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Topological graph theory through matroid theory

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A standard statement in undergraduate graph theory is that ‘deletion and contraction are dual operations’. But this statement is only partially true, and completing it will take us on a route through graphs in surfaces to the world of delta-matroids. Along the way we’ll see some of this general setting’s advantages and challenges.

Some classical graph theory

Let’s begin with some undergraduate graph theory. Suppose we have a graph $G = (V, E)$ and an edge e of G . We can *delete* the edge e by simply removing it. The resulting graph is denoted by $G \setminus e$. We can also *contract* e by removing e and then identifying its endvertices, resulting in a graph denoted by G/e .

The operations of deletion and contraction are not entirely independent of each other. A graph G^* is an *abstract dual* of G if it has the same edge set as G , and a set of edges in G forms a spanning tree (a tree in G that includes all vertices of G) if and only if its complement forms a spanning tree in G^* .

Deletion and contraction are dual operations, and related through the beautiful identities

$$(G/e)^* = G^* \setminus e \quad \text{and} \quad (G \setminus e)^* = G^*/e. \quad (1)$$

However, there is a catch: these identities are not valid for all graphs. In fact the situation is worse: not all graphs have abstract duals!

So in (1) do we have a fundamental graph theoretic result that is not truly a result about graphs? If so, in what setting does it properly reside? To find out there are two routes we can follow: an algebraic route and a topological route.

Following the topological route

Only some graphs have abstract duals and there is a topological characterisation of those that do: they can be drawn on the plane. A connected graph \mathbb{G} is said to be a *plane graph* if it has been drawn on the plane in such a way that its edges don’t intersect, and is *planar* if it admits such a plane drawing.

Plane graphs have another type of dual. If \mathbb{G} is a plane graph, then its *geometric dual* \mathbb{G}^* is the plane graph obtained by placing a vertex in each face of \mathbb{G} and joining two of these vertices by an edge whenever the corresponding faces share an edge in \mathbb{G} .

In 1933, H. Whitney proved that a graph has an abstract dual if and only if it is planar. Moreover, the geometric and abstract duals of a plane graph (essentially) coincide. It follows that (1) is valid for planar graphs. Can we drop planarity?

An *embedded graph* \mathbb{G} comprises a graph drawn on any closed surface so that its edges don’t intersect and its faces are discs (so cutting the surface open along the graph results in a collection of discs). We can form the *geometric dual* \mathbb{G}^* of an embedded graph \mathbb{G} just as we did for plane graphs. We can also delete and contract edges.

If e is an edge of \mathbb{G} then $\mathbb{G} \setminus e$ is formed by removing e from the drawing. This may create redundant handles in the surface. If it does, remove them so the faces remain discs. And \mathbb{G}/e is formed by contracting the edge e to a point which forms a new vertex of the graph. This may result in a pinch-point (a point whose neighbourhood looks like a diaboloid). If it does, resolve it by splitting the vertex and the pinch-point in two.

This gives our topological completion of (1):

$$(\mathbb{G}/e)^* = \mathbb{G}^* \setminus e \quad \text{and} \quad (\mathbb{G} \setminus e)^* = \mathbb{G}^*/e, \quad (2)$$

identities that, unlike in (1), apply to \mathbb{G} without any restriction. Now let’s consider the algebraic route.

Following the algebraic route

Only planar graphs have abstract duals: if G is a non-planar graph, then there is no graph H whose spanning trees are the complements of the spanning trees of G . But there is a combinatorial structure having ‘spanning tree like’ objects that are the complements of the spanning trees of G .

The edge sets of the spanning trees of a graph satisfy an *exchange property*: if T_1 and T_2 are spanning trees of a graph G and e is an edge of $T_1 \setminus T_2$, then there is an edge f of $T_2 \setminus T_1$ so that $T_1 \setminus \{e\} \cup \{f\}$ is also a spanning tree of G . This exchange property should look familiar from linear algebra as the bases of a vector space satisfy the same property.

In 1935 Whitney introduced matroids in an attempt to find a combinatorial abstraction of the exchange property. A *matroid* is a pair (E, \mathcal{B}) comprising a finite set E and a non-empty collection \mathcal{B} of subsets of E , called *bases*, that satisfy the exchange property. So examples of matroids come from graphs and from vector spaces. In the (connected) graph case, E is the edge set of the graph and \mathcal{B} comprises the edge sets of its spanning trees.

Every matroid M has a *dual* M^* , formed by taking the complements of all the bases. So a non-planar graph does not have a dual, but its matroid does. Moreover, there are notions of deletion and contraction in matroids, which are consistent with the definitions for graphs: for *all* matroids

$$(M/e)^* = M^* \setminus e \quad \text{and} \quad (M \setminus e)^* = M^* / e, \quad (3)$$

and we have a second way to complete (1).

All roads lead to Rome

Starting with graphs we can move to embedded graphs or to matroids. Are these really two different directions, or artefacts of some higher theory?

The key to answering this question is to think of spanning trees in a different way. Take an embedded graph, choose a spanning tree T and imagine standing just to one side of one of its edges. We can walk along the side of this edge until we reach a vertex, continue walking round the side of this vertex to the next edge, then walking along that edge. By continuing in this way, we end up walking around the boundary of the spanning tree and will return to our starting place having visited every vertex. In a plane graph, the only subgraphs for which this is possible are the spanning trees, but for embedded graphs in general, there will be other subgraphs with this property. These subgraphs are the *spanning quasi-trees* of an embedded graph and they satisfy a modified version of the exchange property.

The pair $D(\mathbb{G}) := (E, \mathcal{F})$, where E is the edge set of \mathbb{G} and \mathcal{F} the edge sets of its spanning quasi-trees then gives an abstraction of an embedded graph that turns out to be an example of a *delta-matroid* (see “The definition of a delta-matroid”).

As spanning trees are just plane spanning quasi-trees, delta-matroids arise through the topological route of dropping the planarity condition when forming the matroid of a graph. There’s also an algebraic

route to delta-matroids: it’s not hard to see that a matroid is a delta-matroid in which every set in \mathcal{F} has the same size. Delta-matroids arise by loosening the requirements so that bases are no longer forced to have the same size. In fact, one can use Euler’s formula to show that these two approaches agree.

Compatibility and minors

Many results in graph theory, not just the duality result in (1), turn out to be special cases of results in matroid theory. This is beneficial in two ways. First, graph theory can serve as an excellent guide for studying matroids. W. Tutte famously observed that, “if a theorem about graphs can be expressed in terms of edges and circuits alone it probably exemplifies a more general theorem about matroids.” Second, insights from matroid theory can lead to new results about graphs. Both areas have benefited from this relationship (see [2] for some examples).

A similar relationship, proposed in [1], holds between topological graph theory and delta-matroid theory. Such a relationship is hinted at by observing that basic parameters associated with each type of object agree. For example a delta-matroid $D = (E, \mathcal{F})$ is said to be *even* if the sizes of the sets in \mathcal{F} all have the same parity. A delta-matroid being even corresponds to an embedded graph being orientable. Similarly the genus of an embedded graph is given by the difference between the sizes of the largest and smallest sets in \mathcal{F} .

The definition of a delta-matroid

A *delta-matroid*, $D = (E, \mathcal{F})$, is a pair consisting of a set E , and a non-empty collection \mathcal{F} of its subsets that satisfies the *Symmetric Exchange Axiom*:

$(\forall X, Y \in \mathcal{F}) (\forall u \in X \Delta Y) (\exists v \in X \Delta Y)$ such that $(X \Delta \{u, v\}) \in \mathcal{F}$.

Here $X \Delta Y := (X \cup Y) \setminus (X \cap Y)$ is the symmetric difference of sets.

Three groups introduced Delta-matroids independently in the 1980’s: A. Bouchet in 1987; R. Chandrasekaran and S. Kabadi in 1988, under the name of pseudo-matroids; and A. Dress and T. Havel in 1986, under the name of metroids. Each group had a different motivation.

Possibly the most important relations come from deletion, contraction and duality. These may be extended from matroids to delta-matroids in a way that is consistent with their meaning in embedded graphs. In other words, if $D(\mathbb{G})$ denotes the delta-matroid associated with an embedded graph \mathbb{G} , then for every edge e of \mathbb{G} ,

$$D(\mathbb{G}/e) = D(\mathbb{G})/e, \quad D(\mathbb{G} \setminus e) = D(\mathbb{G}) \setminus e$$

and

$$D(\mathbb{G}^*) = D(\mathbb{G})^*.$$

This last duality relation can be strengthened. Rather surprisingly, there is a way to form the geometric dual \mathbb{G}^* of an embedded graph one edge at a time. This leads to the *partial duals* \mathbb{G}^A of \mathbb{G} in which the geometric dual of \mathbb{G} is only formed with respect to a subset A of edges of \mathbb{G} (so $\mathbb{G}^E = \mathbb{G}^*$ and $\mathbb{G}^\emptyset = \mathbb{G}$). This unexpected extension of duality arose from knot theory and is due to S. Chmutov, appearing in 2009.

On the other hand, in 1987 A. Bouchet introduced a fundamental operation on a delta-matroid called a *twist*. Given a delta-matroid $D = (E, \mathcal{F})$ and a subset A of E , the twist $D * A$ is formed by replacing each set in \mathcal{F} with its symmetric difference with A . Since $D * E = D^*$ a twist can be considered as a ‘partial dual’ of a delta-matroid. Again the two concepts of partial duals align: $D(\mathbb{G}^A) = D(\mathbb{G}) * A$.

The identities above enable the use of geometric insights from embedded graphs to study delta-matroids and vice versa. Below we give an illustration of how bouncing between the areas can advance them both. Our examples revolve around the concept of minors, which is key in structural graph and matroid theory. If D' can be obtained from D by a sequence of deletions and contractions, then D' is called a *minor* of D . Minors for graphs and embedded graphs are defined similarly, but isolated vertices may also be deleted.

But before going further let’s pause to address the natural question of whether embedded graphs and delta-matroids are just two descriptions of the same thing. Although every embedded graph gives rise to a delta-matroid, most delta-matroids do not arise in this way. Furthermore, it is possible that different embedded graphs give rise to the same delta-matroid. This mismatch between the areas is well understood, but it does mean that care must be taken when moving between them. Results in one area may not directly give results in the other, rather, as we will see, they point you in the right direction.

The Tutte polynomial

Many graph parameters satisfy recursive *deletion-contraction relations* (i.e., linear relations involving G , $G \setminus e$, and G/e). For example, if $a(G)$ is the number of ways to direct the edges of a graph so that it contains no directed cycles, then $a(G) = a(G \setminus e) + a(G/e)$. The *Tutte polynomial*, T , is a function from graphs to $\mathbb{Z}[x, y]$. It associates a 2-variable polynomial $T(G; x, y)$ to a graph G . It is a universal deletion-contraction invariant in the sense that any graph parameter with deletion-contraction relations can be obtained from it (for example, $a(G) = T(G; 2, 0)$). Because of this it has applications in a diverse variety of topics including codes, network reliability, chip-firing and the sandpile model, knot theory, and statistical physics.

The Tutte polynomial can be defined through a recursive deletion-contraction relation. A consequence of this and (I) is that it satisfies a duality relation,

$$T(G; x, y) = T(G^*; y, x), \quad (4)$$

where G^* is the abstract dual. This identity is surprisingly important in the theory of the Tutte polynomial. But here, we shall use it as another springboard into delta-matroids.

Two extensions of the Tutte polynomial

Once again, since not all graphs have abstract duals, in (4) we find ourselves in the situation where we have a fundamental graph theoretic identity that does not apply to all graphs. But once again, we can complete the result by taking either an algebraic route or a topological route.

For the algebraic route, the Tutte polynomial can be extended to matroids (as was done by H. Crapo in 1969), and the identity $T(M; x, y) = T(M^*; y, x)$ holds for *all* matroids. The definition of the Tutte polynomial of a matroid is, more or less, a word for word lifting of its definition for graphs. Such an approach does not work in the topological setting.

Finding a version of the Tutte polynomial for embedded graphs is a long-standing problem and one that is not entirely settled. The situation is surprisingly subtle, several candidates have been proposed, and it’s not completely clear what the correct definition should be. Some of the latest

research shows that there is a family of Tutte polynomials for embedded graphs, each arising from a slightly different notion of an ‘embedded graph’. Moreover, all of the previously defined candidate polynomials can be recovered from members of this family.

In this family the ‘Tutte polynomial’ for the type of embedded graph we consider here is a two-variable polynomial $R(\mathbb{G}; x, y)$, commonly called the *ribbon graph polynomial*. It is a specialisation of a well-known polynomial of B. Bollobás and O. Riordan from 2001.

The ribbon graph polynomial has many properties analogous to those of the Tutte polynomial. Like the Tutte polynomial, it has a recursive deletion-contraction definition, so from (2) we can deduce the topological analogue of (4): for any embedded graph \mathbb{G} we have $R(\mathbb{G}; x, y) = R(\mathbb{G}^*; y, x)$.

Completing the polynomial picture

Again the topological and algebraic routes have taken us to two different generalisations and we are left asking if we can complete the picture by showing that they both arise from a common object. Again we can answer this by appealing to delta-matroids.

The above relationships between matroids, embedded graphs and delta-matroids enable us to translate between the three settings. Applying this to the graph polynomials results in a *Tutte polynomial for delta-matroids*, denoted by $R(D; x, y)$ for a delta-matroid D . Again this has a recursive deletion-contraction definition, and $R(D; x, y) = R(D^*; y, x)$. And again, since the polynomials coincide when D comes from an embedded graph or is a matroid, our two approaches are united by delta-matroids. Let’s push the delta-matroid theory to see what it tells us about our various polynomials.

Connectivity and separability

Take two connected graphs, pick a vertex of each and merge these vertices together to form a single larger graph. Graphs arising in this way are called *separable*. The new graph is connected, but in a sense its matroid is not. Taking the union of spanning trees of the original graphs yields a spanning tree of the new graph and every spanning tree of the new graph arises in this way. Intuitively, the two parts of the matroid corresponding to the original graphs have no influence on each other. More generally we say

that a matroid M is the direct sum of matroids M_1 and M_2 (with disjoint sets of elements) if the bases of M are precisely the sets that may be formed by taking the union of a basis of M_1 and a basis of M_2 . Matroids arising from a non-trivial direct sum are also called *separable*. These ideas extend mutatis mutandis to the quasi-trees of an embedded graph and to delta-matroids.

Splitter theorems and inductive tools

Chain and splitter theorems are useful tools in inductive proofs in structural matroid theory. Chain theorems allow us to delete or contract various elements of a matroid without reducing the connectivity; splitter theorems tell us that if N is a connected minor of a connected matroid M , then we can move from M to N by deleting and contracting elements without reducing the connectivity. All the key concepts extend from matroids to delta-matroids suggesting that these results may be more widely applicable to delta-matroids and indeed they are.

Bouchet proved if D is an even, non-separable delta-matroid with element e , then either D/e or $D \setminus e$ is non-separable. As a corollary one obtains that if G is embedded in an orientable surface, then for each edge e of G , either G/e or $G \setminus e$ is non-separable.

By extending Bouchet’s result to a larger class of delta-matroids, in 2017 C. Chun, D. Chun and S. Noble extended the corollary to all embedded graphs and also proved that if D is a non-separable even delta-matroid with non-separable minor D' , then for every $e \in E(D) - E(D')$ either D/e or $D \setminus e$ is non-separable and contains D' as a minor.

Irreducible graph polynomials

The Tutte polynomial of a separable graph factorizes into the product of the Tutte polynomials of its constituent graphs. That this is really an artefact of the separability of the corresponding matroid was confirmed by C. Merino, A. de Mier and M. Noy who proved in 2001 that a matroid is non-separable if and only if its Tutte polynomial is irreducible. Again, all of the key concepts in this result have counterparts in embedded graphs and delta-matroids and we can prove that the separability of an even delta-matroid corresponds to the irreducibility of its Tutte polynomial.

A key step in the proof involves using the chain theorem to show that if D is an even delta-matroid with at least two elements, then the coefficient of x in $R(D)$ is non-zero if and only if D is non-separable. One direction follows easily from the facts that the constant term of $R(D)$ is zero and that $R(D_1 \oplus D_2) = R(D_1)R(D_2)$. The polynomial $R(D)$ has a recursive definition akin to that of the Tutte polynomial, but unlike the Tutte polynomial, some of the coefficients may be negative. However, one can show that in the recursive definition of R no cancellation involving the coefficient of x occurs. But if D is non-separable, then at least one of D/e and $D \setminus e$ is non-separable, giving a simple inductive proof.

Series-parallel graphs

Using the splitter theorem, we can say more about the coefficient of x in $R(D)$, generalizing another result on the Tutte polynomial of a graph. A plane graph is *series-parallel* if it can be built by starting from a cycle with two edges and then repeatedly subdividing or adding an edge in parallel with an existing edge.

T. Brylawski proved that G is series-parallel if and only if the coefficient of x in $T(G)$ is one. Again, this is really a result about embedded graphs and delta-matroids. We say that an embedded graph is series-parallel if it is the partial dual of a plane series-parallel graph and that a delta-matroid is series-parallel if it is the delta-matroid of a series-parallel embedded graph.

We can prove that an even delta-matroid D is series-parallel if and only if the coefficient of x in $R(D)$ is ± 1 . In one direction this follows from the fact that D contains an element e so that one of D/e and $D \setminus e$ is separable and the other is series-parallel. The other direction follows from the splitter theorem, because any even delta-matroid that is not series-parallel has a minor belonging to a small class of specific delta-matroids. For each of these the coefficient of x in $R(D)$ is ± 2 .

Is this the whole picture?

We started this article with the partial results about graphs given by (1) and argued that its incompleteness is an artefact of a higher theory. We then illustrated the benefits of the resulting relationship between topological graph theory and delta-matroid theory,

showing that each area can be advanced by bouncing between the two. (Our example considered the Tutte polynomial, but that’s not too relevant. We could have chosen a different one.) However, an astute reader may have noticed that we also ended the article with examples that were themselves partial results — they were restricted to even delta-matroids, or, correspondingly, orientable surfaces.

It turns out these partial results are themselves artefacts of a higher theory. Completing them requires a move into objects called multi-matroids. At this level of generality, the hierarchy seems to stop. But the picture is more complicated than it first appears. Some results in topological graph theory seem to genuinely reside in the world of delta-matroids, while others belong to multi-matroids. We do not yet understand why and when this split happens, and understanding it and its implications in topological graph theory is an active area of research.

FURTHER READING

- [1] C. Chun, I. Moffatt, S. Noble and R. Rueckriemen, *Matroids, Delta-matroids and Embedded Graphs*, *J. Combin. Theory, Series A*, 167 (2019) 7–59.
- [2] J. Oxley, *On the interplay between graphs and matroids*, *Surveys in Combinatorics*, 2001, London Math. Soc. Lecture Notes 288 2001 pp. 199–239.

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