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# **Solving Incomplete Markets Models by Derivative Aggregation**

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# Solving Incomplete Markets Models by Derivative Aggregation

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## **Abstract**

This article presents a novel computational approach to solving models with both uninsurable idiosyncratic and aggregate risk that uses projection methods, simulation and perturbation. The approach is shown to be both as efficient and as accurate as existing methods on a model based on Krusell and Smith (1998), for which prior solutions exist. The approach has the advantage of extending straightforwardly, and with reasonable computational cost, to models with a greater range of diversity between agents, which is demonstrated by solving both a model with heterogeneity in discount-rates and a lifecycle model with incomplete markets.

JEL Codes: C63, E21, E32

Keyword: Idiosyncratic Risk, Business Cycles, Numerical Methods

# 1 Introduction

Any researcher interested in questions regarding the interaction of aggregate and individual outcomes in the economy will, at some point, want to solve models that include both aggregate and individual uncertainty. Though some models of this type may have analytical solutions, in most cases they must be solved computationally using numerical approaches. Such methods have been available since at least the pioneering contribution of Krusell and Smith (1998) (henceforth K & S). There have been many refinements to the available algorithms since, but the solution of models with anything other than the most rudimentary shock structures remains computationally expensive and slow.

This paper presents a novel algorithm that uses a combination of projection methods, simulation and perturbation to solve the model. The algorithm is in principle applicable to any model of incomplete markets with a clear relationship between individual and aggregate states. To demonstrate the high level of accuracy and low computational cost of the approach I use it to solve the model described in Den Haan et al. (2010), for which it is shown to be comparable to the explicit aggregation (XPA) algorithm of Den Haan and Rendahl (2010). The latter was found to be one of the best in terms of accuracy and performance in Den Haan (2010). I then demonstrate that the algorithm is easily extended to both an economy in which individuals have differing preferences, and a life-cycle economy with a large number of generations, whilst remaining reasonably accurate and computationally feasible.

The remainder of this paper is structured as follows: section 2 presents the benchmark model used, section 3 discusses existing solution approaches and section 4 introduces the new algorithm that I propose. Section 5 presents parameters and results from the baseline model, a model with added preference heterogeneity and a lifecycle model, and section 6 concludes.

## 2 The Benchmark Model

The benchmark model solved to illustrate the technique is that presented in Den Haan et al. (2010).

### 2.1 The Production Technology

The economy is a production economy with competitive goods, labour and physical capital markets. Firms in the economy face the production function

$$Y_t = a_t K_t^\alpha (\bar{l} L_t)^{1-\alpha} \quad (1)$$

where  $a_t$  is aggregate productivity,  $K_t$  is aggregate capital,  $L_t$  is employment and  $\bar{l}$  is the time endowment per employed person.  $a_t$  follows an exogenous stochastic process, hence aggregate output is uncertain.

Firms hire capital and labour to maximise profits each period. The firms' first order conditions yield a rental rate of capital,  $r_t$ , and wage per unit of time worked,  $w_t$ , of

$$r_t = \alpha a_t \left( \frac{K_t}{\bar{l} L_t} \right)^{\alpha-1} \quad (2)$$

$$w_t = (1 - \alpha) a_t \left( \frac{K_t}{\bar{l} L_t} \right)^{1-\alpha} \quad (3)$$

### 2.2 Households

The economy is populated by a continuum of infinitely-lived agents of measure one, indexed on the unit interval. Each agent  $i$  may be employed or unemployed in any period, but cannot choose which: the individual employment process is exogenous. There is a government which pays unemployment benefits  $\mu \bar{l} w_t$  to

each unemployed agent, so that  $\mu$  is the replacement rate of unemployment benefits. The government runs a balanced budget and finances the benefits by levying a proportional income tax  $\tau_t = \frac{\mu(1-L_t)}{L_t}$ .

Insurance markets are incomplete. Specifically, agents cannot insure against loss of income due to unemployment. The only opportunity for partial risk mitigation is to invest in production capital. Agents have identical utility functions and maximise expected lifetime utility subject to their budget constraint. Agent  $i$ 's problem is thus

$$\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} E \left[ \sum_{t=0}^{\infty} \left( \beta^t \frac{(c_t^i)^{1-\gamma} - 1}{1-\gamma} \right) \right] \quad (4)$$

$$\text{s.t. } c_t^i + k_{t+1}^i = (1 + r_t - \delta)k_t^i + [(1 - \tau_t)\bar{l}e_t^i + \mu(1 - e_t^i)]w_t \quad (5)$$

$$k_{t+1}^i \geq 0 \quad (6)$$

where  $c_t^i$  is consumption,  $k_t^i$  is individual capital holdings and  $e_t^i \in \{0, 1\}$  is 1 for employed agents and 0 for the unemployed.  $\beta$  is the per-period discount rate,  $\gamma$  is the coefficient of relative risk aversion and  $\delta$  the depreciation rate of physical capital.

Solving the individuals' maximisation problem yields first-order condition

$$\beta E \left[ (c_{t+1}^i)^{-\gamma} (1 + r_{t+1} - \delta) \right] = (c_t^i)^{-\gamma} - \phi_t^i \quad (7)$$

where  $\phi_t^i \geq 0$  is the multiplier on the borrowing constraint.

### 2.3 Effects of Incomplete Markets

Agents' inability to fully insure against idiosyncratic income shocks, coupled with their differing history of such shocks, imply that they will hold differing amounts of capital when they reach time  $t$ . Aggregate capital is therefore the mean of the non-degenerate capital distribution:

$$K_t = \int_{i=0}^1 k_t^i di \tag{8}$$

In order to calculate the expectation in eq. (7), agents must form expectations over  $r_{t+1}$ , which in turn depends on  $K_{t+1}$ . By eq. (8) the latter depends on the individual capital distribution in  $t + 1$  and hence on the individual capital distribution in period  $t$ . This distribution is therefore part of the state that determines individual choices. Since the distribution and, by implication, the state are infinite dimensional, common methods for solving forward-looking models do not apply.

## 3 Prior Solution Approaches

The contribution of Krusell and Smith (1998) was to show that restricting the number of state variables that individuals consider to a finite (and small) number allows an approximate solution to be found numerically. Two key properties of this solution stand out: First, individuals' predictions are very accurate, implying that they would be happy to stick with their prediction mechanism rather than trying to find a better one. Second, one aggregate state variable, the mean capital held, is sufficient, so that the economy behaves much like a representative agent one. This second property was termed *approximate aggregation* by Krusell and Smith (1998).

### 3.1 The Underlying Assumption

To state the assumption both ascribed to individuals and used to solve the model more formally:

Let  $\Omega = \{[0, 1] \rightarrow \mathbb{R}^+ \times \{0, 1\}\}$  be the set of all distributions of individuals over capital holdings and employment levels. The economy is a transformation  $T : \Omega \times \mathbb{R} \rightarrow \Omega$  that takes such a distribution and an exogenous shock in period  $t$  and produces a new distribution in period  $t + 1$ . Further, for any vector  $S_t$  of aggregate variables, let  $S_\Omega : \Omega \rightarrow \mathbb{R}^n$  be the function that calculates  $S_t$  from  $\omega_t \in \Omega$ .

**Assumption 1** *For the subset  $\Omega^* \subseteq \Omega$  of mappings that are actually realised there exists a threshold  $\epsilon \ll 1$ , a probability threshold  $p \gg 0$  and a forecasting function  $S^e : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  s.t.  $\Pr(|\frac{S^e(S_\Omega(\omega), a)}{S_\Omega(T(\omega, a))} - 1| < \epsilon | \omega \in \Omega^*) > p$ . In other words, for any wealth distribution  $\omega$  that might be realised the forecasting function  $S^e$  provides forecasts of  $t+1$  period aggregates accurate to within relative difference  $\epsilon$  with probability  $p$ .*

Note that, in the limit as  $\epsilon \rightarrow 0$  and  $p \rightarrow 1$ ,  $S^e(S_\Omega(\omega), a) \rightarrow S_\Omega(T(\omega, a))$ , so that the agents would be able to predict with perfect accuracy the value of next period's state variables.

### 3.2 The Individual Problem

Under this assumption, individuals consider a finite vector  $S_t^1$  of aggregate variables when forming expectations over  $t + 1$ -period prices. They therefore need to predict  $K_{t+1}$  based on the current aggregate state. They need to find

$$K_{t+1}^e(S_t, a_t) \tag{9}$$

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<sup>1</sup> $K_t$  must be one of the values in  $S_t$  since it is the aggregate variable required to calculate  $t$ -period prices.



Agents also expect to be alive in  $t + 2$ , however, and therefore need to form expectations over  $t + 2$  period prices. To use the function above to predict them requires that they have expectations over all members of  $S_{t+1}$ . They must therefore expand their forecasts to predict all members of  $S$ :

$$S_{t+1}^e(S_t, a_t) \tag{10}$$

Given these expectations over aggregate outcomes they can then solve their individual problem to determine the individual capital transition rule

$$k_{t+1}(k_t, e_t, S_t, a_t; S_{t+1}^e) \tag{11}$$

### 3.3 The Modeller's Problem

As presented above, individuals form expectations over aggregate outcomes and can consequently solve their individual optimisation problem. The researcher solving the model cannot follow this apparently two-stage approach, since aggregate outcomes are aggregates of individual choices. The 'first' stage, solving the aggregate problem, therefore depends on the 'second' stage as well. The two functions must be solved for simultaneously.

The approach taken by K & S, and also followed in this paper, is to make an initial guess regarding  $S_{t+1}^e$ , say  $S_{t+1}^0$ , and then to iterate over the following two steps until successive iterations yield results within a given small margin of error of each other over the whole range of the two functions:

1. Find  $k_{t+1}^s \equiv k_{t+1}(k_t, e_t, S_t, a_t; S_{t+1}^s)$
2. Calculate the implied aggregate transition to get  $S_{t+1}^{s+1}(S_t, a_t)$

A number of solution methods follow this broad outline and differ only in the choice of variables in  $S_t$  and the method for performing the two steps above.

K & S allow the first two moments of the capital distribution in  $S_t$ , although they found that the first is sufficient. They use value function iteration on a grid in step 1, and solve step 2 by simulating an economy populated by a large number of individuals for a large number of periods and then estimating log-linear aggregate transition rules for  $S_{t+1}^{s+1}$ .

Carroll (2006) replaces the value function iteration in step 1 with backward iteration on the Euler equation. This approach is much less computationally intensive because it does not require a numerical optimisation step. He calls this approach the *Method of Endogenous Gridpoints*.

Young (2010) uses a discrete representation of the entire capital distribution, rather than a large number of individuals, in step 2. This removes one of the potential issues encountered in the K & S approach, namely that the individual stochastic shock realisation used during simulation affect the solution.

Den Haan and Rendahl (2010) use Carroll's approach in step 1, but replace step 2 by approximating the individual transition rule with a (piecewise) linear function, so that aggregation of the function equates to calculating the function value at the average capital holdings. This does away with the need for simulation altogether, but requires the introduction of an additional aggregate state variable so that the capital held by employed and unemployed agents can be tracked separately, because the individual transition rules differ between the two groups. Hence  $S_t = (K_t^e, K_t^u)$ .

The approach of Reiter (2010) is conceptually close to the approach I propose here in that he uses a discrete representation of the steady state distribution from the model without aggregate uncertainty as a reference to perform the aggregation step.  $S_t$  includes summary statistics of the distribution. He independently adjusts the reference distribution to match each point on a grid over  $(S_t, a_t)$  and then finds the  $S_{t+1}$  at that point using fixed-point methods.

An approach that does not follow the outline above but is also related to my approach is Reiter (2009). He also first takes a discrete approximation of the steady state distribution under the assumption of no aggregate uncertainty. He then considers each of the points in this discrete approximation as a separate variable, derives a system of equations that describes the economy as a function of these variables and finds a first-order approximation of that solution using a perturbation approach.

## 4 The Derivative Aggregation Approach

I propose to solve the model by setting  $S_t = (K_t)$  and taking a first-order approximation<sup>2</sup> of the aggregate transition rule by directly aggregating the derivatives of the individuals' transition rules.

In more detail, I define a grid of points over all individual and aggregate states  $\{k_t, e_t, K_t, a_t\}$ . Starting from some initial guess for  $K_{t+1}^0$  I iterate over the following two steps<sup>3</sup> until convergence of both  $K_{t+1}^s$  and  $k_{t+1}^s$ :

1. Solve for  $k_{t+1}^s \equiv k_{t+1}(k_t, e_t, S_t, a_t; K_{t+1}^s)$  using the method of endogenous grids introduced by Carroll (2006). This involves reverse-time iteration on the Euler eq. (7).
2. Under some assumptions, outlined below, on the distribution of agents over wealth, I calculate the derivative  $\left. \frac{dK_{t+1}^s}{dK_t} \right|_{K_t=\bar{K}^s}$  by aggregating individuals'  $\left. \frac{\partial k_{t+1}^s}{\partial K_t} \right|_{K_t=\bar{K}^s}$  at a point  $\bar{K}^s$ . I use this to calculate  $\frac{\partial \log(K_{t+1}^s)}{\partial \log(K_t)}$  and then set  $K_{t+1}^{s+1}$  to be the implied log-linear approximation of the aggregate transition rule around  $\bar{K}^s$ .

Step 2. above is the primary contribution that this paper makes.

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<sup>2</sup>Theoretically higher order approximations are also possible. The additional overhead would not be prohibitive.

<sup>3</sup>Here,  $t$  signifies the period of the economy, and  $s$  the iteration step

The wealth distributions assumed in step 2) may give rise to the concern that the results depend on the particular choices made. However, Assumption 1 already states that the prices on which individual decisions depend can be predicted to a high degree of accuracy using only a small set of moments of the wealth distribution. The precise distribution of wealth, by implication, has limited importance. The fact that approximate aggregation holds justifies this assumption.

I next show that the derivative of the aggregate capital transition rule can be determined from the individual transition function under the assumption of a capital distribution and a scheme for adjusting that distribution as overall wealth changes. I then discuss suitable choices for the two distributions. Results from using this approach are presented in the next section.

## 4.1 Approximating the Aggregate Transition Rule

**Proposition 1** *Assume*

1. *an aggregate productivity level  $\bar{a}$*
2. *a mapping  $\omega \in \Omega$  of individuals to capital holdings and employment status, consistent with  $\bar{a}$*
3. *a second mapping  $\delta : [0, 1] \mapsto [0, 1]$  that specifies what part of any change in aggregate wealth each individual receives, where  $\int_0^1 \delta(i) di = 1$*
4. *an individual savings function  $k_{t+1}(k_t, e_t, K_t, a_t)$  that is continuous and differentiable in both  $k_t$  and  $K_t$ .*

*Then the rate of change of aggregate future capital with respect to aggregate capital,  $\frac{dK_{t+1}}{dK_t}$ , as aggregate capital changes by adjusting individual wealth according*

to 3., is given by

$$\frac{dK_{t+1}}{dK_t} = \int_0^1 \left[ \frac{\partial k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})}{\partial k_t^i} \delta(i) + \frac{\partial k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})}{\partial K_t} \right] di$$

**Proof** From eq. (8) we have

$$K_t = \int_0^1 k_t^i di \quad (12)$$

$$\Rightarrow K_{t+1} = \int_0^1 k_{t+1}^i di = \int_0^1 k_{t+1}(k_t^i, e_t^i, K_t, \bar{a}) di \quad (13)$$

where the latter equation is obtained by iterating eq. (8) forward one period and making the individual transition rule explicit.

By assumption (3), if  $K_t$  changes by a small amount  $\Delta$ ,  $k_t^i$  changes by  $\delta(i)\Delta$ .

Using the definition of the derivative, and conditional on  $a_t = \bar{a}$ ,

$$\frac{dK_{t+1}}{dK_t} = \lim_{\Delta \rightarrow 0} \frac{K_{t+1}(K_t + \Delta) - K_{t+1}(K_t)}{(K_t + \Delta) - K_t} \quad (14)$$

$$= \lim_{\Delta \rightarrow 0} \frac{\int_0^1 [k_{t+1}(k_t^i + \delta(i)\Delta, e_t^i, K_t + \Delta, \bar{a})] di - \int_0^1 [k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})] di}{\Delta} \quad (15)$$

$$= \lim_{\Delta \rightarrow 0} \frac{\int_0^1 [k_{t+1}(k_t^i + \delta(i)\Delta, e_t^i, K_t + \Delta, \bar{a}) - k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})] di}{\Delta} \quad (16)$$

Both numerator and denominator tend to 0 as  $\Delta \rightarrow 0$ , hence by l'Hôpital's rule:

$$\frac{dK_{t+1}}{dK_t} = \lim_{\Delta \rightarrow 0} \frac{\int_0^1 \left[ \frac{\partial k_{t+1}(k_t^i + \delta(i)\Delta, e_t^i, K_t + \Delta, \bar{a})}{\partial k_t^i} \delta(i) + \frac{\partial k_{t+1}(k_t^i + \delta(i)\Delta, e_t^i, K_t + \Delta, \bar{a})}{\partial K_t} \right] di}{1} \quad (17)$$

$$= \int_0^1 \left[ \frac{\partial k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})}{\partial k_t^i} \delta(i) + \frac{\partial k_{t+1}(k_t^i, e_t^i, K_t, \bar{a})}{\partial K_t} \right] di \quad (18)$$

□

Note that the partial derivatives of individual future capital  $k_{t+1}$  w.r.t. current aggregate capital  $K_t$  and current individual capital  $k_t$  are not well defined at any point where the borrowing constraint just ceases to bind: the left derivative is 0 whereas the right derivative is positive. I always take the left derivative for points where  $k_{t+1} = 0^4$ , and ipso facto arrive at the left derivative of the aggregate transition rule. Since the set of points exactly on the boundary is either of measure 0 ( $k_t > 0$ ), or  $\delta(i) = 0$  in the case that the boundary is at  $k_t = 0$ , the aggregate right derivative should only differ infinitesimally, justifying the approximation.

## 4.2 Choosing Suitable Distributions

The natural initial guess for the wealth distribution  $\omega$  is the steady-state distribution of the economy with no aggregate risk, but with the same structure of idiosyncratic risk. The idiosyncratic variation in income for individuals is far greater than the fluctuations caused by aggregate productivity changes. Thus, the change in behaviour of individuals in reaction to the aggregate risk can be expected to be quite small, and in consequence the steady-state distribution without aggregate risk might be quite close to realized distributions under aggregate uncertainty.

For  $\delta$  there are two choices that immediately stand out as satisfying the adding-up constraint:  $\delta(i) = 1$  and  $\delta(i) = k^i/K$ . The former would mean that, given an economy with a slightly higher level of aggregate capital, each individual would have the same amount of extra wealth. The latter, on the other hand, would distribute additional wealth in proportion to existing wealth. In the model economy some individuals are liquidity constrained in any given period, hence starting the next period with no capital. This is inconsistent with

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<sup>4</sup>The algorithm uses a discrete approximation of the distribution, and it is not possible to tell whether a point where  $k_{t+1} = 0$  but where the next point has positive next-period wealth is on or before the boundary, hence the only consistent choice is 0.

the uniform distribution. My initial choice is therefore  $\delta(i) = k^i/K$ .

## 5 Derivative Aggregation in Practice

### 5.1 The Baseline Model

For the baseline model I use the parametrisation of Den Haan et al. (2010) so that results may be compared. In this parametrisation aggregate productivity follows a two-state Markov chain with values  $\{1 - \xi, 1 + \xi\}$ . The unemployment rate is determined by the aggregate productivity level and is 10% in the low-productivity state and 4% in periods of high productivity. By implication, individual state transition probabilities depend both on current and next period aggregate productivity.

		$a_{t+1}$		$1 - \xi$		$1 + \xi$	
		$e_{t+1}^i$		0	1	0	1
$a_t$	$e_t^i$						
$1 - \xi$	0	0.525	0.35	0.03125	0.09375		
	1	0.038889	0.836111	0.002083	0.122917		
$1 + \xi$	0	0.09375	0.03125	0.291667	0.583333		
	1	0.009115	0.115885	0.024306	0.850694		

Table 1: Transition Probabilities in the baseline case (Source: Den Haan et al., 2010)

The joint transition matrix of individual and aggregate states is show in Table 1. The values taken by other parameters are listed in Table 2.

Parameter	$\beta$	$\gamma$	$\alpha$	$\delta$	$\bar{l}$	$\mu$	$\xi$
Value	0.99	1	0.36	0.025	1/0.9	0.15	0.01

Table 2: Parameter values in the baseline case (Source: Den Haan et al., 2010)

### 5.1.1 How the Solution Is Evaluated

After solving the model using the algorithm described above, I use the following procedure to assess the results:

I first simulate the economy for 10000 periods, starting from the distribution used to determine the aggregate derivative above, using the individual transition rule obtained. I do this by using the simulation procedure introduced by Young (2010), which uses the individual transition rules to move from one discrete representation of the distribution to the next. I collect the aggregate capital time series that results from this approach.

As a second step, I simulate the same series of shocks<sup>5</sup> but under the assumption that the aggregate transition rule obtained from the solution is correct. In other words, in the second simulation the economy is treated as a representative agent economy with the assumed aggregate transition rule, and the individual transition rules are ignored. I again obtain a time series of aggregate capital levels.

Finally, to compare the two time series I take the relative difference at each point, thus obtaining 10000 error terms, which are in effect the errors the individuals would perceive (ex post) from forecasting the next 10000 periods when the economy is in the initial state.

This is also the procedure used in Den Haan (2010).

### 5.1.2 Initial Results

The average relative error is .27503% (XPA: .105%), with a bias of .27%, and the maximum error is .845% (XPA: .343%). One period ahead predictions have an  $R^2$  of 0.99999, and the maximum one period ahead forecast error is .04%.

The 1-period-ahead results suggest that agents would not find their forecasts

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<sup>5</sup>I use the sequence used in Den Haan and Rendahl (2010)



unacceptable, but compared to XPA over the longer forecast period the accuracy is not good.

### 5.1.3 Refining the Algorithm

I have made three assumptions in the approach outlined: that a log-linear approximation is a good fit, that the steady-state distribution from no-aggregate-risk economy is a suitable distribution and that distributing additional wealth in proportion to existing wealth is appropriate. The first assumption draws strong support from prior research cited above. The third assumption is economically attractive, and no other straightforward choices present themselves. The second assumption is most questionable. Observation of the generated time-series also shows that, though the aggregate capital level implied by this distribution is within the range of those realised in the economy, the implied  $K_{t+1}$  at that point is on the margin of those actually realised during simulation.

I therefore modify the algorithm as follows: in each iteration of updating the aggregate transition rule (i.e. step 2) I simulate the economy for 10 periods in the good aggregate state, and calculate the implied aggregate transition rule for the good state as above on the resulting distribution. Then I simulate the economy for 10 periods in the bad aggregate state and do the same for the implied aggregate transition rule in that state. Importantly, I keep the final state of these simulations as the starting point for the next iteration. Finally, I update the aggregate transition rule by taking the weighted average of the previously assumed values and the newly calculated ones, where the weights are  $\{0.95, 0.05\}$  respectively. This ‘damping’ of the updates to the aggregate transition rule prevents wild swings and is a common procedure in recursive methods (see, for example, Den Haan and Rendahl, 2010).

This introduces two significant changes: Firstly, the aggregate transition rule thus derived is a weighted average of the transition rules implied by all

the past distributions used to calculate it, weighted towards more recent values. Secondly, the distributions used in successive iterations of the calculation are not the same. This may hinder convergence. It also means, however, that, as more iterations are performed, the distribution moves closer to one that is likely to occur under uncertainty, which should aid accuracy.

#### 5.1.4 Refined Results

Re-running the simulations above, the average relative error is .119% (XPA: .105%), and the maximum error is .551% (XPA: .343%). One period ahead predictions have an  $R^2$  of 0.99999, and the maximum one period ahead forecast error is .02%. The goodness-of-fit is now comparable to that of XPA.

The model was solved in 60 seconds<sup>6</sup> (XPA: 195 seconds<sup>7</sup>).

## 5.2 Introducing Preference Heterogeneity

In the economy discussed so far, agents differ only in their income. Their preferences are identical. In this section, I introduce preference heterogeneity to the model, and demonstrate that the solution methodology introduced above is still efficient and produces viable results.

### 5.2.1 Stochastic Rate of Time Discount

The preferences used are identical to the heterogeneous discount rate scenario used by K & S: the rate at which agents discount future utility,  $\beta$ , can now take one of three values:  $\{0.9858, 0.9894, 0.9930\}$ . Agents, being infinitely-lived, are considered to represent ‘dynasties’, so that the rate of time discount of each agent may vary as generations change. This is emulated by allowing each agents’

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<sup>6</sup>Of this, 17s were used to solve the model with no aggregate risk and 11s to find the steady-state distribution. The remaining 32 seconds were used to solve the model with aggregate risk using the procedure outlined above.

<sup>7</sup>Note that the XPA algorithm was run on the same machine, implemented using the same technology and using the same sequence of shocks as was used for derivative aggregation

$\beta$  to change between periods. The transition probabilities are the same as those used in K & S, where they were chosen to satisfy three constraints:

- the average duration of the upper and lower values are 50 periods, approximating the lifespan of a generation.
- at any time 80% of the population have the median value and 10% each of the outer values
- agents never jump between extremes

The preferences are uncorrelated with the individual employment or aggregate productivity states. The joint state transition probability matrix is given in the appendix.

### 5.2.2 Results

Using the algorithm as described above, the solution converges in 448 seconds<sup>8</sup>.

The solution is again evaluated as described in section 5.1.1. The mean relative error between the two time series generated is 0.14%, the maximum 0.65% and the bias of the error is  $-0.05\%$ . The one period ahead predictions have a mean error of  $.007\%$ , with a maximum discrepancy of  $.04\%$ . The  $R^2$  is 0.99998.

### 5.2.3 Scalability

The model with heterogeneous discount rates increased the number of individual states by a factor of 3 relative to the baseline specification. The time required to solve the model increased by a factor of 7.5. The algorithm consists of a number of steps, some of which are  $O(n)$  with respect to the number of individual states

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<sup>8</sup>Of this, 82 seconds were required to solve the model with no aggregate risk and 33 seconds find the steady-state distribution in that economy

$n$ , and some of which are  $O(n^2)$ <sup>9</sup>. Further, the number of steps required to converge to a solution may differ.

The baseline model's solution converged in 228 iterations averaging  $44ms$  each. The heterogeneous beta model was solved in 509 iterations of  $168ms$  each - close to an  $O(n)$  increase per step.

By comparison, explicit aggregation without additional assumptions on the functional form of the relationship between discount factor and individual transition function would require increasing the number of aggregate states by a factor of 3 also<sup>10</sup>. Thus the time per step is at least  $O(n^2)$  for that algorithm.

Thus the derivative aggregation approach is likely to be a better choice when considering problems where individuals can experience a high number of states, either because there are many values in one dimension - for instance, a more complete description of the individual income process - or agents differ in multiple dimensions.

### 5.3 A Life-Cycle Model

Another dimension of heterogeneity among economic agents is age. The potential for the age-related concerns to affect economic decisions has been formally acknowledged at least since Modigliani and Brumberg (1954). The computational solution of macroeconomic models with relatively complete descriptions of the life-cycle was pioneered by Auerbach and Kotlikoff (1987). Lifecycle version of incomplete markets models have also been solved (see, for example, Heer and Maussner, 2009).

From a theoretical perspective the solution methodology presented in this paper lends itself to solving life-cycle models. Finding the initial distribution

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<sup>9</sup>The transition from all possible current to all possible future states, for instance, includes  $n$  in two dimensions

<sup>10</sup>This arises from the need to track the mean capital held by all agents in each individual state separately.

no longer requires discovery of the steady state without uncertainty: under the assumption of no (or constant) initial endowments, the fixed point of aggregate capital can be determined in each iteration simply by simulating one whole generation under the assumption of a given level of capital, determining the actual aggregate capital implied by that simulation, and finding the fixed point of capital.

In this section I extend the baseline model to include a life-cycle dimension. Agents now live for 55 periods, with no mortality in earlier years. They enter the economy with no wealth. Their productivity remains constant over the first 40 years of life and then drops to 0, so that they retire. Thus the model provides a rudimentary approximation of the economic life of agents entering the economy at age 20, retiring at 60 and living to an age of 75.

Agents now maximise expected lifetime utility. Thus the problem of agent  $i$  of age  $a$  can be expressed as

$$\max_{\{c_{a+s,t+s}^i, k_{a+s,t+s+1}^i\}_{s=0}^{55-a}} E \left[ \sum_{t=0}^{\infty} \left( \beta^t \frac{(c_{a,t}^i)^{1-\gamma} - 1}{1-\gamma} \right) \right] \quad (19)$$

$$\text{s.t. } c_{a,t}^i + k_{a+1,t+1}^i = (1 + r_t - \delta)k_{a,t}^i + [(1 - \tau_t)\bar{l}\epsilon_{a,t}^i + \mu(1 - \epsilon_{a,t}^i)]w_t \quad (20)$$

$$k_{a+1,t+1}^i \geq 0 \quad (21)$$

$$k_{56,t} = 0 \quad (22)$$

All other aspects of the economy - employment, production, government - remain as before. This is of course a grossly simplified depiction of the life-cycle, ignoring in particular life-cycle variations in expected income and uncertainty over the time of death. Thus it serves only to indicate whether the approach presented here can feasibly solve such models.

### 5.3.1 Results

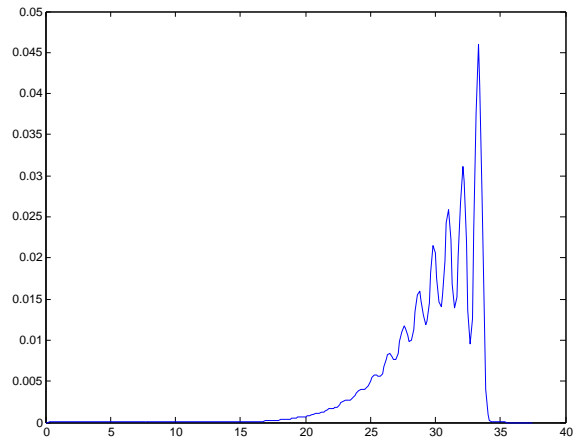
The model converged to a solution in 122s.

The “aggregate only” and “individual” simulations described above were performed with the identical shock sequence. The mean relative forecast error is now 0.27%, with a maximum of 0.89%. For one-period-ahead forecasts, the mean is 0.03% with a maximum error of 0.1%. The  $R^2$  in this case is 0.9998.

### 5.3.2 An Observation on the Life-Cycle Model

Two issues that arise in this specification of the life-cycle model are illustrated in fig. 1 depicting the distribution of wealth within the age 40 cohort, which is about to retire: the distribution is not smooth, and there is a clear maximum level of wealth which is also the point of maximal density. Both issues arise from the same two features of the model: there are only a finite number of stochastic paths that agents’ idiosyncratic shocks can take over their lifetime (2 potential states each period of working life, so there are  $2^{40}$  paths), so that the probability mass of any particular path is non-zero. Since all agents initially have 0 wealth the probability mass of their initial wealth level is also non-zero, namely 1. But then any point along the path for a given realisation of aggregate shocks also has non-zero mass, and the maximum level of wealth attained is shared by all those who have the same, most favourable, sequence of income shocks.

These issues are straightforward to address, principally by allowing for non-uniform bequests in the model, but also by providing a less parsimonious description of the income process with more states. These changes would add to computation time, but neither renders the derivative aggregation approach inapplicable. Since the effort required is principally on the construction of the model and would not provide additional insight regarding the topic of this paper it is left to future research.



*Student Version of MATLAB*

Figure 1: Distribution of Wealth Amongst Agents Aged 40

## 6 Conclusion

I have presented a new approach to solving incomplete market models with aggregate uncertainty, and demonstrated that the solutions arrived at using this approach are as accurate as existing methods. The solution to the baseline model is found more quickly than, and as accurately as, using the Explicit Aggregation approach presented in Den Haan and Rendahl (2010), previously found to be among the fastest by Den Haan (2010).

Importantly, I have also demonstrated that the approach extends to models with more state variables and that it scales reasonably well to that scenario. Explicit aggregation would require additional assumptions on functional form or additional aggregate variables, making it less scalable. I finally went on to solve a rudimentary lifecycle model exhibiting within-cohort heterogeneity and incomplete markets. Once again the time required to find the solution was reasonable, and the aggregate transition rules obtained provided good forecasts of the actual economy, especially over short time horizons.

The approach as presented constructs a first-order approximation of the aggregate transition rule. The procedure could be extended to higher order approximations without great overhead should the first-order solution prove a poor fit.

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