On Testing for Bubbles During Hyperinflations*

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Abstract

We consider testing for the presence of rational bubbles during hyperinflations via an analysis of the non-stationarity properties of relevant observable time series. The test procedure is based on a Markov regime-switching model with independent stochastic changes in its intercept, error variance and autoregressive coefficients. This model formulation allow us to disentangle fundamentals-driven changes in the drift, bubble-driven explosiveness, and volatility changes that may be fundamentals-driven and/or bubble-driven. The testing methodology is illustrated by applying it to data from hyperinflations in Argentina, Brazil, Germany and Poland.

Keywords: Bootstrap; Bubbles; Explosiveness; Markov-switching autoregressive model; Unit-root test.

1 Introduction

The question of whether hyperinflations are accompanied, or made worse, by speculative behavior, which may be characterized as a speculative price bubble, has attracted the interest of economic theorists and practitioners alike. However, even though it is clear what the fundamentals are in hyperinflation periods, assessing whether a bubble is present in the data is not an easy task. This is partly because episodes of high inflation are typically accompanied by actions of monetary authorities aimed at stabilizing prices and, as a result, the evolution and volatility of prices may be driven by different factors, one of which is pure speculative behavior. Our objective in this paper is to consider how to assess the possibility of the existence of bubbles in a way which takes into account the impact changes in monetary policy may have on the evolution and stochastic properties of prices.

There are many possible reasons why researchers may wrongly interpret what is observed as bubbles when unaccounted movements in prices are the result of changes in the underlying fundamentals. One is the possibility of confusing bubbles with expected changes in policy that have not materialized (see Hamilton (1986)). Another, and possibly more critical, is interpreting as bubbles deviations of prices from incorrect fundamentals that ignore policy changes that took place (see Driffill and Sola (1998)). These difficulties are well understood in the literature and, while there is little a researcher can do to account for difficulties of the former kind, there is some room for action in the latter case. Nevertheless, indirect tests for bubbles, which are extensively used in the literature, have not advanced much in this direction, which is what we attempt to do in this paper.

Indirect tests for the existence of rational bubbles rely on assessing the non-stationarity and explosiveness properties of time series of prices and of observable fundamentals. Two well-known approaches of this type are the Markov-switching unit-root tests of Hall et al. (1999) and the recursive unit-root tests of Phillips et al. (2011) and Phillips et al. (2015). The key idea behind these approaches is to distinguish between periods in which the time series of interest, typically prices and their underlying fundamentals, are difference-stationary (i.e., have an autoregressive unit root) from periods in which they are explosive (i.e., have an autoregressive root greater than one). If there are no sub-periods in which prices are explosive (which, in turn, implies that fundamentals are always difference-stationary), then the existence of bubbles can be ruled out since price explosiveness is a necessary (although by no means sufficient) condition for the existence of bubbles. If, on the other hand, both fundamentals and prices are explosive in the same sub-periods and difference-stationary in all others, then price explosiveness may be entirely driven by fundamentals, even though the possibility of a bubble cannot be ruled out (if such a price bubble exists, its explosive behavior will be indistinguishable from that induced by the fundamentals). Lastly, if the fundamentals are always difference-stationary but prices are difference-stationary in some sub-periods and explosive in others, then in the sub-periods associated with explosive behavior prices evolve in a way that is potentially consistent with the presence of a (big enough) bubble – although other fundamentals-based interpretations of these patterns are also possible (see, e.g., Hamilton (1986)).

As already mentioned, an additional complication, which has not been addressed successfully in the literature on indirect tests for bubbles, is that changes in the structure of the generating mechanism of the fundamentals that drive prices will typically induce changes in prices themselves. Hence, price movements that may be interpreted as bubbles on the basis of an indirect test may in fact be entirely the result of changes in the behavior of the fundamentals which have not been taken into consideration (see, e.g., Driffill and Sola (1998)). For example, during periods of high inflation, monetary authorities may occasionally intervene in the money market in an attempt to stabilize the rate of price increases. Even though such interventions may have no effect on the explosive behavior of fundamentals (money expansion), they are likely to have an effect on prices. A researcher who does not allow for the effects of a changing rate of monetary expansion in their analysis may incorrectly interpret the induced price movements as evidence in favor of the existence of a price bubble. In addition, it is not unreasonable to expect changes in the volatility of prices to be driven by changes in the underlying fundamentals, changes in the size of a bubble, whenever the latter is present, or both. Even when using indirect tests for bubbles, these different sources of potential changes and explosiveness need to be accounted for in order to avoid confounding the effects of bubbles with those of fundamentals.

In this paper, we tackle the difficulties that arise when testing for the existence of bubbles in an environment in which changes in the growth rate and volatility of fundamentals may affect the evolution of prices in ways that mimic the explosive behavior of a bubble. We do so by considering an indirect test for bubbles based on a model specification that allows us to disentangle fundamentals-driven changes in the drift of prices, bubble-driven explosiveness, and volatility changes that may be fundamentals-driven and/or bubble-driven. Our proposed testing strategy is a generalization of that of Hall et al. (1999) and is based on a Markov-switching autoregressive model in which stochastic changes in its intercept, in its error variance and in the roots of the characteristic equation associated with it are governed by separate (independent) Markov processes.

We motivate our approach in the next section of the paper by considering a simple model of hyperinflation with stochastic changes in the mean and volatility of the monetary growth rate. In Section 3, we describe the Markov regime-switching model that is used to construct unit-root tests (against an explosive alternative), discuss a bootstrap procedure for obtaining p-values and/or critical values for such tests, and present some simulation results relating to the properties of the tests. The proposed methodology is illustrated in Section 4 by analyzing hyperinflation episodes in Argentina, Brazil, Germany and Poland. Section 5 summarizes and concludes.

2 A simple model with changing rates of monetary expansion

To fix ideas and motivate our approach, consider a simple discrete-time model of inflationary dynamics consisting of a rule that describes money expansion, given by

$$\Delta m_t = \mu + \sigma \varepsilon_t, \tag{1}$$

and a Cagan-type equation for money demand, which, in equilibrium, takes the form

$$m_t - p_t = -\beta \mathbb{E}_t(\Delta p_{t+1}). \tag{2}$$

Here, m_t and p_t are the natural logarithms of the nominal money stock and the price level, respectively, at time t, $\{\varepsilon_t\}$ are random shocks such that $\mathbb{E}_{t-1}(\varepsilon_t) = 0$ and $\mathbb{E}(\varepsilon_t^2) = 1$, μ, σ and β are positive parameters, Δ is the first-difference operator, and $\mathbb{E}_t$ denotes conditional expectation given information available at time t.

Reorganizing (2) gives the forward-looking price equation

$$p_t = \frac{1}{1+\beta} m_t + \frac{\beta}{1+\beta} \mathbb{E}_t(p_{t+1}), \tag{3}$$

which admits the "fundamental" solution $p_t^F = \beta\mu + m_t$ as well as solutions which involve an additional "bubble" component. More specifically, if the existence of bubbles cannot be ruled out, $p_t = p_t^F + B_t$ is a solution to (3) for any bubble B_t which is explosive in conditional expectation and satisfies the equation

$$B_t = \frac{\beta}{1+\beta} \mathbb{E}_t(B_{t+1}). \tag{4}$$

The bubble process $\{B_t\}$ is typically assumed to be independent of the money-supply process $\{m_t\}$. Such bubbles may arise when individuals do not have confidence in the ability of monetary authorities to control the rate of inflation and expect increases in the price level higher than those that are justified by the underlying fundamentals (which is likely to happen in times of a hyperinflation).

Let us consider next a more general setup which allows for a stabilization plan in the form of reductions in the rate of monetary expansion. To be more specific, suppose that, instead of (1), money supply obeys the equation

$$\Delta m_t = \mu_0 + (\mu_1 - \mu_0) S_t + [\sigma_0 + (\sigma_1 - \sigma_0) S_t] \varepsilon_t, \tag{5}$$

where S_t is a binary random variable taking values 0 and 1, and μ_0, μ_1, σ_0 and σ_1 are positive parameters. This implies that the slope of the trend in m_t is μ_0 when $S_t = 0$ and μ_1 when $S_t = 1$; correspondingly, the standard deviation of the shocks is either σ_0 or σ_1, depending on the value of S_t. The random variables $\{S_t\}$ are assumed to be independent of $\{\varepsilon_t\}$ and to form a temporally homogeneous Markov chain with transition probabilities $P_{ij} = \mathbb{P}(S_{t+1} = j | S_t = i)$, $i, j = 0, 1$ $(0 < P_{ij} < 1)$. Assuming $\mu_0 < \mu_1$, one may think of the state associated with $S_t = 0$ as representing a stabilization state, with transitions into and out of it being governed by the conditional probabilities P_{ij}.

To obtain the solution for the path of prices under (5), let

$$p_t^F = \begin{cases} \kappa_0 + \gamma_0 m_t, & \text{if } S_t = 0, \\ \kappa_1 + \gamma_1 m_t, & \text{if } S_t = 1. \end{cases}$$

Then, in view of (3), κ_0, γ_0, κ_1 and γ_1 are such that, conditionally on $S_t = 0$,

$$\kappa_0 + \gamma_0 m_t = \frac{1}{1+\beta} m_t + \frac{\beta}{1+\beta} \left\{ P_{00}[\kappa_0 + \gamma_0(\mu_0 + m_t)] + (1 - P_{00})[\kappa_1 + \gamma_1(\mu_1 + m_t)] \right\},$$

while, conditionally on $S_t = 1$,

$$\kappa_1 + \gamma_1 m_t = \frac{1}{1+\beta} m_t + \frac{\beta}{1+\beta} \left\{ (1 - P_{11})[\kappa_0 + \gamma_0(\mu_0 + m_t)] + P_{11}[\kappa_1 + \gamma_1(\mu_1 + m_t)] \right\}.$$

These equations imply that

$$\kappa_0 = \frac{\beta}{1+\beta} \left[P_{00}(\kappa_0 + \mu_0) + (1 - P_{00})(\kappa_1 + \mu_1) \right], \tag{6}$$

$$\kappa_1 = \frac{\beta}{1+\beta} \left[(1 - P_{11})(\kappa_0 + \mu_0) + P_{11}(\kappa_1 + \mu_1) \right], \tag{7}$$

and $\gamma_0 = \gamma_1 = 1$. Thus, reorganizing (6) and (7), we have

$$\kappa_0 = \frac{\mu_0[P_{00} + \beta(1 - P_{11})] + \mu_1(1+\beta)(1 - P_{00})}{[(1+2\beta)/\beta] - P_{00} - P_{11}},$$

$$\kappa_1 = \frac{\mu_0 (1+\beta)(1 - P_{11}) + \mu_1[P_{11} + \beta(1 - P_{00})]}{[(1+2\beta)/\beta] - P_{00} - P_{11}}.$$

The fundamental solution for the path of prices may, therefore, be expressed as

$$p_t^F = \kappa_0 + (\kappa_1 - \kappa_0) S_t + m_t, \tag{8}$$

and thus, in view of (5) and (8),

$$\Delta p_t^F = \mu_0 + (\mu_1 - \mu_0) S_t + (\kappa_1 - \kappa_0)(S_t - S_{t-1}) + [\sigma_0 + (\sigma_1 - \sigma_0) S_t] \varepsilon_t, \tag{9}$$

with

$$\kappa_1 - \kappa_0 = \frac{(\mu_0 - \mu_1)(1 - P_{00} - P_{11})}{[(1+2\beta)/\beta] - P_{00} - P_{11}}.$$

The stochastic difference equation (9) describing the dynamics of prices inherits the autoregressive unit root present in the money-supply process, as well as the volatility changes the latter is subject to. In addition, the drift in prices undergoes discrete stochastic shifts driven by the changes in the slope of the trend in money supply. It is worth emphasizing that such shifts are present in prices despite the absence of bubbles.

As mentioned already, if the existence of bubbles cannot be ruled out, solutions to (3) are the sum of the fundamental solution p_t^F satisfying (8) and any bubble B_t satisfying (4). Since a bubble is defined only via (4), there is, in principle, an infinite variety of bubble processes. An empirically plausible class of stochastic, positive and periodically collapsing rational bubbles is described in Evans (1991). In our setting, these bubbles are of the form

$$B_{t+1} = \frac{1+\beta}{\beta} B_t \zeta_{t+1} \mathbb{I}(B_t \leq \tau) + \left[\delta + \frac{1+\beta}{\beta q}\left(B_t - \frac{\beta\delta}{1+\beta}\right)\eta_{t+1}\right]\zeta_{t+1}\mathbb{I}(B_t > \tau), \tag{10}$$

where τ and δ are positive parameters such that $(1+\beta)\tau > \delta\beta$, $\{\zeta_t\}$ are independent, identically distributed (i.i.d.) positive random variables, independent of $\{\varepsilon_t\}$, such that $\mathbb{E}_t(\zeta_{t+1}) = 1$, $\{\eta_t\}$ are i.i.d. Bernoulli random variables, independent of $\{\zeta_t\}$ and $\{\varepsilon_t\}$, such that $\mathbb{P}(\eta_t = 1) = 1 - \mathbb{P}(\eta_t = 0) = q$ $(0 < q \leq 1)$, and $\mathbb{I}(\cdot)$ is the indicator function (equal to 1 if its argument is true and 0 otherwise). Hence, a bubble grows at mean rate $(1+\beta)/\beta$ as long as $B_t \leq \tau$; when eventually $B_t > \tau$, it grows at the faster mean rate $(1+\beta)/(\beta q)$, but collapses to the mean value δ with probability $1-q$ each period. Assuming $\log \zeta_t$ is normally distributed, the maximum time span of such a collapsing bubble in a sample of T observations can be shown to be at most of order $\log T$ in probability (Phillips et al. (2011)), which is very short compared to the full sample size. The methodology discussed in the next section, based on a regime-switching autoregressive model, is well-suited to detecting periodically collapsing bubbles.

3 Testing for explosiveness in a model with independent stochastic changes

As illustrated in the context of the simple monetary model considered in the previous section, changes in the drift of prices, under high inflation, will typically be driven by changes in monetary policy, as will shifts in the volatility of prices. In the presence of a price bubble, however, it is reasonable to expect changes in the volatility of prices to be affected not only by changes in the underlying fundamentals associated with stabilization programmes pursued by monetary authorities but also by the size of the bubble.

In the test procedure of Hall et al. (1999), changes in the parameters of an augmented Dickey–Fuller regression (with homoskedastic errors) are assumed to be governed by a single two-state Markov chain. As a result, such changes may not necessarily be attributable to changes in the monetary policy stance, in the sense of the example discussed in Section 2, as they may reflect changes in policy, the existence of explosive episodes associated with bubbles, or both. The assumption of a state-independent error variance, although restrictive, is advisable in this context in order to ensure that filtering algorithms do not confuse periods during which there is genuine explosive behavior (due to an autoregressive root greater than one) with periods which are spuriously explosive due to large volatility induced by a bubble. This difficulty may be overcome to some extent by using a specification in which changes in the error variance are driven by a Markov process independent of the Markov process that governs changes in the other parameters of the model (see, e.g., Shi (2013)). However, even this specification does not allow us to disentangle fundamentals-driven changes in the drift, bubble-driven explosiveness, and volatility changes that may be fundamentals-driven and/or bubble-driven. In order to do so, we consider here a model in which changes in the intercept, the error variance and the roots of the characteristic equation are governed by separate Markov processes.[1]

3.1 Test procedure

For a time series $\{X_t\}$ of length T (X_t being p_t or m_t in our setting), our procedure for detecting explosiveness is based on a Markov-switching autoregressive model (of order

[1] In the case of indirect tests for bubbles based on recursive unit-root tests, the importance of considering potential changes in volatility has been highlighted by Harvey et al. (2016) and Monschang and Wilfling (2021), among others, who demonstrated that neglected heteroskedasticity can have deleterious effects on the properties of tests. Harvey et al. (2019a) and Harvey et al. (2019b) discuss recursive-type tests which allow for non-stochastic volatility subject to a countable number of jumps.

$K+1$) parameterized in the form

$$\Delta X_t = \alpha_0 + (\alpha_1 - \alpha_0)S_{1,t} + [\phi_0 + (\phi_1 - \phi_0)S_{3,t}]X_{t-1} + \sum_{k=1}^{K}[\psi_{0,k} + (\psi_{1,k} - \psi_{0,k})S_{3,t}]\Delta X_{t-k} + [\omega_0 + (\omega_1 - \omega_0)S_{2,t}]u_t, \quad (11)$$

for some non-negative integer K.[2] Here, $\{u_t\}$ are i.i.d. random variables with $\mathbb{E}(u_t) = 0$ and $\mathbb{E}(u_t^2) = 1$, $\{S_{1,t}\}$, $\{S_{2,t}\}$ and $\{S_{3,t}\}$ are latent random variables, independent of $\{u_t\}$, taking values in the set $\{0, 1\}$, and α_0, α_1, ϕ_0, ϕ_1, $\psi_{0,k}$, $\psi_{1,k}$, ω_0 and ω_1 are unknown parameters. The random variables $\{S_{1,t}\}$, $\{S_{2,t}\}$ and $\{S_{3,t}\}$ determine the prevailing state/regime at any given time and are assumed to form temporally homogeneous Markov chains, independent of one another, with transition probabilities

$$P_{ij}^{(r)} = \mathbb{P}(S_{r,t+1} = j | S_{r,t} = i) \quad (i, j = 0, 1;\ r = 1, 2, 3). \quad (12)$$

The composite regime process $\{\xi_t = (S_{1,t}, S_{2,t}, S_{3,t})\}$ may, therefore, be viewed as a temporally homogeneous Markov chain with values in the three-fold Cartesian product of $\{0, 1\}$ and transition probabilities

$$\mathbb{P}[\xi_{t+1} = (h, l, d) | \xi_t = (i, j, b)] = P_{ih}^{(1)} \cdot P_{jl}^{(2)} \cdot P_{bd}^{(3)} \quad (i, j, b, h, l, d = 0, 1). \quad (13)$$

It is worth noting that, although $\{\xi_t\}$ is an eight-state Markov chain, its transition matrix contains only six independent parameters that need to be estimated (this figure would rise to 56 if the components of ξ_t were assumed to be correlated with one another).

The key feature of the model in (11)–(13) is that changes in the error variance, regardless of whether they may be due to changes in the fundamentals or to the size of a bubble, are governed by $S_{2,t}$, while changes in the intercept are driven by $S_{1,t}$. At the same time, changes in the roots of the characteristic equation associated with the model, which are directly related to the possible presence of a bubble, are governed by $S_{3,t}$. In the regimes associated with $\xi_t = (i, j, 0)$ and $\xi_t = (i, j, 1)$, with $i, j = 0, 1$, $\{X_t\}$ is explosive if $\phi_0 > 0$ and $\phi_1 > 0$, respectively. Such explosive behavior is necessary (but not sufficient) for the existence of a rational bubble in the associated regime.

[2] As usual, the sum on the right-hand side of (11) is understood to be empty when $K = 0$.

The parameters of model (11)–(13) can be estimated by the method of maximum likelihood using a recursive algorithm analogous to that discussed in Hamilton (1994, pp. 692–694). Hence, a natural test of the unit-root null hypothesis $\phi_i = 0$ $(i = 0, 1)$ against the explosive alternative $\phi_i > 0$ will reject for large values of the statistic

$$\mathcal{T}_i = \frac{\hat{\phi}_i}{s(\hat{\phi}_i)} \quad (i = 0, 1),$$

where $s(\hat{\phi}_i)$ is the estimated standard error of $\hat{\phi}_i$ (here and elsewhere, a hat over a parameter signifies its maximum-likelihood estimator). As is common in the literature, maximum-likelihood estimates will be obtained under the assumption that u_t is normally distributed.

Like Hall et al. (1999), we propose to obtain p-values and/or critical values for right-tailed tests based on $\mathcal{T}_0$ and $\mathcal{T}_1$ by means of a parametric bootstrap procedure. This involves generating bootstrap data $\{X_t^*\}$ according to the recursive relation

$$X_t^* = X_{t-1}^* + \hat{\alpha}_0 + (\hat{\alpha}_1 - \hat{\alpha}_0)S_{1,t}^* + \sum_{k=1}^{K}[\hat{\psi}_{0,k} + (\hat{\psi}_{1,k} - \hat{\psi}_{0,k})S_{3,t}^*]\Delta X_{t-k}^* \\ + [\hat{\omega}_0 + (\hat{\omega}_1 - \hat{\omega}_0)S_{2,t}^*]u_t^*, \tag{14}$$

for $t = K + 2, \ldots, T$, setting $X_t^* = X_t$ for $t = 1, \ldots, K + 1$. Here, $\{u_t^*\}$ are i.i.d. standard normal random variables (independent of $X_1, \ldots, X_T$) and

$$S_{r,t}^* = \mathbb{I}(\mathbb{P}(S_{r,t} = 1|I_t; \hat{\vartheta}) > 0.5) \quad (r = 1, 2, 3), \tag{15}$$

$\mathbb{P}(S_{r,t} = 1|I_t; \hat{\vartheta})$ being the inferred probability that $S_{r,t} = 1$, given data $I_t = \{X_1, \ldots, X_t\}$ available through time t, and based on the maximum-likelihood estimate $\hat{\vartheta}$ of the vector of parameters of model (11)–(13). The bootstrap value $\mathcal{T}_i^*$ of $\mathcal{T}_i$ $(i = 0, 1)$ is then obtained by applying the definition of $\mathcal{T}_i$ to the bootstrap data $X_1^*, \ldots, X_T^*$ in place of the original data $X_1, \ldots, X_T$. Repeating the above two steps a large number of times, say N, yields a set of $\mathcal{T}_i^*$ values, and the bootstrap p-value for a right-tailed test based on $\mathcal{T}_i$ is computed as the proportion of the $\mathcal{T}_i^*$ values that are greater than the observed value of $\mathcal{T}_i$. Hence, the hypothesis of a unit root $(\phi_i = 0)$ is rejected in favor of explosiveness $(\phi_i > 0)$, at a given level of significance α $(0 < \alpha < 1)$, if the bootstrap p-value does not exceed α.

Equivalently, the test rejects if the observed value of $\mathcal{T}_i$ exceeds the $\lceil(1-\alpha)N\rceil$-th largest of the $\mathcal{T}_i^*$ values, $\lceil(1-\alpha)N\rceil$ being the smallest integer greater than or equal to $(1-\alpha)N$.

Note that the bootstrap procedure described above uses the inferred regimes $S_{r,t}^*$ from (15) in each of the N bootstrap samples generated according to (14). This ensures that regime shifts in the bootstrap data mimic shifts in the observed data as closely as possible. Although the classification rule $\mathbb{P}(S_{r,t}=1|I_t;\hat{\vartheta}) \gtrless 0.5$ used to define the indicators $S_{r,t}^*$ is somewhat arbitrary, its use is unlikely to be problematic in practice; in our experience, inferred probabilities about the prevailing regime at each sample date are rarely near the non-informative 0.5 value. Also note that the indicators $S_{r,t}^*$ may be constructed using smoothed probabilities $\mathbb{P}(S_{r,t}=1|I_T;\hat{\vartheta})$ based on the full set of data in place of the filtered probabilities $\mathbb{P}(S_{r,t}=1|I_t;\hat{\vartheta})$; smoothed and filtered probabilities typically differ very little from each other in applications. Finally, it is worth emphasizing that bootstrap data are generated in such a way that they satisfy the constraints of the hypotheses being tested ($\phi_0=0$ and $\phi_1=0$) even though the observed data may not. Restricting the bootstrap data-generating mechanism so that it satisfies the hypotheses under test is important for ensuring that the bootstrap procedure provides an accurate approximation to the null sampling distribution of $\mathcal{T}_0$ and $\mathcal{T}_1$, and yields tests that reject with high probability when there is explosiveness in the observed data.

3.2 Simulations

To get some insight into the properties of the proposed testing methodology, we report some limited Monte Carlo simulation results. The experimental design is chosen so as to reflect the characteristics of real-world time series such as those analyzed in Section 4. All results are obtained from 1,000 Monte Carlo replications, using $N=200$ bootstrap replications in each of these to compute p-values for right-tailed unit-root tests based on the statistics $\mathcal{T}_0$ and $\mathcal{T}_1$.[3] The proportion of Monte Carlo replications in which the unit-root null hypothesis $\phi_i=0$ $(i=0,1)$ is rejected in favor of the explosive alternative $\phi_i>0$,

[3]Using a larger number of bootstrap replications increases the computational cost of the Monte Carlo experiments considerably without any substantial changes in the results. Hall (1986) provides a theoretical explanation of the ability of bootstrap procedures to yield good results without the need to employ a large number of replications.

at the 5% significance level, in each of three experiments is shown in Table 1.

Table 1: Rejection Frequencies of Right-Tailed Tests

	MC.1		MC.2		MC.3
Null	$T = 60$	$T = 120$	$T = 60$	$T = 120$	$T = 60$
$\phi_0 = 0$	0.1548	0.1377	0.1250	0.2145	0.8033
$\phi_1 = 0$	0.0875	0.0623	1.0000	1.0000	0.9508

In the first simulation experiment (MC.1), artificial data are generated according to the Markov-switching model (11)–(13) with $K = 1$, normally distributed u_t, and the following parameter values:

$$\alpha_0 = 0.1, \quad \alpha_1 = 0.8, \quad \phi_0 = 0, \quad \phi_1 = 0, \quad \psi_{0,1} = -0.4, \quad \psi_{1,1} = -0.2,$$
$$P_{00}^{(1)} = 0.9, \quad P_{11}^{(1)} = 0.7, \quad P_{00}^{(2)} = 0.6, \quad P_{11}^{(2)} = 0.85, \quad P_{00}^{(3)} = 0.8, \quad P_{11}^{(3)} = 0.5,$$
$$\omega_0 = \sqrt{0.01}, \quad \omega_1 = \sqrt{0.35}.$$

Under this design, there is a unit root (and no explosiveness) in all eight Markov regimes. The test for $\phi_0 = 0$ has rejection frequencies in excess of the nominal level for the relatively modest sample sizes considered. By contrast, the rejection frequencies of the test for $\phi_1 = 0$ are reasonably close to the nominal level, especially for the larger of the two sample sizes.

In the second simulation experiment (MC.2), the data-generating process is the same as the one used in experiment MC.1 except for $\phi_1 = 0.05$. This implies that the process has a unit root in regimes associated with $S_{3,t} = 0$ and an explosive root in regimes associated with $S_{3,t} = 1$ (the largest root of its characteristic equation, conditionally on $S_{3,t} = 1$, being 1.0419). The test procedure has impressive power to detect explosiveness, the test based on $\mathcal{T}_1$ correctly rejecting the hypothesis $\phi_1 = 0$ in favor of $\phi_1 > 0$ in all Monte Carlo replications, even for the smaller of the two sample sizes. As in the first experiment, the test based on $\mathcal{T}_0$ rejects the hypothesis $\phi_0 = 0$ more frequently than the nominal level of the test implies, especially when $T = 120$.[4]

[4]One possible, albeit computationally expensive, way to reduce the discrepancy between the nominal level of the tests and their actual probability of incorrectly rejecting the null hypothesis is to rely on a nested double-bootstrap procedure to adjust the p-values (see, e.g., Davison and Hinkley (1997, pp. 175–180)).

In the final simulation experiment (MC.3), we revisit the monetary model discussed in Section 2 and consider the properties of the test procedure in the presence of periodically collapsing bubbles of the well-known type proposed by Evans (1991). More specifically, the tests are applied to artificial data generated as $X_t = p_t^F + B_t$, where p_t^F satisfies (9) and B_t satisfies (10). The parameter values used are:

$$\mu_0 = 0.1, \quad \mu_1 = 0.2, \quad \sigma_0 = \sqrt{0.0005}, \quad \sigma_1 = \sqrt{0.001}, \quad P_{00} = P_{11} = 0.85,$$
$$\beta = 8, \quad \tau = 4, \quad \delta = 0.2, \quad q = 0.7.$$

In addition, $\{\varepsilon_t\}$ are i.i.d. with a standard normal distribution, $\{\log \zeta_t\}$ are i.i.d. with a normal distribution having mean $-10^{-8}/2$ and standard deviation 10^{-4} (so that the mean of ζ_t is 1), and the initial value of B_t is 0.01.[5] We only consider samples of size $T = 60$ in this case to avoid having a large number of erupting bubbles, which seems desirable in our context (unlike explosive episodes in asset prices, hyperinfaltions are very limited in number). Despite the relatively small sample size, the test procedure based on the Markov-switching model (11)–(13) rejects the unit-root hypothesis in favor of explosiveness with very high frequency. The fact that evidence against the unit-root hypothesis is found in all Markov regimes is not very surprising. The bubble process satisfying (10) is explosive in conditional expectation even during periods of slow growth, and this is reflected in the rejection frequencies of the tests, which are always high but more so in regimes associated with the rapidly expanding phase of the bubble. By varying the parameters determining the dynamics of the bubble process and of the fundamentals, it is obviously possible to obtain realizations of $\{X_t\}$ for which the explosiveness generated by the bubble during its slow-growth phase would be undetectable by a test based on $\mathcal{T}_0$ or $\mathcal{T}_1$ as a consequence of the observable time series behaving much like a difference-stationary process in the regime associated with the slowly expanding bubble.

[5]It is worth reiterating that bubbles can in principle take an infinite number of forms, one of which is (10). We focus on this specification since it has become the benchmark in the literature. Other useful specifications include those of Fukuta (1998) and Rotermann and Wilfling (2018).

4 Empirical results

We illustrate the methodology discussed in the previous section by analyzing hyperinflation episodes in Argentina, Brazil, Germany and Poland. More specifically, we consider Argentina's 1983:1 to 1989:12 hyperinflation, the Brazilian hyperinflation of 1988:8 to 1994:3, and the 1921:1 to 1923:12 German and Polish hyperinflations. In all four cases, the analysis is based on monthly observations on the logarithm of prices and the logarithm of money supply.[6] The objective is to assess whether any explosiveness found in prices may be attributable to rational bubbles once we account for changes in the drift and volatility of the prices that may arise as a result of different stabilization attempts by the monetary authorities. Since the time-series properties of the underlying economic fundamentals (i.e., money supply) are unknown, it is imperative that these be considered together with the properties of prices.

Maximum-likelihood estimation results for the Markov-switching model (11)–(13), with $K = 4$, are reported in Table 2 and Table 3 for (the logarithmically transformed) prices and money supply, respectively.[7] Figures in parentheses are estimated standard errors (obtained from the negative Hessian of the log-likelihood function). Figures in square brackets are bootstrap p-values for tests of the null hypotheses $\phi_0 = 0$ and $\phi_1 = 0$ against the alternatives $\phi_0 > 0$ and $\phi_1 > 0$, respectively (computed from $N = 1,000$ bootstrap replications). As a check for neglected non-linear dependence in the errors of the model due to conditional heteroskedasticity, approximate p-values for Ljung–Box statistics based on the first 4 and 8 estimated autocorrelations of squared standardized residuals are also reported (labelled $LB_{\hat{u}^2}(4)$ and $LB_{\hat{u}^2}(8)$, respectively); these reveal no significant signs of autocorrelation in the squared standardized residuals from any of the fitted models. Lastly, plots of the inferred probabilities of a high-drift regime (i.e., $\mathbb{P}(S_{1,t} = 1|I_t; \hat{\vartheta})$) and a large-root regime (i.e., $\mathbb{P}(S_{3,t} = 1|I_t; \hat{\vartheta})$) for the four countries are shown in Figures 1–4.[8]

[6] Argentinian data are taken from Hall et al. (1999); Brazilian data are taken from the macroeconomic database of the Institute of Applied Economic Research (www.ipeadata.gov.br); German and Polish data are taken from Sargent (1983).

[7] Estimates of the transition probabilities and of the coefficients $\psi_{0,k}$ and $\psi_{1,k}$, $k = 1, 2, 3, 4$, are not reported in order to save space, but are available upon request.

[8] The corresponding plots associated with changes in volatility ($\mathbb{P}(S_{2,t} = 1|I_t; \hat{\vartheta})$) are available upon

Table 2: Parameter Estimates – Prices (standard errors in parentheses, bootstrap p-values in square brackets)

Parameter	Argentina	Brazil	Germany	Poland
α_0	0.0528	0.0530	0.4447	-1.8523
	(0.0065)	(0.0068)	(0.1791)	(0.0374)
α_1	0.1292	0.1398	1.0983	-1.7246
	(0.0113)	(0.0099)	(0.1890)	(0.0364)
ω_0	0.0174	0.0158	0.0932	0.0106
	(0.0016)	(0.0021)	(0.0125)	(0.0015)
ω_1	0.3425	0.0285	2.1274	0.3982
	(0.0911)	(0.0069)	(0.8740)	(0.1499)
ϕ_0	-0.0131	0.0097	-0.0409	0.1737
	(0.0014)	(0.0018)	(0.0244)	(0.0035)
	[0.9394]	[0.0089]	[0.8325]	[0.0044]
ϕ_1	0.0237	0.0161	0.0350	0.1886
	(0.0039)	(0.0015)	(0.0154)	(0.0036)
	[0.2060]	[0.0071]	[0.1967]	[0.0029]
$LB_{\hat{u}^2}(4)$	0.8889	0.9858	0.6520	0.2058
$LB_{\hat{u}^2}(8)$	0.9825	0.9992	0.6348	0.6251

In the case of Argentina, the estimated value of ϕ_1 for prices is positive, indicating that regimes associated with $S_{3,t} = 1$ are potentially explosive. However, the relatively large p-value for a right-tailed test of $\phi_1 = 0$ reveals no compelling evidence against the unit-root hypothesis, which rules out the existence of a bubble. More importantly, there is much stronger evidence in favor of explosiveness in money supply, the estimates of both ϕ_0 and ϕ_1 being significantly positive. This suggests that hyperinflation episodes were primarily driven by the monetary policy pursued at the time. It is also interesting to note that hyperinflation periods are associated with higher drifts in prices.

The inferred probabilities of a large-root regime identify three episodes of potential explosiveness in prices, namely 1985:4 to 1985:9, 1987:11 to 1988:8, and 1989:1 to 1989:12. These findings are similar to those reported in Hall et al. (1999). However, as mentioned

request.

Table 3: Parameter Estimates – Money (standard errors in parentheses, bootstrap p-values in square brackets)

Parameter	Argentina	Brazil	Germany	Poland
α_0	-0.0716	0.2083	0.1978	-0.0314
	(0.0490)	(0.0244)	(0.2791)	(0.1295)
α_1	-0.0052	0.5486	0.3358	0.0279
	(0.0480)	(0.0300)	(0.2829)	(0.1251)
ω_0	0.0491	0.0294	0.0166	0.0088
	(0.0046)	(0.0062)	(0.0023)	(0.0026)
ω_1	0.4485	0.1137	4.1909	0.1201
	(0.1507)	(0.0158)	(1.4650)	(0.0486)
ϕ_0	0.0097	-0.0203	-0.0427	0.0063
	(0.0034)	(0.0996)	(0.0142)	(0.0101)
	[0.0302]	[0.8418]	[0.9293]	[0.3595]
ϕ_1	0.0098	-0.0171	-0.0095	0.0357
	(0.0038)	(0.0029)	(0.0154)	(0.0166)
	[0.0265]	[0.8150]	[0.6653]	[0.1736]
$LB_{\hat{u}^2}(4)$	0.8748	0.1639	0.9976	0.9804
$LB_{\hat{u}^2}(8)$	0.9866	0.5887	0.2713	0.9977

above, the hypothesis of a unit root in regimes associated with either $S_{3,t} = 1$ or $S_{3,t} = 0$ cannot be rejected at usual levels of significance, a finding which is inconsistent with the presence of a rational price bubble. This conclusion is different from the one reached by Hall et al. (1999) and is the result of relying on a more flexible model formulation incorporating independent Markov changes in its parameters, a feature which is of particular importance when considering inflationary periods during which monetary authorities actively try to contain growth in prices.

In the case of Brazilian prices, we find significantly positive drifts in the regimes associated with $S_{1,t} = 0$ and $S_{1,t} = 1$; in addition, the small p-values for tests of $\phi_0 = 0$ and $\phi_1 = 0$ indicate that there is strong evidence in favor of explosiveness in the regimes associated with both $S_{3,t} = 0$ and $S_{3,t} = 1$. The estimated drifts in money supply are much higher than those in prices, but the unit-root hypothesis cannot be rejected in any of the

Markov regimes. These findings are consistent with the presence of a rational price bubble in the entire sample period, the filtered probabilities associated with $S_{3,t}$ identifying periods with different degrees of explosiveness (as measured by ϕ_0 and ϕ_1). Nevertheless, it is possible that the very high rates of monetary expansion were in part responsible for the eventual explosiveness of prices.[9]

The results for the German price series are similar to those for Argentina, in the sense that we find significantly positive drifts and very weak evidence against the unit-root hypothesis. The latter hypothesis cannot be rejected for money supply either, the two drift coefficients for which are positive but insignificantly different from zero. The filtered probabilities identify the period from 1923:5 to 1923:12 as being associated with potential explosiveness, a period which is not associated with explosiveness in money supply. However, as mentioned already, the p-value for a right-tailed test for $\phi_1 = 0$ is relatively high, so these findings are not consistent with the presence of a rational bubble during the German hyperinflation.[10]

Finally, the results for Polish prices reveal negative drifts, while the hypothesis of a unit root is very firmly rejected in favor of explosiveness regardless of whether $S_{3,t} = 0$ or $S_{3,t} = 1$. Since no significant evidence of explosive behavior is found in the money supply, the results point towards a price bubble being present during the whole of the sample period. As in the case of Brazil, the filtered probabilities associated with $S_{3,t}$ separate the sample into periods with different degrees of explosiveness.[11]

5 Concluding Remarks

This paper has considered how to detect the presence of rational bubbles during hyperinflations via an analysis of the integration properties of relevant observable time series.

[9] The analysis of de Holanda Barbosa and da Silva Filho (2015), based on a framework that is not directly comparable to ours, supports the claim that the Brazilian hyperinflation was caused by fundamentals via an increasing fiscal deficit financed by money.

[10] Using a different methodology, Blackburn and Sola (1996) conclude that the empirical evidence is consistent with the existence of a purely collapsing stochastic bubble.

[11] Hooker (2000) concludes, using a different approach, that the empirical evidence does not support the existence of a rational bubble during either the German or the Polish hyperinflations.

We have discussed a generalization of the test procedure of Hall et al. (1999) which allows for independent stochastic changes in the drift of the time series of interest, in its volatility and in the roots of its autoregressive characteristic equation. The former two types of changes may or may not be related to the presence of a bubble and it is, therefore, important to separate these from changes in the integration properties of the series that are driven by the explosive behavior of a bubble. Unlike the case of bubbles in financial time series, monetary authorities are expected to pursue an active policy during hyperinflation episodes, something which affects the time series under consideration and must be taken into account when formulating the model used to test the no-bubbles hypothesis. In the proposed methodology, this is achieved by constructing tests of the hypothesis of a unit root against explosive alternatives in the context of an autoregressive model with stochastic parameter variation, with changes in different parameters being governed by independent Markov processes. Such tests have substantial power to detect explosive behavior, regardless of whether explosiveness is an intrinsic characteristic of the data or due to the presence of a rational bubble. In view of the fact that structural change and regime shifts are endemic, both in the sense of changing policy regimes and in the sense of changes in the general economic structure, the framework considered here provides a reasonable and meaningful way of accounting for such changes in the economic situations discussed in this paper.

References

Blackburn, K. and Sola, M.: 1996, Market fundamentals versus speculative bubbles: A new test applied to the German hyperinflation, *International Journal of Finance and Economics* **1**, 303–317.

Davison, A. C. and Hinkley, D. V.: 1997, *Bootstrap Methods and their Application*, Cambridge University Press, Cambridge.

de Holanda Barbosa, F. H. and da Silva Filho, T. N. T.: 2015, Bubble, weak and strong hyperinflation: Theory and empirical evidence, *EconomiA* **16**, 145–156.

Driffill, J. and Sola, M.: 1998, Intrinsic bubbles and regime-switching, *Journal of Monetary Economics* **42**, 357–373.

Evans, G. W.: 1991, Pitfalls in testing for explosive bubbles in asset prices, *American Economic Review* **81**, 922–930.

Fukuta, Y.: 1998, A simple discrete-time approximation of continuous-time bubbles, *Journal of Economic Dynamics and Control* **22**, 937–954.

Hall, P.: 1986, On the number of bootstrap simulations required to construct a confidence interval, *Annals of Statistics* **14**, 1453–1462.

Hall, S. G., Psaradakis, Z. and Sola, M.: 1999, Detecting periodically collapsing bubbles: A Markov-switching unit root test, *Journal of Applied Econometrics* **14**, 143–154.

Hamilton, J. D.: 1986, On testing for self-fulfilling speculative price bubbles, *International Economic Review* **27**, 545–552.

Hamilton, J. D.: 1994, *Time Series Analysis*, Princeton University Press, Princeton.

Harvey, D. I., Leybourne, S. J., Sollis, R. and Taylor, A. M. R.: 2016, Tests for explosive financial bubbles in the presence of non-stationary volatility, *Journal of Empirical Finance* **38**, 548–574.

Harvey, D. I., Leybourne, S. J. and Zu, Y.: 2019a, Sign-based unit root tests for explosive financial bubbles in the presence of deterministically time-varying volatility, *Econometric Theory* **36**, 1–48.

Harvey, D. I., Leybourne, S. J. and Zu, Y.: 2019b, Testing explosive bubbles with time-varying volatility, *Econometric Reviews* **38**, 1131–1151.

Hooker, M. A.: 2000, Misspecification versus bubbles in hyperinflation data: Monte Carlo and interwar European evidence, *Journal of International Money and Finance* **19**, 583–600.

Monschang, V. and Wilfling, B.: 2021, Sup-ADF-style bubble-detection methods under test, *Empirical Economics* **61**, 145–172.

Phillips, P. C. B., Shi, S. and Yu, J.: 2015, Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500, *International Economic Review* **56**, 1043–1077.

Phillips, P. C. B., Wu, Y. and Yu, J.: 2011, Explosive behavior in the 1990s Nasdaq: When did exuberance escalate asset values?, *International Economic Review* **52**, 201–226.

Rotermann, B. and Wilfling, B.: 2018, A new stock-price bubble with stochastically deflating trajectories, *Applied Economics Letters* **25**, 1091–1096.

Sargent, T. J.: 1983, The ends of four big inflations, *in* R. E. Hall (ed.), *Inflation: Causes and Effects*, University of Chicago Press, Chicago, pp. 41–98.

Shi, S.-P.: 2013, Specification sensitivities in the Markov-switching unit root tests for bubbles, *Empirical Economics* **45**, 697–713.

Figure 1: Filtered Probabilities – Argentina

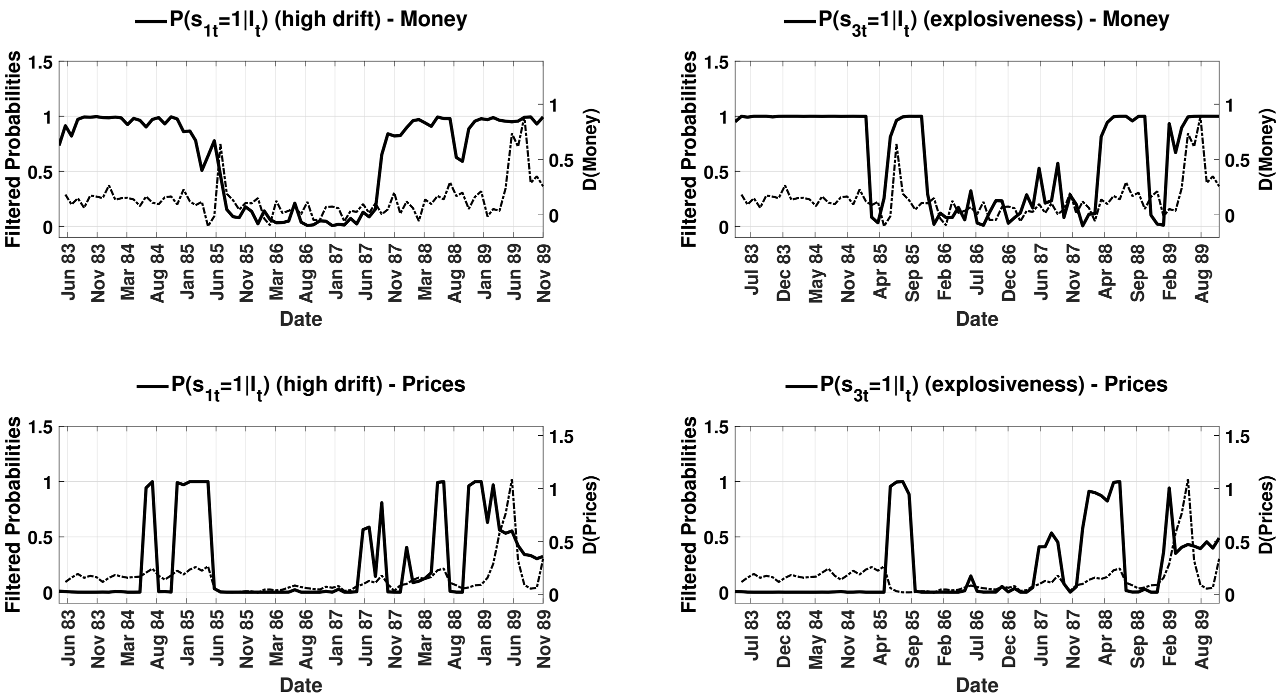

Figure 2: Filtered Probabilities – Brasil

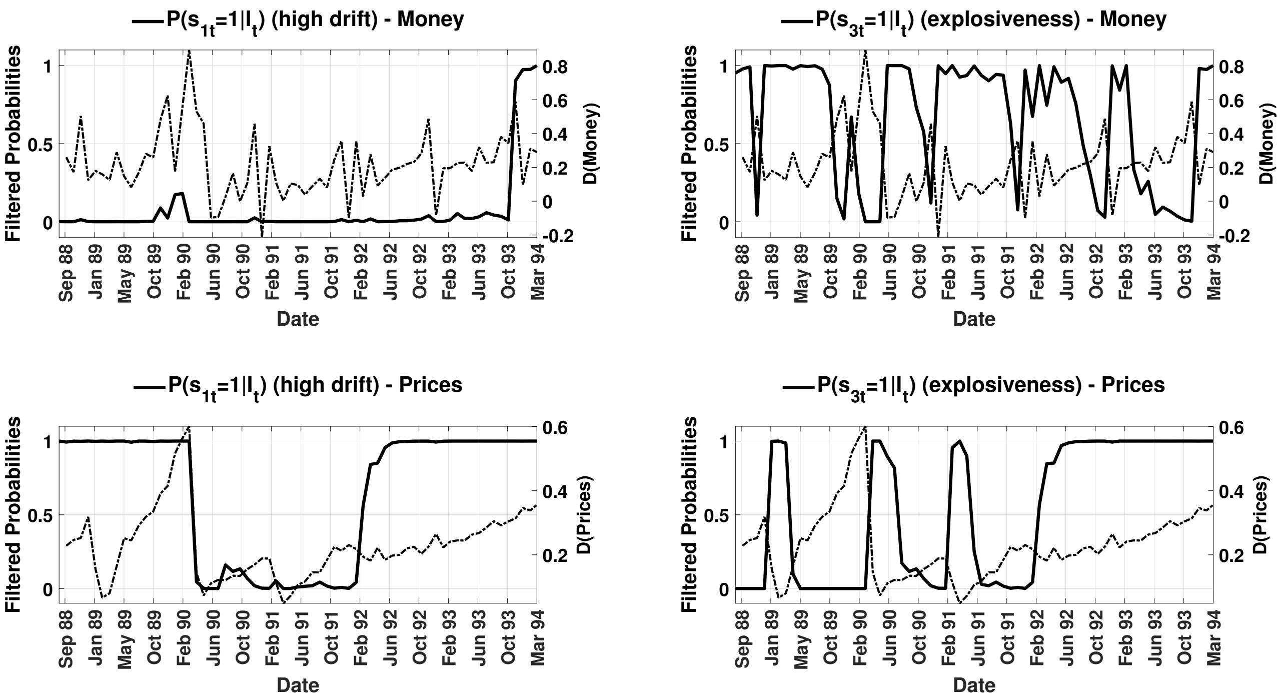

Figure 3: Filtered Probabilities – Germany

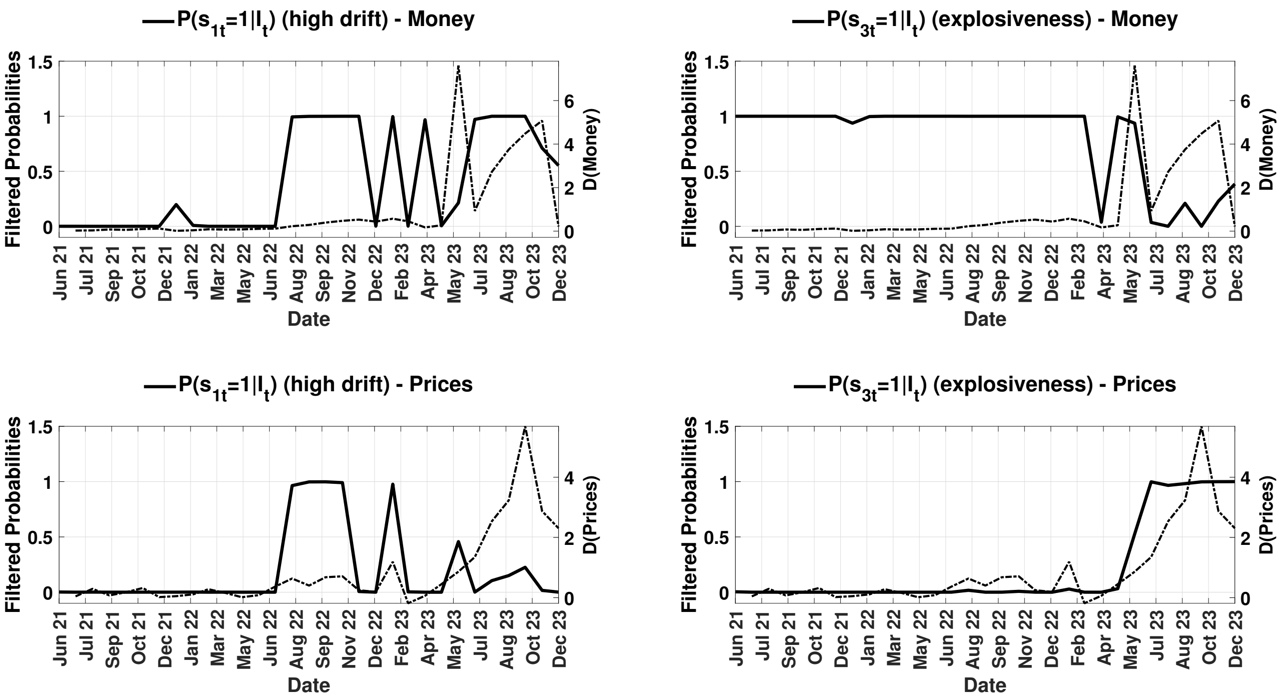

Figure 4: Filtered Probabilities – Poland