem 2.5 *Let $d \geq 2$ denotes a fixed spatial dimension and Ψ a strictly positive definite zonal kernel on S^{d-1} for which the Fourier expansion coefficients $\widehat{\psi}_m$ are strictly positive for all $m \geq 0$. Define*

$$N_{\psi} = \left\{ f \in L_2(S^{d-1}) : \|f\|_{\psi}^2 = \sum_{m=0}^{\infty} \sum_{n=1}^{N_{m,d}} \frac{|\widehat{f}_{n,m}|^2}{\widehat{\psi}_m} < \infty \right\}, \tag{2.7}$$

where $\|\cdot\|_{\psi}$ is a norm induced by the inner-product

$$(f, g)_{\psi} := \sum_{m=0}^{\infty} \sum_{n=1}^{N_{m,d}} \frac{\widehat{f}_{n,m} \widehat{g}_{n,m}}{\widehat{\psi}_m}.$$

Then N_{ψ} is a real Hilbert space with inner product $(\cdot, \cdot)_{\psi}$ and reproducing kernel $\Psi(\xi, \eta)$.

Remark 2.6 We note that if there exist positive constants $c_1 < c_2$ such that

$$\frac{c_1}{(1+m^2)^{\lambda}} \leq \widehat{\psi}_m \leq \frac{c_2}{(1+m^2)^{\lambda}}, \quad m \geq 0, \tag{2.8}$$

where $\lambda > \frac{d-1}{2}$, then the space N_{ψ} defined in the previous theorem is norm equivalent to the Sobolev space

$$H^{\lambda}(S^{d-1}) := \left\{ f \in L_2(S^{d-1}) \cap C(S^{d-1}) : \sum_{m=0}^{\infty} \sum_{n=1}^{N_{m,d}} (1+m^2)^{\lambda} |\widehat{f}_{n,m}|^2 < \infty \right\}.$$

We let Ψ_{d-1} denotes the class of continuous functions ψ associated to the strictly positive definite zonal kernel $\Psi(\xi, \eta)$ via (2.6). The class Ψ_{d-1} is important in the field of spatial statistics where certain elements can be proposed as correlation functions that induce geodesically isotropic covariance functions. The notation used in

the spatial statistics literature can easily be retrieved from that presented above. Specifically, the addition theorem for spherical harmonics [16, equation (1.24)] states

$$\sum_{n=1}^{N_{m,d}} \mathcal{Y}_{m,n}(\xi)\mathcal{Y}_{m,n}(\eta) = \frac{N_{m,d}}{\omega_{d-1}} \frac{P_m^{(d-2)/2}(\xi^T \eta)}{P_m^{(d-2)/2}(1)}, \quad \xi, \eta \in S^{d-1}, \quad (2.9)$$

where P_m^λ denotes the λ -Gegenbauer polynomial of degree m , ω_{d-1} denotes the surface area of S^{d-1} and $N_{m,d}$ denotes the dimension of the space of spherical harmonics of degree m given by

$$N_{0,d} = 1 \quad \text{and} \quad N_{m,d} = (2m + d - 2) \frac{(m + d - 3)!}{(d - 2)!m!}. \quad (2.10)$$

Using this, we can see that if ψ belongs to Ψ_{d-1} , then it satisfies

$$\psi(\cos(\theta)) = \sum_{m=0}^{\infty} b_m \frac{P_m^{(d-2)/2}(\cos \theta)}{P_m^{(d-2)/2}(1)}, \quad \theta \in [0, \pi], \quad (2.11)$$

where following Daley and Porcu [9], the sequence $\{b_m\}_{m \geq 0}$ in (2.11) is called the $(d - 1)$ -Schoenberg sequence. This sequence is related to the spherical Fourier coefficients (2.6), using (2.9), by

$$b_m = \frac{N_{m,d}}{\omega_{d-1}} \widehat{\psi}_m, \quad m = 0, 1, 2, \dots \quad (2.12)$$

In the framework of Gaussian random fields $Z = \{Z(\xi) : \xi \in S^{d-1}\}$ that are continuously indexed over S^{d-1} , we suppose Z has constant mean and covariance function $\mathcal{C}(\xi_1, \xi_2) = \text{cov}\{Z(\xi_1), Z(\xi_2)\}$, for $\xi_1, \xi_2 \in S^{d-1}$. If ψ belongs to Ψ_{d-1} and additionally satisfies $\psi(1) = \psi(\cos(0)) = 1$, then it can be proposed as a correlation function that induces a geodesically isotropic covariance function via

$$\mathcal{C}(\xi_1, \xi_2) = \sigma^2 \psi(\xi_1^T \xi_2), \quad \xi_1, \xi_2 \in S^{d-1}, \quad (2.13)$$

where $\sigma^2 > 0$ is the variance of Z . A wide range of parametric families of correlation functions have been proposed for applications in spatial statistics, see for instance, [8, 10, 15] and [11].

A source of functions from Ψ_{d-1} can easily be accessed by taking $\phi \in \Phi_d$, then using the relation $\|\xi - \eta\| = \sqrt{2 - 2\xi^T \eta}$, for $\xi, \eta \in S^{d-1}$, one can define its restriction to S^{d-1} by

$$\Psi(\xi, \eta) = \psi(\xi^T \eta) = \phi\left(\sqrt{2 - 2\xi^T \eta}\right). \quad (2.14)$$

In this regard, we have the following formula [23, Theorem 4.1] which links the spherical Fourier coefficients of Ψ to the radial Fourier transform $\widehat{\phi}(z)$,

$$\widehat{\psi}_m := (2\pi)^{\frac{d}{2}} I\left(m + \frac{d-2}{2}\right) \quad \text{where} \quad I(\gamma) = \int_0^\infty z J_\gamma^2(z) \widehat{\phi}(z) dz. \quad (2.15)$$

In the spatial statistic literature, the above approach is known as the chordal distance based model. It has received constructive criticism around its flexibility, mainly

due to the fact that the chordal (Euclidean) distance underestimates the angular distance between points on the sphere (see [26] and [15] for further discussion).

3 The generalised Wendland functions

We will investigate a family of parameterised basis functions that is generated by a truncated power function. Specifically, we choose a support parameter $\epsilon > 0$ and define

$$\phi_{\mu,0}^{(\epsilon)}(r) := (1 - \epsilon r)_+^\mu = \begin{cases} (1 - \epsilon r)^\mu & \text{for } 0 \leq r \leq \frac{1}{\epsilon}; \\ 0 & \text{for } r \geq \frac{1}{\epsilon}, \end{cases}$$

and consider

$$\phi_{\mu,\alpha}^{(\epsilon)}(r) := \frac{1}{2^{\alpha-1}\Gamma(\alpha)} \int_{\epsilon r}^1 \phi_{\mu,0}^{(1)}(t) t (t^2 - (\epsilon r)^2)^{\alpha-1} dt, \quad r \geq 0, \tag{3.16}$$

where $\mu > 0$, $\alpha > 0$ and $\Gamma(\cdot)$ denote the Gamma function. We note that $\phi_{\mu,\alpha}^{(\epsilon)}$ is compactly supported on $[0, \frac{1}{\epsilon}]$.

This class is a subclass of the Buhmann functions originally introduced in [4] (see also [34]). This specific class was first studied by Zastavnyi in [33]. Applications of this function class as covariance functions of Gaussian random fields are discussed in [3].

For a certain selection of the μ and α parameters, we can recover, from formula (3.16), both the original Wendland functions and also the so-called missing Wendland functions. It is for this reason that elements of this family are referred to as the generalised Wendland functions. The original Wendland functions are recovered when the space dimension d is odd, α is a positive integer k and $\mu := \ell = \frac{d+1}{2} + k$, i.e. the smallest integer that still allows positive definiteness. In this case, one can show that

$$\phi_{\ell,k}^{(\epsilon)}(r) = p_k(\epsilon r)(1 - \epsilon r)_+^{\ell+k},$$

where p_k is a polynomial of degree k . The missing Wendland functions are recovered when the space dimension d is even, and $\mu := \ell = \frac{d}{2} + k$, i.e. once again the smallest integer that still allows positive definiteness. The missing Wendland functions have two polynomial components, one with a logarithmic multiplier $L(r) := \log\left(\frac{r}{1+\sqrt{1-r^2}}\right)$ and one with a square root multiplier $S(r) := \sqrt{1-r^2}$ (see [29]).

The d -dimensional Fourier transform of $\phi_{\mu,\alpha}^{(\epsilon)}$ was computed in [5] and is given by

$$\widehat{\phi_{\mu,\alpha}^{(\epsilon)}}(z) = \frac{C_{\lambda,\mu}}{\sqrt{2\pi}\epsilon^d} {}_1F_2\left[\begin{matrix} \lambda \\ \lambda + \frac{\mu}{2} \end{matrix}; -\left(\frac{z}{2\epsilon}\right)^2\right], \tag{3.17}$$

where

$$\lambda := \frac{d+1}{2} + \alpha \quad \text{and} \quad C_{\lambda,\mu} := \frac{2^\lambda \Gamma(\lambda) \Gamma(\mu+1)}{\Gamma(2\lambda + \mu)} \tag{3.18}$$

and where ${}_1F_2(a; b, c; z)$ denotes the hypergeometric function (see [1, 15.1.1]). Hypergeometric functions will feature heavily in this work and so we briefly remind the reader that a general hypergeometric function is defined by

$${}_pF_q \left[\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n z^n}{(b_1)_n \cdots (b_q)_n n!}, \tag{3.19}$$

where

$$(c)_n := c(c + 1) \cdots (c + n - 1) = \frac{\Gamma(c + n)}{\Gamma(c)} \text{ for } n \geq 1, \tag{3.20}$$

denotes the Pochhammer symbol, with $(c)_0 := 1$.

It is known (see [5]) that $\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) > 0$ if and only if $\mu \geq \lambda$. Thus, with such a choice, $\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(r)$ induces a strictly positive definite and compactly supported radial kernel on \mathbb{R}^d . The following formula, taken from [14], provides the asymptotic behaviour for ${}_1F_2$ hypergeometric functions for large argument t

$${}_1F_2 \left[\begin{matrix} \alpha \\ \rho\alpha & \rho\alpha + \frac{1}{2} \end{matrix}; -\frac{t^2}{4} \right] = \frac{\Gamma(2\rho\alpha)}{\Gamma(2\rho\alpha - 2\alpha)} \frac{1}{t^{2\alpha}} \left[1 + O(t^{-2}) \right] + \frac{\Gamma(2\rho\alpha)}{\Gamma(\alpha)} 2^{1-\alpha} \frac{1}{t^{2\rho\alpha-\alpha}} \left[\cos \left(t + \frac{\pi}{2} (\alpha - 2\alpha\rho) \right) + O(t^{-1}) \right].$$

Setting $\alpha := \lambda$, $\rho\alpha := \lambda + \frac{\mu}{2}$ and $t := \frac{z}{\epsilon}$, we deduce that

$${}_1F_2 \left[\begin{matrix} \lambda \\ \lambda + \frac{\mu}{2} & \lambda + \frac{\mu+1}{2} \end{matrix}; -\frac{z^2}{4\epsilon^2} \right] = \frac{\Gamma(2\lambda + \mu)}{\Gamma(\mu)} \left(\frac{\epsilon}{z} \right)^{2\lambda} \left[1 + O \left(\frac{\epsilon^2}{z^2} \right) \right] + \frac{\Gamma(2\lambda + \mu)}{\Gamma(\lambda)} \frac{1}{2^{\lambda-1}} \left(\frac{\epsilon}{z} \right)^{\lambda+\mu} \left[\cos \left(\frac{z}{\epsilon} - \frac{\pi}{2} (\lambda + \mu) \right) + O \left(\frac{\epsilon}{z} \right) \right] \tag{3.21}$$

as $\frac{z}{\epsilon} \rightarrow \infty$.

Using (3.17) and (3.18), we can also deduce that

$$\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) = \epsilon^{-d} \left[\frac{2^\lambda \Gamma(\lambda) \mu}{\sqrt{2\pi}} \left(\frac{\epsilon}{z} \right)^{2\lambda} \left[1 + O \left(\frac{\epsilon^2}{z^2} \right) \right] + \sqrt{\frac{2}{\pi}} \Gamma(\mu + 1) \left(\frac{\epsilon}{z} \right)^{\lambda+\mu} \left[\cos \left(\frac{z}{\epsilon} - \frac{\pi}{2} (\lambda + \mu) \right) + O \left(\frac{\epsilon}{z} \right) \right] \right]. \tag{3.22}$$

For $\mu > \lambda > 1$, it is the first term that determines the asymptotic decay and we can conclude that

$$\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \sim \frac{2^\lambda \Gamma(\lambda) \mu}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{z^{2\lambda}} \text{ as } \frac{z}{\epsilon} \rightarrow \infty. \tag{3.23}$$

For the case $\mu = \lambda > 1$, the first and second terms need to be considered for the asymptotic decay and we can conclude that

$$\widehat{\phi}_{\lambda,\alpha}^{(\epsilon)}(z) \sim \frac{2^\lambda \Gamma(\lambda + 1)}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{z^{2\lambda}} \left[1 + \frac{1}{2^{\lambda-1}} \cos \left(\frac{z}{\epsilon} - \pi\lambda \right) \right] \text{ as } z \rightarrow \infty. \tag{3.24}$$

To investigate the native space, we assume $\epsilon > 0$ to be fixed. We know from the definition of the asymptotics that $\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \sim g(z)$ is equivalent to

$$\gamma(z) := \frac{\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z)}{g(z)} \rightarrow 1 \quad \text{for } z \rightarrow \infty.$$

Since both $\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z)$ and $g(z)$ are strictly positive and continuous for $z > 0$, we can deduce that for any arguments larger than some $z_0 > 0$, the infimum, $c_1 > 0$, and supremum, $c_2 > 0$, of γ are well defined. This proves that

$$c_1 \leq \frac{\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z)}{\frac{2^\lambda \Gamma(\lambda)\mu}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{z^{2\lambda}}} \leq c_2, \quad z \geq z_0, \quad \mu > \lambda > 1, \tag{3.25}$$

and

$$c_1 \leq \frac{\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z)}{\frac{2^\lambda \Gamma(\lambda+1)}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{z^{2\lambda}} \left(1 + \frac{1}{2^{\lambda-1}} \cos\left(\frac{z}{\epsilon} - \pi\lambda\right)\right)} \leq c_2, \quad z \geq z_0, \quad \mu = \lambda > 1. \tag{3.26}$$

If we assume that $z_0 > 1$ then, for any $z \geq z_0$, we have that

$$0 < (1 + z^2) \left(1 - \frac{1}{z_0^2}\right) = z^2 - \left(\frac{z^2 - z_0^2 + 1}{z_0^2}\right) \leq z^2 \leq (1 + z^2),$$

and thus

$$0 < \frac{1}{(1 + z^2)^\lambda} \leq \frac{1}{z^{2\lambda}} \leq \left(1 - \frac{1}{z_0^2}\right)^{-\lambda} \cdot \frac{1}{(1 + z^2)^\lambda}, \quad \lambda > 1, \quad z \geq z_0 > 1.$$

From (3.25) we can now deduce that

$$\frac{\tilde{c}_1 \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda} \leq \widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \leq \frac{\tilde{c}_2 \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda}, \quad z \geq z_0, \quad \mu > \lambda > 1,$$

where

$$\tilde{c}_1 := \frac{2^\lambda \Gamma(\lambda)\mu}{\sqrt{2\pi}} c_1, \quad \text{and} \quad \tilde{c}_2 := \frac{2^\lambda \Gamma(\lambda)\mu}{\sqrt{2\pi}} \left(1 - \frac{1}{z_0^2}\right)^{-\lambda} c_2.$$

The above inequality can be extended to $z \geq 0$ (albeit with potentially different constants) since $\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z)$ is strictly positive and continuous on $[0, z_0]$. For the case $\mu = \lambda$, we can perform the same first step on (3.26) to show

$$\frac{\tilde{c}_1 \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda} \left(1 + \frac{\cos\left(\frac{z}{\epsilon} - \pi\lambda\right)}{2^{\lambda-1}}\right) \leq \widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \leq \frac{\tilde{c}_2 \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda} \left(1 + \frac{\cos\left(\frac{z}{\epsilon} - \pi\lambda\right)}{2^{\lambda-1}}\right),$$

for $z \geq z_0$. Now using $-1 \leq \cos(t) \leq 1$ and that $\lambda > 1$, we can simplify the above to write

$$\frac{c_1^* \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda} \leq \widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \leq \frac{c_2^* \epsilon^{2\alpha+1}}{(1 + z^2)^\lambda}, \quad z \geq z_0 > 1,$$

where c_1^* and c_2^* are positive constants given by

$$c_1^* := \left(1 - \frac{1}{2^{\lambda-1}}\right) \tilde{c}_1, \quad \text{and} \quad c_2^* := \left(1 + \frac{1}{2^{\lambda-1}}\right) \tilde{c}_2.$$

Again, the inequalities can be extended to $z \geq 0$ due to the continuity of the positive Fourier transform on $[0, z_0]$.

Summarising, we have proven that for $\mu \geq \lambda > 1$, there exist positive constants $\kappa_1 < \kappa_2$ such that

$$\frac{\kappa_1 \epsilon^{2\alpha+1}}{(1+z^2)^\lambda} \leq \widehat{\phi_{\mu,\alpha}^{(\epsilon)}}(z) \leq \frac{\kappa_2 \epsilon^{2\alpha+1}}{(1+z^2)^\lambda}, \quad z \geq 0. \tag{3.27}$$

We remark that the constants κ_1 and κ_2 are not necessarily fully independent of the choice of ϵ ; however, the given bounds allow us to use Theorem 2.3 and Remark 2.4 to deduce that when $\mu \geq \lambda$, $\Phi(\mathbf{x}, \mathbf{y}) = \phi_{\mu,\alpha}^{(\epsilon)}(\|\mathbf{x} - \mathbf{y}\|)$ defines a reproducing kernel of a Hilbert space that is norm equivalent to the Sobolev space $H^\lambda(\mathbb{R}^d)$.

For details on the influence of the scaling parameter on the bounding constants, we refer the reader to [32] Lemma 1, [6] Lemma 4.8 and also the recent paper [28].

4 Generalised Wendland functions for the sphere

In this section, we will consider the restriction of $\phi_{\mu,\alpha}^{(\epsilon)}(r)$ to the $(d - 1)$ -dimensional unit sphere S^{d-1} . Using (2.14) and (2.6), we can write

$$\phi_{\mu,\alpha}^{(\epsilon)}\left(\sqrt{2 - 2\xi^T \eta}\right) = \Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta) = \sum_{m=0}^{\infty} \sum_{n=1}^{N_{m,d}} \widehat{\psi_{\mu,\alpha}^{(\epsilon)}}(m) \mathcal{Y}_{m,n}(\xi) \mathcal{Y}_{m,n}(\eta). \tag{4.28}$$

We note that the support condition of the restriction can be recast in terms of angular distance θ (between $\xi, \eta \in S^{d-1}$) as

$$0 \leq \sqrt{2 - 2\cos(\theta)} \leq \frac{1}{\epsilon} \implies 1 - \frac{1}{2\epsilon^2} \leq \cos(\theta) \leq 1.$$

Thus, we need only consider the range $\epsilon \geq \frac{1}{2}$ for the support parameter; the case $\epsilon = \frac{1}{2}$ ensures that the restricted function is globally supported on the entire sphere, whereas $\epsilon > \frac{1}{2}$ ensures that it is supported on a spherical cap of radius $\theta = \cos^{-1}\left(1 - \frac{1}{2\epsilon^2}\right)$.

Theorem 4.7 *The spherical Fourier coefficients of the generalised Wendland function $\Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta)$, with $\epsilon \geq \frac{1}{2}$, are given by*

$$\widehat{\psi_{\mu,\alpha}^{(\epsilon)}}(m) = (2\pi)^{\frac{d-1}{2}} \frac{C_{\lambda-\frac{1}{2},\mu}}{\sqrt{2\pi}\epsilon^{d-1}} {}_3F_2\left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix}; \frac{1}{4\epsilon^2}\right], \tag{4.29}$$

where, in analogy with (3.18), we have that

$$\lambda := \frac{d + 1}{2} + \alpha \quad \text{and} \quad C_{\lambda-\frac{1}{2},\mu} := \frac{2^{\lambda-\frac{1}{2}}\Gamma\left(\lambda - \frac{1}{2}\right)\Gamma(\mu + 1)}{\Gamma(2\lambda + \mu - 1)}. \tag{4.30}$$

Proof In view of (2.15), we need to evaluate

$$\begin{aligned} I(\gamma) &= \int_0^\infty z J_\gamma^2(z) \widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) dz \\ &= \frac{C_{\lambda,\mu}}{\sqrt{2\pi}\epsilon^d} \int_0^\infty z J_\gamma^2(z) {}_1F_2\left[\lambda + \frac{\mu}{2} \quad \lambda + \frac{\mu+1}{2}; -\left(\frac{z}{2\epsilon}\right)^2\right] dz \quad \text{using (3.17)}. \end{aligned}$$

The following identity is from [20, Section 6.2 (41)]

$$(J_\gamma(z))^2 = (2^\gamma \Gamma(\gamma + 1))^{-2} z^{2\gamma} {}_1F_2\left[\gamma + \frac{1}{2} \quad \gamma + \frac{1}{2}; -z^2\right]$$

and it allows us to express $I(\gamma)$ as an integral involving the product of two ${}_1F_2$ hypergeometric functions.

$$\begin{aligned} I(\gamma) &= \frac{C_{\lambda,\mu}}{\sqrt{2\pi}\epsilon^d 2^{2\gamma} \Gamma(\gamma + 1)^2} \\ &\int_0^\infty z^{2\gamma+1} {}_1F_2\left[\gamma + \frac{1}{2} \quad \gamma + \frac{1}{2}; -z^2\right] {}_1F_2\left[\lambda + \frac{\mu}{2} \quad \lambda + \frac{\mu+1}{2}; -\left(\frac{z}{2\epsilon}\right)^2\right] dz. \end{aligned} \tag{4.31}$$

The next identity holds for $\alpha, \beta > 0, p \leq q + 1$, and it is taken from [27, 2.22.2.1]

$${}_{p+1}F_{q+1}\left[\begin{matrix} a_1 & \cdots & a_p & \alpha \\ b_1 & \cdots & b_q & \alpha + \beta \end{matrix}; z\right] = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} {}_pF_q\left[\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}; zt\right] dt. \tag{4.32}$$

Setting $p = 0, q = 1, \alpha = \lambda, b_1 = \lambda + \frac{\mu}{2}$ and $\beta = \frac{\mu+1}{2}$, we have

$$\begin{aligned} &{}_1F_2\left[\lambda + \frac{\mu}{2} \quad \lambda + \frac{\mu+1}{2}; -\left(\frac{z}{2\epsilon}\right)^2\right] \\ &= \frac{\Gamma\left(\lambda + \frac{\mu+1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{\mu+1}{2}\right)} \int_0^1 t^{\lambda-1} (1-t)^{\frac{\mu-1}{2}} {}_0F_1\left[\begin{matrix} - \\ \lambda + \frac{\mu}{2} \end{matrix}; -\left(\frac{z}{2\epsilon}\right)^2 t\right] dt. \end{aligned}$$

Substituting this into (4.31) gives

$$I(\gamma) = \frac{C_{\lambda,\mu}\Gamma\left(\lambda + \frac{\mu+1}{2}\right)}{\sqrt{2\pi}\epsilon^d 2^{2\gamma} \Gamma(\gamma + 1)^2 \Gamma(\lambda)\Gamma\left(\frac{\mu+1}{2}\right)} \int_0^1 t^{\lambda-1} (1-t)^{\frac{\mu-1}{2}} \mathcal{F}_{\gamma,\mu,\lambda}(t) dt, \tag{4.33}$$

where

$$\mathcal{F}_{\gamma,\mu,\lambda}(t) = \int_0^\infty z^{2\gamma+1} {}_0F_1\left[\begin{matrix} - \\ \lambda + \frac{\mu}{2} \end{matrix}; -\left(\frac{\sqrt{t}}{2\epsilon}\right)^2 z^2\right] {}_1F_2\left[\begin{matrix} \gamma + \frac{1}{2} \\ \gamma + 1 \quad 2\gamma + 1 \end{matrix}; -z^2\right] dz. \tag{4.34}$$

The next identity, which holds for $a < b$ is taken from [21, 5.1b]

$$\begin{aligned} & \int_0^\infty z^{s-1} {}_0F_1\left[\begin{matrix} - \\ 1 + \rho \end{matrix}; -a^2 z^2\right] {}_1F_2\left[\begin{matrix} \alpha \\ \beta \quad 1 + \nu \end{matrix}; -b^2 z^2\right] dz \\ &= \frac{1}{2} b^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\alpha - \frac{s}{2}\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma(\alpha) \Gamma\left(\beta - \frac{s}{2}\right) \Gamma\left(1 + \nu - \frac{s}{2}\right)} {}_3F_2\left[\begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu \\ 1 + \frac{s}{2} - \alpha, 1 + \rho \end{matrix}; \frac{a^2}{b^2}\right] \\ &+ \frac{1}{2} \frac{a^{2\alpha-s} \Gamma(1 + \rho) \Gamma(1 + \nu) \Gamma(\beta) \Gamma\left(\frac{s}{2} - \alpha\right)}{b^{2\alpha} \Gamma(1 + \nu - \alpha) \Gamma\left(1 + \rho + \alpha - \frac{s}{2}\right) \Gamma(\beta - \alpha)} {}_3F_2\left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu \\ 1 + \alpha - \frac{s}{2}, 1 + \rho + \alpha - \frac{s}{2} \end{matrix}; \frac{a^2}{b^2}\right]. \end{aligned}$$

We observe that, since $\epsilon \geq \frac{1}{2}$ and $0 < t < 1$, this identity can be used to evaluate $\mathcal{F}_{\gamma,\mu,\lambda}(t)$. Specifically, setting

$$s = 2(\gamma + 1), \quad a = \frac{\sqrt{t}}{2\epsilon}, \quad b = 1, \quad \rho = \lambda + \frac{\mu}{2} - 1, \quad \alpha = \gamma + \frac{1}{2}, \quad \beta = \gamma + 1 \quad \text{and} \quad \nu = 2\gamma,$$

we find that

$$\begin{aligned} \mathcal{F}_{\gamma,\mu,\lambda}(t) &= \frac{1}{2} \frac{\Gamma(2\gamma + 1) \Gamma(\gamma + 1)^2 \Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma(0) \Gamma(\gamma)} {}_3F_2\left[\begin{matrix} \gamma + 1, 1, -\gamma + 1 \\ \frac{3}{2}, \lambda + \frac{\mu}{2} \end{matrix}; \frac{t}{4\epsilon^2}\right] \\ &+ \frac{\epsilon}{\sqrt{t}} \frac{\Gamma\left(\lambda + \frac{\mu}{2}\right) \Gamma(2\gamma + 1) \Gamma(\gamma + 1) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\frac{1}{2}\right)} {}_3F_2\left[\begin{matrix} \frac{1}{2} + \gamma, \frac{1}{2}, \frac{1}{2} - \gamma \\ \frac{1}{2}, \lambda + \frac{\mu-1}{2} \end{matrix}; \frac{t}{4\epsilon^2}\right]. \end{aligned} \tag{4.35}$$

We observe that the first term vanishes due to the appearance of $\Gamma(0)$ in the denominator of the multiplier. Furthermore, the ${}_3F_2$ function in the second term collapses to a ${}_2F_1$ variety due to the duplication of $\frac{1}{2}$ in the defining parameters. Thus, we are left with

$$\mathcal{F}_{\gamma,\mu,\lambda}(t) = \frac{\epsilon}{\sqrt{t}} \frac{\Gamma\left(\lambda + \frac{\mu}{2}\right) \Gamma(2\gamma + 1) \Gamma(\gamma + 1)}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(\lambda + \frac{\mu-1}{2}\right)} {}_2F_1\left[\begin{matrix} \frac{1}{2} + \gamma, \frac{1}{2} - \gamma \\ \lambda + \frac{\mu-1}{2} \end{matrix}; \frac{t}{4\epsilon^2}\right]. \tag{4.36}$$

Inserting this into (4.33), we find that

$$\begin{aligned}
 I(\gamma) &= \frac{C_{\lambda,\mu} \Gamma\left(\lambda + \frac{\mu+1}{2}\right)}{\sqrt{2\pi} \epsilon^d 2^{2\gamma} \Gamma(\gamma + 1)^2 \Gamma(\lambda) \Gamma\left(\frac{\mu+1}{2}\right)} \frac{\epsilon \cdot \Gamma\left(\lambda + \frac{\mu}{2}\right) \Gamma(2\gamma + 1) \Gamma(\gamma + 1)}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(\lambda + \frac{\mu-1}{2}\right)} \\
 &\quad \times \int_0^1 (1-t)^{\frac{\mu-1}{2}} t^{\lambda-\frac{3}{2}} {}_2F_1\left[\begin{matrix} \frac{1}{2} + \gamma, \frac{1}{2} - \gamma \\ \lambda + \frac{\mu-1}{2} \end{matrix}; \frac{t}{4\epsilon^2}\right] dt, \\
 &= \frac{C_{\lambda,\mu} \left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda + \frac{\mu}{2}\right)}{\sqrt{2\pi} \epsilon^{d-1} \Gamma(\lambda) \Gamma\left(\frac{\mu+1}{2}\right)} \int_0^1 (1-t)^{\frac{\mu-1}{2}} t^{\lambda-\frac{3}{2}} {}_2F_1\left[\begin{matrix} \frac{1}{2} + \gamma, \frac{1}{2} - \gamma \\ \lambda + \frac{\mu-1}{2} \end{matrix}; \frac{t}{4\epsilon^2}\right] dt,
 \end{aligned}$$

where, in the final line, we have used the duplication formula for the Gamma function [1, 6.1.18]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{4.37}$$

with $z = \gamma + \frac{1}{2}$. Using (4.32) to replace the integral in the last equation, where we set $p = 2, q = 1, \alpha = \lambda - \frac{1}{2}, \beta = \frac{\mu+1}{2}, a_1 = \frac{1}{2} + \gamma, a_2 = \frac{1}{2} - \gamma, b_1 = \lambda + \frac{\mu-1}{2}$ and $z = \frac{1}{4\epsilon^2}$, we find

$$I(\gamma) = \frac{C_{\lambda,\mu} \left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda - \frac{1}{2}\right)}{\pi \sqrt{2} \epsilon^{d-1} \Gamma(\lambda)} {}_3F_2\left[\begin{matrix} \frac{1}{2} + \gamma, \frac{1}{2} - \gamma, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix}; \frac{1}{4\epsilon^2}\right]. \tag{4.38}$$

Recalling the definition of $C_{\lambda,\mu}$ (3.18), the multiplier of the ${}_3F_2$ hypergeometric function is given by

$$\frac{2^\lambda \Gamma(\lambda) \Gamma(\mu + 1)}{\underbrace{\Gamma(2\lambda + \mu)}_{C_{\lambda,\mu}}} \frac{\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda - \frac{1}{2}\right)}{\pi \sqrt{2} \epsilon^{d-1} \Gamma(\lambda)} = \frac{1}{2\pi \cdot \epsilon^{d-1}} \frac{2^{\lambda-\frac{1}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \Gamma(\mu + 1)}{\underbrace{\Gamma(2\lambda + \mu - 1)}_{C_{\lambda-\frac{1}{2},\mu}}}.$$

Thus, appealing to (2.15), we can conclude that

$$\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m) = (2\pi)^{\frac{d-1}{2}} \frac{C_{\lambda-\frac{1}{2},\mu}}{\sqrt{2\pi} \epsilon^{d-1}} {}_3F_2\left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix}; \frac{1}{4\epsilon^2}\right].$$

□

5 Asymptotic decay of $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$

In this section, we investigate the asymptotic behaviour of the spherical Fourier coefficients of the generalised Wendland function. We take two approaches to this. First, we make use of the upper and lower bounds on the Euclidean Fourier transform (3.27) together with the Equation (2.15) to derive tight asymptotic bounds. Second, in the case where d is odd, we provide the precise asymptotic decay rate of the spherical

Fourier coefficients by using known large parameter asymptotic results for hypergeometric series. In addition, we present numerical evidence which suggests that the same precise asymptotic decay rate holds when d is even and leave this as an open problem.

5.1 Tight asymptotic bounds

The connection of the decay of the Fourier transform and the decay of the spherical Fourier coefficients was studied in [24]. We employ their results to prove the following:

Theorem 5.8 *There exist positive constants $A_1 < A_2$ such that the spherical Fourier coefficients of the generalised Wendland functions $\Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta)$ defined in (4.28) satisfy*

$$\frac{A_1}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}} \leq \widehat{\psi_{\mu,\alpha}^{(\epsilon)}}(m) \leq \frac{A_2}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}}, \tag{5.39}$$

for all m provided that $\alpha > 0$ and $\mu \geq \frac{d+1}{2} + \alpha$. As in the Euclidean case, this asymptotic behaviour implies that the zonal kernel $\Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta)$ is the reproducing kernel for a Hilbert space that is norm equivalent to the Sobolev space $H^{\lambda-\frac{1}{2}}(S^{d-1})$.

Proof We recall that the spherical Fourier coefficients of the generalised Wendland functions are given by

$$\widehat{\psi_{\mu,\alpha}^{(\epsilon)}}(m) = (2\pi)^{\frac{d}{2}} I\left(m + \frac{d-2}{2}\right), \text{ where } I(\gamma) = \int_0^\infty z \widehat{\phi_{\mu,\alpha}^{(\epsilon)}}(z) J_\gamma^2(z) dz,$$

and further, from (3.27), that there exists positive constants $\kappa_1 < \kappa_2$ such that

$$\frac{\kappa_1 \epsilon^{2\alpha+1}}{(1+z^2)^\lambda} \leq \widehat{\phi_{\mu,\alpha}^{(\epsilon)}}(z) \leq \frac{\kappa_2 \epsilon^{2\alpha+1}}{(1+z^2)^\lambda}, \quad z \geq 0, \quad \lambda = \frac{d+1}{2} + \alpha, \text{ and } \alpha > 0.$$

The radial function whose d -dimensional Fourier transform (2.2) coincides with $(1+z^2)^{-\lambda}$, where $\lambda > \frac{d}{2}$, is the Matern basis function (see [12] Section 3.1.1) and is given by

$$\mathcal{M}_\lambda(r) = \frac{1}{2^{\lambda-1} \Gamma(\lambda)} r^{\lambda-\frac{d}{2}} \mathcal{K}_{\lambda-\frac{d}{2}}(r), \tag{5.40}$$

where \mathcal{K}_ν denotes the modified Bessel function of the second kind of order ν [1, Equation (9.6.22)].

Using the positivity of the integrand $I(\gamma)$ and the bounds on $\widehat{\phi_{\mu,\alpha}^{(\epsilon)}}(z)$, it follows that

$$\kappa_1 \epsilon^{2\alpha+1} \widehat{\psi}_\lambda(m) \leq \widehat{\psi_{\mu,\alpha}^{(\epsilon)}}(m) \leq \kappa_2 \epsilon^{2\alpha+1} \widehat{\psi}_\lambda(m), \tag{5.41}$$

where $\widehat{\psi}_\lambda(m)$ are the spherical Fourier coefficients of the Matern function restricted to S^{d-1} given by

$$\widehat{\psi}_\lambda(m) = (2\pi)^{\frac{d}{2}} \int_0^\infty z \widehat{\mathcal{M}}_\lambda(z) J_{m+\frac{d-2}{2}}^2(z) dz = (2\pi)^{\frac{d}{2}} \int_0^\infty \frac{z}{(1+z^2)^\lambda} J_{m+\frac{d-2}{2}}^2(z) dz.$$

Upper and lower bounds for the integral

$$\int_0^\infty \frac{z}{(1+z^2)^\lambda} J_\nu^2(z) dz,$$

were given in [24, Proposition 4.1]; setting the parameter $\nu = m + \frac{d-2}{2}$, they provide, for $m \rightarrow \infty$, that

$$\frac{(2\pi)^{\frac{d}{2}} C_\lambda 2^{-2\lambda}}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}} \left[1 + O\left(\frac{1}{m + \frac{d-2}{2}}\right)\right] \leq \widehat{\psi}_\lambda(m) \leq \frac{(2\pi)^{\frac{d}{2}} C_\lambda}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}} \left[1 + O\left(\frac{1}{m + \frac{d-2}{2}}\right)\right],$$

where, following an application of (4.37), the constant above is given by

$$C_\lambda = \frac{\Gamma(2\lambda - 1)}{2^{2\lambda-3} \Gamma^2(\lambda)} = \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)}.$$

The higher order terms on the bound can be removed by multiplying the lower bound of the inequality by $\frac{1}{2}$ and the upper bound by 2. Doing so, and using (5.41), we have that

$$\frac{A'_1}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}} \leq \widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m) \leq \frac{A'_2}{\left(m + \frac{d-2}{2}\right)^{2\lambda-1}} \quad \text{as } m \rightarrow \infty,$$

where

$$0 < A'_1 = \frac{1}{2^{2\lambda-1}} \frac{(2\pi)^{\frac{d}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \kappa_1 \epsilon^{2\alpha+1}}{\Gamma(\lambda) \sqrt{\pi}} < 4 \frac{(2\pi)^{\frac{d}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \kappa_2 \epsilon^{2\alpha+1}}{\Gamma(\lambda) \sqrt{\pi}} = A'_2.$$

Since all $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$ are positive and finite, the constants A'_1, A'_2 can be further adapted to A_1, A_2 , possibly depending on ϵ, μ and α , such that (5.39) holds not only for $m > m_0$ for some $m_0 \geq 1$ but for all m . □

5.2 Precise asymptotic decay rate

Theorem 5.9 *Let $d \geq 3$ denote an odd positive integer. The spherical Fourier coefficients of the generalised Wendland functions $\Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta)$ ($\xi, \eta \in S^{d-1}$) defined in (4.28) exhibit the following precise asymptotic decay*

$$\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m) \sim (2\pi)^{\frac{d-1}{2}} \frac{2^{\lambda-\frac{1}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \mu}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{\left(m + \frac{d-1}{2}\right)^{2\lambda-1}}, \tag{5.42}$$

provided that $\alpha > 0$ and $\mu \geq \frac{d+1}{2} + \alpha$.

Proof The hypergeometric function appearing in the expression (4.29) for the spherical Fourier coefficients is given by

$${}_3F_2 \left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix} ; \frac{1}{4\epsilon^2} \right]. \tag{5.43}$$

When $d \geq 3$ is odd then, due to the appearance of a negative integer in the upper coefficient list, the series terminates and so it collapses to a hypergeometric polynomial of the form

$${}_3F_2 \left[\begin{matrix} -n, n + c, a \\ b_1, b_2 \end{matrix} ; z \right], \quad n \in \mathbb{Z}_+. \tag{5.44}$$

In the case where $d \geq 2$ is even, the series does not terminate.

A survey of the literature in this area shows that there are very few known asymptotic results that apply to general ${}_3F_2$ hypergeometric functions for large parameters. Most of the known results apply to the case where the series terminates and, in this regard, we are fortunate that the limiting behaviour of (5.44) as $n \rightarrow \infty$ is covered in [13] where it is shown that provided none of the hypergeometric parameters (a, b_1, b_2, c) in (5.44) coincides with zero or with a negative integer, then with the following definitions

$$2\alpha = a - b_1 - b_2 + \frac{1}{2} \quad \text{and} \quad z = \sin^2(\theta/2) \in (0, 1), \tag{5.45}$$

we have the following asymptotic results.

◊ For $z \in (0, 1)$:

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} -n, n + c, a \\ b_1, b_2 \end{matrix} ; z \right] &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1 - a)\Gamma(b_2 - a)} \frac{1}{z^a} \frac{1}{(n + c)^{2a}} \left[1 + O\left(\frac{1}{(n + c)}\right) \right] \\ &+ \frac{\Gamma(b_1)\Gamma(b_2)(n + c)^{2\alpha}}{\sqrt{\pi}\Gamma(a)} \left[\frac{(\sin^2(\theta/2))^\alpha \cos\left(\left(n + \frac{c}{2}\right)\theta + \pi\alpha\right)}{(\cos^2(\theta/2))^{\alpha + \frac{c}{2}}} \right] + O\left((n + c)^{2\alpha - 1}\right). \end{aligned} \tag{5.46}$$

◊ For $z = 1$:

$$\begin{aligned}
 {}_3F_2 \left[\begin{matrix} -n, n+c, a \\ b_1, b_2 \end{matrix} ; 1 \right] &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1-a)\Gamma(b_2-a)} \frac{1}{(n+c)^{2a}} \left[1 + O\left(\frac{1}{(n+c)}\right) \right] \\
 &+ \frac{(-1)^n \Gamma(b_1)\Gamma(b_2)\Gamma(n+2c+4\alpha)}{\Gamma\left(c+2\alpha+\frac{1}{2}\right)\Gamma(a)\Gamma(n+c)} \left[1 - \frac{(c+4\alpha)\left(c+2\alpha-\frac{1}{2}\right)}{(n+c)} + O\left(\frac{1}{(n+c)^2}\right) \right].
 \end{aligned} \tag{5.47}$$

In our case, $n = m + \frac{d-3}{2}$, $c = 1$, $a = \lambda - \frac{1}{2}$, $b_1 = \lambda + \frac{\mu-1}{2}$, $b_2 = \lambda + \frac{\mu}{2}$ and $z = \frac{1}{4\epsilon^2}$. Since $\lambda, \mu > \frac{1}{2}$, it is straightforward to check that these parameters satisfy the conditions associated with the above asymptotic formulae. Thus, we can set

$$2\alpha := -\left(\lambda + \mu - \frac{1}{2}\right) \quad \text{and} \quad \sin^2\left(\frac{\theta}{2}\right) := \frac{1}{4\epsilon^2},$$

and employ (5.46) and (5.47) to yield the following asymptotic results.

◊ For $\epsilon > \frac{1}{2}$:

$$\begin{aligned}
 &{}_3F_2 \left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix} ; \frac{1}{4\epsilon^2} \right] \\
 &= \frac{\Gamma\left(\lambda + \frac{\mu-1}{2}\right)\Gamma\left(\lambda + \frac{\mu}{2}\right)(2\epsilon)^{2\lambda-1}}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)\left(m + \frac{d-1}{2}\right)^{2\lambda-1}} \left[1 + O\left(\frac{1}{m + \frac{d-1}{2}}\right) \right] \\
 &+ \frac{\Gamma\left(\lambda + \frac{\mu-1}{2}\right)\Gamma\left(\lambda + \frac{\mu}{2}\right)2\epsilon(4\epsilon^2 - 1)^{\frac{1}{2}(\lambda+\mu-\frac{3}{2})}}{\sqrt{\pi}\Gamma\left(\lambda - \frac{1}{2}\right)\left(m + \frac{d-1}{2}\right)^{\lambda+\mu-\frac{1}{2}}} \\
 &\times \left[\cos\left(\left(m + \frac{d-2}{2}\right)\theta - \frac{\pi}{2}\left(\lambda + \mu - \frac{1}{2}\right)\right) \right] + O\left(\frac{1}{\left(m + \frac{d-1}{2}\right)^{\lambda+\mu+\frac{1}{2}}}\right).
 \end{aligned} \tag{5.48}$$

We observe that since $\mu \geq \lambda$, we have that

$$\lambda + \mu + \frac{1}{2} > \lambda + \mu - \frac{1}{2} \geq 2\lambda - \frac{1}{2} > 2\lambda - 1,$$

and so, in this case, the precise asymptotic decay of the hypergeometric function is determined by the first term of (5.48) and is given by

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix} ; \frac{1}{4\epsilon^2} \right] \\
 & \sim \frac{\Gamma\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda + \frac{\mu}{2}\right) (2\epsilon)^{2\lambda-1}}{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1}{2}\right) \left(m + \frac{d-1}{2}\right)^{2\lambda-1}} = \frac{\Gamma(2\lambda + \mu - 1) \epsilon^{2\lambda-1}}{\Gamma(\mu)} \frac{1}{\left(m + \frac{d-1}{2}\right)^{2\lambda-1}} \\
 & = \frac{2^{\lambda-\frac{1}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \mu \epsilon^{2\lambda-1}}{C_{\lambda-\frac{1}{2}, \mu} \left(m + \frac{d-1}{2}\right)^{2\lambda-1}},
 \end{aligned} \tag{5.49}$$

where the first equality follows by applying the duplication formula (4.37) with $z = \lambda + \frac{\mu-1}{2}$ and $z = \frac{\mu}{2}$, the final equality follows from the definition of $C_{\lambda-\frac{1}{2}, \mu}$ (4.30). The asymptotic result for the spherical coefficients follows, in this case, directly from their definition (4.29).

◇ For $\epsilon = \frac{1}{2}$:

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} -\left(m + \frac{d-3}{2}\right), m + \frac{d-1}{2}, \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2}, \lambda + \frac{\mu}{2} \end{matrix} ; 1 \right] \\
 & = \frac{\Gamma\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda + \frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1}{2}\right) \left(m + \frac{d-1}{2}\right)^{2\lambda-1}} \left[1 + O\left(\frac{1}{m + \frac{d-1}{2}}\right) \right] \\
 & + \frac{(-1)^{m+\frac{d-3}{2}} \Gamma\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda + \frac{\mu}{2}\right) \Gamma\left(m + \frac{d-1}{2} - 2(\lambda + \mu - 1)\right)}{\Gamma(2 - \lambda - \mu) \Gamma\left(\lambda - \frac{1}{2}\right) \Gamma\left(m + \frac{d-1}{2}\right)} \\
 & \times \left[1 - \frac{2(1 - \lambda - \mu)^2}{m + \frac{d-1}{2}} + O\left(\frac{1}{\left(m + \frac{d-1}{2}\right)^2}\right) \right].
 \end{aligned} \tag{5.50}$$

Applying [25, Equation (5.11.12)]

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b}, \tag{5.51}$$

with $z = m + \frac{d-1}{2}$ we find that

$$\frac{\Gamma\left(m + \frac{d-1}{2} - 2(\lambda + \mu - 1)\right)}{\Gamma\left(m + \frac{d-1}{2}\right)} \sim \frac{1}{\left(m + \frac{d-1}{2}\right)^{2(\lambda+\mu-1)}}.$$

Clearly, since $\mu \geq \lambda$ we have that $2(\lambda + \mu - 1) \geq 2(2\lambda - 1) > 2\lambda - 1$, and so the asymptotic behaviour of the hypergeometric function is determined by the first term of (5.50). Mirroring the concluding development in (5.49), we have that

$$\begin{aligned} & {}_3F_2\left[\begin{matrix} -\left(m + \frac{d-3}{2}\right) & m + \frac{d-1}{2} & \lambda - \frac{1}{2} \\ \lambda + \frac{\mu-1}{2} & \lambda + \frac{\mu}{2} \end{matrix}; 1\right] \\ & \sim \frac{\Gamma\left(\lambda + \frac{\mu-1}{2}\right) \Gamma\left(\lambda + \frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1}{2}\right) \left(m + \frac{d-1}{2}\right)^{2\lambda-1}} = \frac{2^{\lambda-\frac{1}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \mu}{C_{\lambda-\frac{1}{2}, \mu}} \frac{\left(\frac{1}{2}\right)^{2\lambda-1}}{\left(m + \frac{d-1}{2}\right)^{2\lambda-1}}, \end{aligned}$$

and the asymptotic result for the spherical coefficients follows from (4.29). □

The above result provides the precise asymptotic decay rate for the spherical Fourier coefficients in the important case of restricting the generalised Wendland functions to the 2-sphere S^2 , where there are enormous practical applications. One immediate application is to investigate the use of the suitably normalised generalised Wendland functions as a correlation model in the simulation of Gaussian random fields on the surface of the earth. In particular, with the precise asymptotic decay rate available, one can employ formula 6 of [2] to investigate the compatibility of the proposed generalised Wendland family with other commonly used correlation models; this is an obvious topic of future research. For the convenience of the reader, we state the decay for the case $d = 3$ specifically

$$\widehat{\psi}_{\mu, \alpha}^{(\epsilon)}(m) \sim \sqrt{2\pi} \frac{2^{\alpha+\frac{3}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \mu \epsilon^{2\alpha+1}}{(m+1)^{2\alpha+3}}, \tag{5.52}$$

with $\alpha > 0, \mu \geq \alpha + 2$.

We have not established precise asymptotics for the case where the generalised Wendland functions are restricted to the circle S^1 and other odd-dimensional spheres; large parameter asymptotics are, to the best of our knowledge, not available for the non-polynomial case of the ${}_3F_2$ hypergeometric function (5.43). The numerical results in Tables 1, 2 and 3 show, for increasingly large values of m , the calcula-

Table 1 Generalised Wendland parameters: $d = 2, \epsilon = 1, \alpha = \frac{1}{2}, \mu = \lambda + 1$

m	Precise formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$	Asymptotic formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$
150	2.2287E-06	2.2060E-06
200	9.4002E-07	9.3297E-07
250	4.8114E-07	4.7839E-07
300	2.7854E-07	2.7712E-07

tion of the coefficients (for odd-dimensional spheres) using formula (4.29) and also the value from the asymptotic formula (5.42). These computations were performed using WOLFRAM MATHEMATICA v12 and they strongly suggest that the established asymptotic behaviour from Theorem 5.9 also holds in the case where $d \geq 2$ is even. These numerical findings lead us to make the following conjecture.

Conjecture 5.10 Let $d \geq 2$ denotes an even positive integer. The spherical Fourier coefficients of the generalised Wendland functions $\Psi_{\mu,\alpha}^{(\epsilon)}(\xi, \eta)$ ($\xi, \eta \in S^{d-1}$) defined in (4.28) exhibit the following precise asymptotic decay

$$\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m) \sim (2\pi)^{\frac{d-1}{2}} \frac{2^{\lambda-\frac{1}{2}} \Gamma\left(\lambda - \frac{1}{2}\right) \mu}{\sqrt{2\pi}} \frac{\epsilon^{2\alpha+1}}{\left(m + \frac{d-1}{2}\right)^{2\lambda-1}}. \tag{5.53}$$

We conclude the paper by drawing the reader’s attention to the close connection between the asymptotic formula for the decay of the Fourier transform of the generalised Wendland functions and that of the associated spherical Fourier coefficients. Recalling that $\lambda = \frac{d+1}{2} + \alpha$, we can define

$$\mathcal{K}_{\mu,\alpha}^{(d)} = \frac{2^{\frac{d+1}{2}+\alpha} \Gamma\left(\frac{d+1}{2} + \alpha\right) \mu \epsilon^{2\alpha+1}}{\sqrt{2\pi}},$$

then revisiting (3.24) and (5.42), we have that

$$\widehat{\phi}_{\mu,\alpha}^{(\epsilon)}(z) \sim \frac{\mathcal{K}_{\mu,\alpha}^{(d)}}{\|z\|^{d+1+2\alpha}} \quad \text{and} \quad \widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m) \sim (2\pi)^{\frac{d-1}{2}} \frac{\mathcal{K}_{\mu,\alpha}^{(d-1)}}{\left(m + \frac{d-1}{2}\right)^{d+2\alpha}}.$$

Table 2 Generalised Wendland parameters: $d = 4, \epsilon = \frac{1}{2}, \alpha = 1, \mu = \lambda$

m	Precise formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$	Asymptotic formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$
100	4.1457E-11	4.0223E-11
200	6.6706E-13	6.5709E-13
400	1.0578E-14	1.0499E-14
800	1.6652E-16	1.6590E-16

Table 3 Generalised Wendland parameters: $d = 6$, $\epsilon = 2$, $\alpha = \frac{1}{2}$, $\mu = \lambda + 4$

m	Precise formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$	Asymptotic formula for $\widehat{\psi}_{\mu,\alpha}^{(\epsilon)}(m)$
100	3.9103E-10	3.9959E-10
200	3.4127E-12	3.4018E-12
300	2.060E-13	2.0493E-13
400	2.7897E-14	2.7754E-14

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Declarations

Conflict of interest The authors declare no competing interests.

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