Investment under Ambiguity with the Best and Worst in Mind

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Abstract

Recent literature on optimal investment has stressed the difference between the impact of risk and the impact of ambiguity - also called Knightian uncertainty - on investors' decisions. In this paper, we show that a decision maker's attitude towards ambiguity is similarly crucial for investment decisions. We capture the investor's individual ambiguity attitude by applying α-MEU preferences to a standard investment problem. We show that the presence of ambiguity often leads to an increase in the subjective project value, and entrepreneurs are more eager to invest. Thereby, our investment model helps to explain differences in investment behavior in situations which are objectively identical.

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1 Introduction

When firms make investment choices, they usually face uncertainty about future cash flows, as has been widely acknowledged by the literature on optimal investment; and every sound investment theory requires an accurate assessment of the uncertainty involved. Researchers have identified two main sources of uncertainty that firms face: risk and ambiguity. While *risk* usually refers to the return volatility of an investment project using a single probability distribution, the notion of *ambiguity*[1] refers to the existence of a multitude of these probability distributions to describe future profits. Whereas the impact of risk on investment decisions has been thoroughly analyzed, the role of ambiguity has only recently drawn the attention of the research community. How is investment actually affected by a perceived change in ambiguity? The answer is not obvious. Standard investment models that allow for ambiguity assume that decision makers are completely averse to ambiguity. So most papers hence postulate that a rise in ambiguity alienates investors, which are consequently less eager to invest. Think, for example, of the development of an oil field. If future oil prices are likely to become less predictable, the project is more likely to be abandoned, since it might not be possible to recover the sunk investment costs. At the very least, it may be postponed.

There are, however, situations in which the assumption of completely ambiguity-averse decision makers is perhaps to excessive. In a survey of successful entrepreneurs Bhidé (2000) shows that those who start businesses are highly self-confident and exhibit a very low degree of ambiguity aversion. When they spot a business opportunity in a new industry, they embrace uncertainty - fully persuaded by the profitability of their investment. That was obviously the case during the new economy bubble, when many entrepreneurs - fully confident of their business idea - invested in the Internet simply because it was new and largely unknown. These two examples suggest that it might be essential to include an individual parameter into the model that captures the entrepreneur's attitude towards the ambiguity he faces. More important, traditional investment models that abstract from the presence of ambiguity cannot distinguish between these two situations, since, in terms of risk, they are objectively identical.[2]

The objective of this paper is to propose a simple investment model that captures personal attitudes towards ambiguity and thus makes it possible to examine the effect of ambiguity

[1] Sometimes ambiguity is also called *Knightian uncertainty*, following the work of Knight (1921). In this study, both terms refer to the same concept and are used interchangeably.

[2] Bhidé (2000) writes that "entrepreneurs who have a high tolerance for ambiguity needed to start promising businesses [...] may not have a risk-seeking disposition". Hence, all throughout this paper, we assume that the riskiness, i.e. volatility, of an investment project is constant and fully known to the decision maker. It should be noted that there might be reasons beyond the entrepreneur's attitude towards ambiguity that can influence his investment decision.

and different ambiguity attitudes on the investment decision. To create a framework for the model, we rely on the irreversible investment theory as formulated by McDonald and Siegel (1986) and Dixit and Pindyck (1994). Also known under the term real option theory, this theory, by applying option-pricing techniques to the investment problem, provides an elegant means of assessing the optimal investment strategy in an uncertain environment. Unlike most papers in the standard irreversible investment literature, this paper does not assume that the entrepreneur has perfect confidence in the perceived probability measure describing future uncertainty. Instead, it assumes that he considers other probability measures to be possible as well; in other words, we extend the model to include ambiguity. Finally, we allow for different attitudes of the entrepreneur towards ambiguity. More precisely, we determine the optimal investment strategy given ambiguity when the preferences of the entrepreneur can be described as a convex combination of the two extreme attitudes towards ambiguity, i.e., considering the best and the worst cases only. Such preferences have been proposed among others by Marinacci (2002) and Olszewski (2007), and are known as α-MEU preferences or α-maxmin expected utility.

In the first part of the paper, we reduce the irreversible investment problem to include only an all-or-nothing decision - the entrepreneur can either go forward with the investment or abandon it altogether. In section 6, we then introduce the possibility to defer the investment project to a later point of time. Unfortunately, the α-maxmin expected utility generally violates the dynamic consistency requirement to solve the resulting intertemporal optimization problem. Nevertheless, we provide solutions to some special cases.

We find that even a very small fraction of optimism from the entrepreneur can change the investment decision significantly. We show that in many cases the threshold for investing, i.e., the required expected value of a project, decreases in the presence of ambiguity. As a consequence, investments are made earlier than when there is no ambiguity. Although the paper looks at the specific case of a firm's investment decisions, the framework analyzed is similar to many decision problems under ambiguity. The results hence have implications to related optimal stopping problems, ranging from firm entry and exit, labor search or wedding decisions.

The model we present in this paper is related to several streams of literature. First, it follows the standard models of irreversible investment under uncertainty, among them that in (Dixit and Pindyck, 1994). Next, it also relates to the literature of decision making under ambiguity. That the distinction between risk and uncertainty is behaviorally meaningful was first shown by the Ellsberg (1961) paradox. Of the various theories that allow for ambiguity, the Choquet expected utility theory by Schmeidler (1989) and the multiple expected utility theory by Gilboa and Schmeidler (1989) are the most prominent. In this paper, we rely on the formulation of the latter, using the continuous time implementation by Chen and Epstein (2002). As such, this paper also owes much to the literature on optimism, overconfidence and decision making in behavioral economics. Heath and Tversky (1991) argue that the subject's attitude towards ambiguity depends on his perceived competence level. They show that those who feel competent react more favorably to ambiguous situations and even seek them out. People tend to overestimate their subjective knowledge or competence (overconfidence), as widely acknowledged in the psychological literature (DeBondt and Thaler, 1995) we should expect to observe some ambiguity-loving behavior when investment decisions are being made. Bhidé (2000) then states that "low ambiguity aversion of the individuals who start promising businesses derives from exceptionally high levels of self-confidence." In fact, the experimental study by Gysler et al. (2002) finds evidence for the alleged close relationship between attitudes towards ambiguity and perceived competence and overconfidence[3].

To my knowledge, this is the first theoretical paper to include ambiguity-loving features into the investment problem in continuous time. As such, it is inspired by the work of Nishimura and Ozaki (2007) who first combined these two streams of research, assuming completely ambiguity averse decision makers. In fact, our model offers a generalization of their work, including ambiguity averse investors as a special case[4].

This paper contributes also to the debate of dynamic optimization under ambiguity. α-MEU preferences are very popular in assessing decisions under ambiguity, since they allow a simple framework to differentiate the magnitude of ambiguity from the decision maker's attitude towards ambiguity. However, as this paper shows, preference orders that can be expressed as a convex combination of more than one ambiguity attitude are generally not dynamically consistent since they cannot be represented in a recursive way. This drawback implies that only special cases of α-maxmin expected utility can be applied to solve dynamic optimization problems.

[3] We should note that there is no universal definition of the terms overconfidence and optimism in the presence of ambiguity. Epstein and Schneider (2008) for example present a setting which allows for the combination of ambiguity averse and overconfident agents.

[4] Other recent works that look at investment decisions or optimal stopping problems under ambiguity are the papers by Asano (2005), Miao and Wang (2007), Trojanowska and Kort (2007), Riedel (2009), and Choi et al. (2009). Asano (2005) applies the optimal stopping problem under ambiguity to environmental policy design. Miao and Wang (2007) consider optimal option exercise under ambiguity. Trojanowska and Kort (2007) analyze the investment decision if the firm's project generates not an infinite profit flow, but ceases to exist after some years. Riedel (2009) formulates a generalization of the solution of optimal stopping problems under maximin preferences. Similar to this paper, Choi et al. (2009) also apply α-MEU preferences to the investment problem. However, they focus on a regime-switching environment.

Finally, by showing that the threshold for new investments decreases in presence of ambiguity loving investors, we also contribute to the search for a reconciliation between the irreversible investment approach and the usual NPV rule for investment. The results by Dixit and Pindyck (1994), stating that the project value must exceed twice its investment cost, have been criticized as implausibly high.

This paper proceeds as follows. In the next section, we present the simple investment problem of the entrepreneur, who must either invest in a project or abandon it. The decision maker's preferences under ambiguity are formulated in section 3. We then describe the Chen and Epstein (2002) model of ambiguity in continuous time, the model we use in this paper. The solution to the simple investment problem is then presented in section 5: first, we derive the value of the investment project, then we turn to the investment decision itself. In section 6, we lift the restriction of immediate investment requirement. Section 7 discusses the problem of dynamic inconsistency of α-MEU preferences. Section 8 offers some concluding remarks and implications of this study.

2 The Firm's Investment Problem

Time t evolves over $[0, \infty)$ and uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$. Let B_t be a standard Brownian motion defined on this probability space, and $\mathbb{F} = \{\mathcal{F}_t\,,\, t \geq 0\}$ its augmented filtration, i.e., the σ-algebra generated by $(B_s)_{s \leq t}$ and the P-null sets.

Think of a risk-neutral entrepreneur who wants to set up a start-up venture and intends to evaluate his investment project in order to decide whether to invest or not. We assume that the entrepreneur's project can be characterized by an uncertain profit flow which follows a geometric Brownian motion:

$$d\pi_t = \mu \pi_t dt + \sigma \pi_t dB_t \tag{1}$$

with π_0 and $\sigma > 0$. Investment costs are denoted I and are constant over time. In the first part of the analysis, we assume that the investor faces an all-or-nothing decision: He can either decide to invest immediately or abandon the project. Later, in section 6, we relax this assumption. The investment problem is hence to choose the optimum between the expected value when investing, denoted by $V(\pi_t)$, less investment costs I, and not investing, which yields a zero profit:

$$F(\pi_t) = \max\{V(\pi_t) - I, 0\} = \max\left\{E_t\left[\int_t^\infty e^{-\rho(s-t)}\pi_s ds\right] - I, 0\right\} \tag{2}$$

where ρ is the decision maker's discount rate. Since the decision maker has the right, but not the obligation to invest, this expression is also called the investment option, and is denoted by $F(\pi_t)$. For this project evaluation to make sense, we assume in addition that $\mu < \rho$ - otherwise the expected project value could get infinitely large (for $\mu \to \rho$ or $\mu > \rho$). In absence of ambiguity, the expected investment value $V(\pi_t)$ is given by

$$V(\pi_t) = \frac{\pi_t}{\rho - \mu}, \tag{3}$$

the usual expected value of an infinite profit stream. If $V(\pi_t)$ exceeds the investment costs I, the entrepreneur engages in the project - otherwise he will not undertake the venture.

3 Decision Making under Ambiguity

How is a decision maker affected by the presence of ambiguity? In his famous urn experiment, Ellsberg (1961) showed not only that there is an impact of *ambiguity* on decisions which is different from *risk*, but his findings suggest that decision makers tend to be ambiguity averse as well. Hence, theory followed experiment, and economists came up with some models that include ambiguity - and the decision maker's attitude towards it - into the expected utility model. One of the most popular models is the Multiple Expected Utility (MEU) theory by Gilboa and Schmeidler (1989) that replaces the usual unique probability distribution with a set of probability distributions[5]. By maximizing utility over the worst possible probability distribution (maximin preferences), they model decision making under complete ambiguity aversion.

However, as noted earlier, in some situations the assumption of complete ambiguity averse decision makers might be too extreme. Especially in the context of start-up investments, the incorporation of ambiguity love from the side of the entrepreneur can be justified (Bhidé, 2000). Moreover, the inclusion of ambiguity loving features has been shown to be behaviorally meaningful by e.g. Heath and Tversky (1991) or Kilka and Weber (1998).

In this paper we hence rely on the so-called $\alpha-$MEU preferences as proposed by Marinacci

[5]The Ellsberg (1961) paradox showed that the conventional approach of choosing a subjective probability distribution in the absence of an objective distribution (Subjective Utility Theory, Savage (1954)) is in conflict with the observed behavior of individuals. In this light, the use of a set of distributions imposes a less rigid framework on the decision maker.

(2002), Ghiradato et al. (2004) and Olszewski (2007). These preferences model decision making under ambiguity by applying a convex combination of two extreme preferences over the set of probability distributions. One part of the weight is attributed to the best possible probability distribution, reflecting ambiguity loving characteristics of the decision maker. The rest of the weight is given to the worst probability distribution, similar to the Gilboa and Schmeidler (1989) model.

Definition 1. (α-maxmin expected utility)*: Let $\mathcal{P}$ denote a compact set of probability distributions and $\alpha \in [0,1]$ and individual parameter describing the decision maker's attitude towards ambiguity. Then the α-maxmin expected value of a stochastic function $f : x \to \mathbb{R}$ is given by*

$$\alpha\text{-}E[f(x)] = \alpha \sup_{p \in \mathcal{P}} E^p \left[f(x)\right] + (1-\alpha) \inf_{p \in \mathcal{P}} E^p \left[f(x)\right] \tag{4}$$

Since α is attributed to the best case, we call this parameter also the degree of optimism[6]. Such a convex combination between the best and the worst case is also known as Hurwitz criterion. Among others, these preferences have been found suitable to model behavior in ambiguous portfolio choice decisions (Ahn et al., 2007).

Several well-known decision criteria are special cases of the α-maxmin expected value as shown in (4). When setting $\alpha = 0$, the α-expected value coincides with those under the maximin preferences of Gilboa and Schmeidler (1989), i.e., pure pessimism. If $\mathcal{P}$ is singleton, the expression (4) reduces to the standard expected value.

4 Ambiguity in Continuous Time

The model of ambiguity in continuous time presented here follows the work of Chen and Epstein (2002), which is based on the recursive multiple priors utility model developed by Epstein and Wang (1994).

[6]In the literature, there is no clear consensus whether α should denote the weight attributed to the best case (Olszewski, 2007; Chateauneuf et al., 2007) or to the worst case (Marinacci, 2002; Ghiradato et al., 2004). A model that is close to our formulation is the *neo-additive capacity* model by Chateauneuf et al. (2007). They model decision making under ambiguity by evaluating a convex combination of the two extreme outcomes, the best and the worst scenario, plus the usual objective additive probability distribution, i.e., expected utility.
It might appear peculiar to the reader that we focus on the two extreme attitudes towards ambiguity only. There are several reasons why we opt for this approach: first, the results are driven by the extremes of the range of uncertainty. Including the entire range between the best and worst case similar to Klibanoff et al. (2005) or attributing some weight to the standard expected utility (Chateauneuf et al., 2007) would basically lead to rather similar, but of course weaker results. Moreover, relying on $\alpha-$MEU preferences does not imply that the decision maker considers these two extreme distributions to be more correct than others. They only represent an assessment of the best and the worst possible scenarios. The $\alpha-$MEU preferences are then a way to combine these assessments into a single preference order.

We define the set of probability distributions $\mathcal{P}$ to be mutually absolutely continuous with respect to P, which we call the objective or reference probability measure. This set can be defined by so-called density generators $\theta = (\theta_t)$, a class of stochastic processes that can be used to generate probability measures $\mathcal{Q}^\theta$ out of the objective P by defining the densities of the probability distributions. Moreover, we assume that the density generators $(\theta_t) \in \Theta$ are restricted to the non-stochastic range $K = [-\kappa, \kappa]$ that thereby defines the objective level of ambiguity. This definition of the set $\mathcal{P}$ translates hence into a constant ambiguity interval around the objective measure P. This specific way to model ambiguity in a continuous time framework ensures that the decision maker's set of priors is rectangular (Epstein and Schneider, 2003b), a necessary assumption to assure a recursive structure of beliefs. For a more detailed discussion and formal derivation of the set $\mathcal{P}$ see appendix A.

It follows from the specification of ambiguity, using Girsanov's theorem (see e.g. Duffie (2001, p. 337)), that a stochastic process $(B_t^\theta)_{0\leq t<\infty}$ defined as

$$(\forall t \geq 0, \forall \theta \in \Theta) \quad B_t^\theta = B_t + \int_0^t \theta_s ds \tag{5}$$

is a standard Brownian motion with respect to $\mathbb{F}$ on $(\Omega, \mathcal{F}, \mathcal{Q}^\theta)$. We use this result (note that (5) is equivalent to $dB_t^\theta = dB_t + \theta_t dt$) to generalize the Ito process of the profit flow given by equation (1) to the general set $\mathcal{P}$:

$$(\forall t \geq 0, \forall \theta \in \Theta) \quad d\pi_t = (\mu - \sigma\theta_t)\pi_t dt + \sigma\pi_t dB_t^\theta$$

Hence, all stochastic processes to describe the profit flow π_t differ only in the drift term from each other. Thus, the multiplicity of measures in $\mathcal{P}$ translates in modeling ambiguity about the drift of the profit flow, which can vary according to the range K. Finally, we recall the solution for π_t:

$$(\forall t \geq 0, \forall \theta \in \Theta) \quad \pi_t = \pi_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t - \sigma\int_0^t \theta_s ds + \sigma B_t^\theta\right) \tag{6}$$

It is important to keep in mind that there is only one observable stochastic profit flow π_t, but many different stochastic differential equations (the set defined in (6)) that can describe it. Ambiguity over this set does not vanish over time, since all θ_t vary within the range K in an independent and indistinguishable way (IID ambiguity, Epstein and Schneider (2003a)). Accordingly, it is not possible to learn the distribution of $\theta_t \in \Theta$, neither to reduce the set $\mathcal{P}$ over time.

5 Optimal Investment under Ambiguity

Now we are ready to analyze the optimal investment decision under ambiguity. First, we present the evaluation of the investment project, given the observable profit level π_t. Then we examine the investor's investment problem, i.e., his decision to carry out the project or to abandon it. Finally, we extend the model by introducing an ambiguous outside option such that the decision to abandon the project is subject to some ambiguous payoff itself. In the comparative statics section, we examine the implications of changes in the degree of ambiguity the entrepreneur faces (the set K) and his level of optimism (α) on the project evaluation and investment decision.

5.1 Project evaluation

According to equation (2), the payoff function $f(x)$ is given by the present value of future profits:

$$f(\pi_t) = E_t\left[\int_t^{\infty} e^{-\rho(s-t)}\pi_s ds\right]$$

Ambiguity is modeled by a set of probability distributions $\mathcal{P} = \{\mathcal{Q}^\theta|\theta \in \Theta\}$, as defined in section 4. The degree of ambiguity is exogenously specified by the interval of the density generators $K = [-\kappa, \kappa]$, which is given by some objective information[7]. For the evaluation problem to be valid, the admissible range of κ must be restricted to $\kappa < (\rho - \mu)/\sigma$; otherwise the denominator could get 0, as we will see in the following. The α-maxmin expected value of the investment project $V(\pi_t|\alpha)$ can then be calculated as

$$V(\pi_t|\alpha) = \alpha \sup_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta}\left[\int_t^{\infty} e^{-\rho(s-t)}\pi_s ds\right] + (1-\alpha) \inf_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta}\left[\int_t^{\infty} e^{-\rho(s-t)}\pi_s ds\right] \tag{7}$$

which has the following solution:

Proposition 1. (Project value)*: Let the level of ambiguity be specified by the set $K = [-\kappa, \kappa]$. Then, given α-MEU preferences and the rectangular structure of beliefs $\mathcal{P}$, the α-maxmin expected value of the investment project with an infinite profit stream π_t is given by:*

$$V(\pi_t|\alpha) = \pi_t\left(\frac{\alpha}{\rho - (\mu + \kappa\sigma)} + \frac{1-\alpha}{\rho - (\mu - \kappa\sigma)}\right) \tag{8}$$

[7] The maximin model of Chen and Epstein (2002) cannot distinguish between ambiguity level and ambiguity attitude, such that κ can be conceived as a measure for both ambiguity level *and* ambiguity attitude. Since $\alpha-$MEU preferences allow for this separation, in this paper κ has a more narrow interpretation as a measure of the objective ambiguity level only.

Proof. See appendix B. □

What can we learn from this expression? First, consider the case without ambiguity, i.e. $\kappa = 0$. Then the term reduces to $V(\pi_t|\alpha) = \pi_t/(\rho - \mu)$. This is identical to the standard expression (3) for the expected present value of an infinite profit stream. If we let ambiguity gradually increase ($\kappa > 0$), the decision maker adds the sum of the two terms of (8), depending on the parameter α. In the case of complete ambiguity aversion, also called pessimism ($\alpha = 0$), the value of the installed investment project coincides with the one under maximin preferences, as analyzed by Nishimura and Ozaki (2007): $V(\pi_t|\alpha = 0) = \frac{\pi_t}{\rho-(\mu-\kappa\sigma)}$.

It is important to note that - although the decision maker is assumed to be risk neutral - the project value under ambiguity is a function of the risk parameter σ, which contrasts to the simple solution under pure risk. In this continuous-time framework, the project value does hence not only depend on the ambiguity itself, but also on the riskiness of the project - even though the decision maker is risk neutral. In fact, the volatility, i.e. risk, of the project *scales* the ambiguity faced by the decision maker ($\kappa\sigma$). The more volatile the project, the higher is the impact on the distortion of the probability distribution caused by ambiguity, and consequently the more ambiguous the whole project.[8]

For notational convenience, we replace the term in the brackets of (8) by the parameter ϕ:

$$\phi = \frac{\alpha}{\rho - (\mu + \kappa\sigma)} + \frac{1-\alpha}{\rho - (\mu - \kappa\sigma)} \tag{9}$$

This gives: $V(\pi_t|\alpha) = \pi_t\phi$.

5.2 Investment decision

After having analyzed the project value, the entrepreneur compares the expected profits of the investment $V(\pi_t|\alpha)$ to the related investment costs I, in order to determine the value of the option to invest:

$$F(\pi_t|\alpha) = \max\{V(\pi_t|\alpha) - I, 0\} = \max\{\pi_t\phi - I, 0\} \tag{10}$$

In case of $\phi\pi_t > I$, investment is carried out. When solving this condition for the current profit

[8] Mathematically this effect can be explained by the assumption that the profit flow follows a geometric Brownian motion (GBM): The density generators θ_t are restricted to move within a constant range, and thus moving the mean of each realization of the Wiener process by a constant term. However, the profit flow itself does not follow a Wiener process, but a GBM. Hence, the movement of the Wiener process is scaled by the constant volatility (σ) of the profits and the level of profits. Consequently not only each realization of the Wiener process is scaled by the volatility, but as well the deviation from the mean due to ambiguity via density generators.

level π_t, it is possible to calculate a critical level of current profits, that must prevail in order to invest.

Corollary 1. (Investment threshold)*: The critical level of profits π^*, that is required in order to invest, is given by:*

$$\pi^* = \frac{I}{\phi} = \frac{I}{\frac{\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{1-\alpha}{\rho-(\mu-\kappa\sigma)}} \tag{11}$$

Put differently, the decision maker invests if and only if the observable profit level π_t exceeds π^*. Otherwise, he will abandon the project.

5.3 Ambiguous outside option

So far we have assumed that the decision to abandon the investment project comes at zero cost. In reality, however, abandoning a project is usually associated with some costs, or possibly gains. Examples of such costs can be write-offs on investments into the development of the project, penalties for non-fulfillment of contracts based on the project, or costs due to severance pays. On the other hand, the firm might benefit from windfall profits by selling some of the know-how related to the development of the project.[9]

We hence lift the assumption of a zero payoff outside option. Instead, we model the value of the outside option X to be random, following a normal distribution $\mathcal{N}(x, \sigma_x)$. In addition, ambiguity about the outside option is modeled in analogy to the profit stream by a set of probability distributions $r \in \mathcal{R} = \{\mathcal{N}(x, \sigma_x) | x \in [-\epsilon, +\epsilon]\}$, which is independent of the ambiguity related to the profits $\mathcal{P}$. This set-up of modeling the outside option is the static counterpart of the model of ambiguity in continuous time as presented in section 4: the firm is again only ambiguous about the mean of X, and not the its variance σ_x. Furthermore, ambiguity is restricted to the range $[-\epsilon, +\epsilon]$, which is the analogous to the κ-ignorance in continuous time.[10] Given the neo-additive preferences and ambiguity specified by the probability space $\mathcal{R}$, we can calculate the α-maxmin expected value of the outside option.

Proposition 2. (Outside option)*: Given α-MEU preferences, and let the objective level of ambiguity be specified by the probability space $\mathcal{R} = \{\mathcal{N}(x, \sigma_x) | x \in [-\epsilon, +\epsilon]\}$. Then the α-maxmin expected value of the outside option X is given by:*

[9] The impact of ambiguous outside options on investment decisions have first been studied by Miao and Wang (2007) in a continuous-time framework.

[10] This method of modeling ambiguity in a static setting is also called $\epsilon-$contamination. For an axiomatic formulation, see Nishimura and Ozaki (2004a) and Kopylov (2008).

$$X(\alpha) = \alpha \sup_{r\in\mathcal{R}} E^r[X] + (1-\alpha) \inf_{r\in\mathcal{R}} E^r[X] = \epsilon(2\alpha - 1)$$

Hence, the more optimistic the decision maker ($\alpha \to 1$), the higher the expected value of the outside option. If $\alpha > 0.5$, the outside option is considered to be positive, otherwise it is expected to be negative, i.e., costly. Similarly to the project value, the impact of the ambiguity attitude increases with the objective level of ambiguity (ϵ) faced by the firm. Including the ambiguous outside option into the decision faced by the firm, the maximization problem of the firm reads as follows:

$$\begin{aligned} F(\pi_t, X|\alpha) &= \max\{V(\pi_t|\alpha) - I, X(\alpha)\} \\ &= \max\{\pi_t\phi - I, \epsilon(2\alpha - 1)\} \end{aligned}$$

If $\pi_t\phi > I + \epsilon(2\alpha - 1)$ investment is carried out. Again, we can solve this condition to obtain a critical level of current profits, that must prevail in order to invest.

Corollary 2. (Investment threshold with ambiguous outside option)*: Let X denote the ambiguous outside option faced by the firm. Then the critical level of profits π^*, that is required in order to invest, is given by:*

$$\pi^* = \frac{I + X}{\phi} = \frac{I + \epsilon(2\alpha - 1)}{\frac{\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{1-\alpha}{\rho-(\mu-\kappa\sigma)}} \tag{12}$$

Since α is an intrinsic parameter of the decision maker's preferences, it applies equally to both the investment project and the outside option.

5.4 Comparative statics

In this section we look at the effects of changes in the perceived level of ambiguity (κ) and the decision maker's attitude towards ambiguity (α) on the expected value of an investment project $V(\pi_t)$, the option to invest $F(\pi_t)$, and the investment threshold π^*. Since most of the interrelations are rather complex, we refer to numerical examples in order to demonstrate the different effects. In the illustrations presented below, we fix the investment parameters, unless otherwise stated, as follows: $\rho = 0.1$, $\mu = 0.05$, $I = 10$, $\pi_t = 1$, and $\sigma = 0.2$. The figures and results do however not depend on the chosen parameter values and are thus robust to changes in these assumptions.[11]

5.4.1 An increase in ambiguity

What happens if the decision maker perceives an increase in ambiguity regarding the future of his project? In such a case, the entrepreneur is likely to consider a larger range of probability distributions to be possible and the set $K = [-\kappa, +\kappa]$ increases. To see the consequences of such a change on the investment evaluation, we analyze the effects of an increase in ambiguity (κ) on the investment value. Since the expected value of the project under ambiguity $V(\pi_t|\alpha)$ is linear in the parameter ϕ, we analyze the derivative of the parameter ϕ with respect to κ:

$$\frac{\partial \phi}{\partial \kappa} = \frac{\sigma \left(\alpha(\rho - (\mu - \kappa\sigma))^2 - (1-\alpha)(\rho - (\mu + \kappa\sigma))^2\right)}{(\rho - (\mu + \sigma\kappa))^2 (\rho - (\mu - \sigma\kappa))^2}$$

This derivative is nonnegative if and only if

$$\alpha \geq \frac{1}{2} - \frac{\kappa\sigma(\rho - \mu)}{(\rho - \mu)^2 + \kappa^2\sigma^2} \tag{13}$$

The term on the right hand side of the inequality can reach values in the range between 0 and 0.5, given the maximal admissible range of ambiguity $\kappa = (\rho - \mu)/\sigma$. In the absence of either risk (i.e. $\sigma = 0$) or ambiguity ($\kappa = 0$), α must be larger than 0.5 such that increasing ambiguity has a positive effect on the expected investment value. For all strictly positive values of both κ and σ, the required level of optimism is smaller, attaining a minimum value of $\alpha \to 0$ in the limit when increasing κ towards its maximal admissible range $(\rho - \mu)/\sigma$. Consequently, we can state the following lemma:

Lemma 1. (Positive effect of ambiguity): *For all positive values of optimism $\alpha > 0$, there exists a threshold level of ambiguity κ^* such that for all degrees of ambiguity $\kappa > \kappa^*$, a perceived increase in ambiguity κ has a positive impact on the expected investment value $V(\pi_t|\alpha)$, and a negative impact on the investment threshold π^*.*

Proof. For $\alpha \geq 0.5$, $\kappa^* = 0$ since we know from condition (13) that $\partial V(\pi_t|\alpha)/\partial\kappa$ is nonnegative for all values of κ if $\alpha \geq 0.5$. For $\alpha \in (0, 0.5)$, κ^* is given by:

$$\kappa^* = \frac{(\rho - \mu)\left(\sqrt{\alpha - \alpha^2} - \frac{1}{2}\right)}{\sigma\left(\alpha - \frac{1}{2}\right)} \tag{14}$$

[11] The choice of the parameter values follows the standard assumptions in the real options literature, see e.g. Dixit and Pindyck (1994), which are motivated by the historical characteristics of the S&P 500 index. We admit that the evolution of a broad stock market index is maybe not appropriate for the assessment of real options in general or start-up investments in this specific case. Still, any assumption can be only exemplarily.

which is the value of κ where $\partial V(\pi_t|\alpha)/\partial\kappa = 0$. When $\alpha \to 0.5$, $\kappa^* \to 0$, i.e. no ambiguity; when $\alpha \to 0$, $\kappa^* \to (\rho - \mu)/\sigma$, the maximal admissible level of ambiguity. □

Intuitively, an increase in ambiguity has two opposed effects on the perceived value of the investment project. On the one side, the perceived downside risk of the project becomes larger. However, the downside risk is limited: The value of the project can never fall below zero. On the other side, the upside potential also grows larger. The crucial difference is that the upside potential is not limited. Since only the joint effect is important for the evaluation of the project, there must be hence always a level of ambiguity where the increase in upside potential outweighs the change in downside risk. Hence, a small fraction of optimism suffices to induce a better picture of the investment project, even for rather pessimistic entrepreneurs.

We can see the positive effect of ambiguity on the expected investment value in figure 1. This graph plots the natural logarithm of the expected value of the investment $V(\pi_t)$ as a function of ambiguity (κ) for different levels of optimism (α)[12]. For most parameter values of optimism, increasing perceived ambiguity leads a decision maker to value the investment project higher than before, especially when ambiguity is already quite high. Only almost completely ambiguity averse decision makers (α close to 0) relate more ambiguity to a lower project value.

[Figure 1 goes here]

In the absence of an outside option, the value of the option to invest $F(\pi_t)$ is just equal to the expected project value less investment costs. Hence, a perceived increase in ambiguity has a similarly positive effect on $F(\pi_t)$. Figure 2 displays the value of the investment project $V(\pi_t)$, and the option to invest $F(\pi_t)$ as a function of the profit level π_t for different degrees of ambiguity. The level of optimism is fixed at $\alpha = 0.5$. The value of both the option to invest as well as the expected project value increase when the perceived ambiguity, i.e. κ, rises. As a consequence, the critical value of current profits π^*, that must prevail so that the decision maker wants to invest, reduces with increasing ambiguity.

[Figure 2 goes here]

The impact of increased ambiguity on the investment threshold is depicted in figure 3, which presents the investment threshold (π^*) as a function of the level of ambiguity (κ) for different values of optimism (α). Analogously to figure 1, increasing ambiguity has a reducing effect on the investment threshold (π^*) for most parameter values of optimism. Only in the case of very

[12] We use the logarithm of the expected investment value because of scaling reasons in the graph.

pessimistic decision makers, i.e., α is close to 0, the presence of ambiguity increases investment threshold. In the absence of ambiguity, the threshold lies at 0.5.

[Figure 3 goes here]

Another implication of figure 3 is that there exists a collection of parameters combinations ($\alpha < 0.5, \kappa$) for which the investment threshold remains unchanged, i.e., $\pi^* = 0.5$. Put differently, for a given level of pessimism, there is an ambiguity level such that ambiguity has no impact on the investment decision. However, the bottom line remains unchanged: for most decision makers, an increase in ambiguity has a positive impact on the investment decision.

Now we turn to the extension of the investment model, and analyze the effect of an increase in the perceived level of ambiguity when the firm faces an ambiguous outside option. First, we consider the impact of an increase in ambiguity on the outside option itself. Taking the derivative of the outside option X with respect to the level of perceived ambiguity (ϵ), we see that the impact of ambiguity is positive if and only if ϵ is strictly positive, and α larger than 0.5. Otherwise the outside option decreases in value:

$$\frac{\partial X}{\partial \epsilon} = 2\alpha - 1$$

What happens to option to invest $F(\pi_t, X)$ if perceived ambiguity rises? Compared to the simple case, *both* the expected investment value and the outside option might be affected by changes in ambiguity. A pessimist ($\alpha = 0$), for example, will evaluate both the project value and the outside option *lower* in the presence of ambiguity. Hence, an increase in ambiguity is feared by such a decision maker - regardless if he will invest or not. The interesting question is how changes in ambiguity impacts the investment decision. Does a higher level of ambiguity induce a pessimistic decision maker to abandon investment, similar to before? Not necessarily, since the outside option gets less attractive as well. Hence, it depends on the relative importance of both the project value and the outside option to see which effect eventually dominates.

[Figure 4 goes here]

This effect can be seen in figure 4, which plots the investment threshold π^* as a function of perceived ambiguity of the project value (κ) for several degrees of ambiguity of the outside option (ϵ). The level of optimism is fixed at $\alpha = 0.2$, i.e. we consider a rather pessimistic decision maker. In general, the shape of the investment threshold curve π^* is similar to the simple case without outside option: at first, the threshold rises, but with a increased level of ambiguity the threshold decreases again. However, as the outside option is getting more ambiguous, i.e., ϵ increases, the threshold level lowers: not investing gets more costly ($X < 0$), such that the firm is induced to invest rather in the project than loosing money by not pursuing the venture.

5.4.2 An increase in optimism

Next, we can look at the effect of an increase in optimism (α) on the subjective project evaluation and investment decision. How is the value of the investment project influenced by an increase in optimism? To see this, we look at the relation between $V(\pi_t)$ and α. Consider the derivative of the coefficient ϕ with respect to α:

$$\frac{\partial \phi}{\partial \alpha} = \frac{1}{\rho - (\mu + \kappa\sigma)} - \frac{1}{\rho - (\mu - \kappa\sigma)} \geq 0$$

This expression is unambiguously positive as long as κ and σ are both strictly positive. Since the value of the investment project is linear in ϕ, increasing optimism always leads to an increase in the perceived value of the project. This is quite intuitive: Future growth opportunities are considered to be more likely, and hence the expected present value of future profits rises.
Similarly, the value of the outside option is positively affected by an increase in the level of optimism, as the derivative of X with respect to α shows:

$$\frac{\partial X}{\partial \alpha} = 2\epsilon \geq 0$$

which is strictly positive if both subjective and objective ambiguity are positive. Again, this effect is in line with the expectation: An optimistic decision maker will also evaluate the outside option higher than a pessimistic decision maker. Again, these positive effects translates directly in the same manner on to the value of the option to invest, $F(\pi_t)$, regardless whether there is an outside option or not.
How is the investment threshold affected by an increase in optimism? First, we look at the situation without outside option. Figure 5 plots the investment threshold (π^*) as a function of the value of optimism (α) for different levels of ambiguity (κ). For all strictly positive degrees of ambiguity, increasing optimism lowers the threshold level. For low values of α, i.e., a pessimist, the threshold is higher than without ambiguity; for high values of α, i.e., an optimist, the threshold is below. Again, this negative relation between optimism and investment threshold is in line with our intuition: The more optimistic the investor, the higher he valuates the project. Thus, the investment threshold decreases when optimism rises. The higher the level

of ambiguity, the more pronounced is this effect.

[Figure 5 goes here]

We finally come to the interesting case with an ambiguous outside option. Since both the value of investing and the value of the outside option increase with rising ambiguity, the joint effect is not obvious: it very much depends on the relative importance of both effects. Figure 6 plots the investment threshold π^* as a function of optimism α for various levels of ambiguity of the outside option (as measured by ϵ). If the objective ambiguity of the outside option is low compared to the objective ambiguity of the investment project (here ϵ close to 0), the positive effect of the investment project is predominant: the more optimistic the investor, the more he is inclined to invest. However, if the outside option is highly ambiguous, the effects are reversed. In the extreme case (here $\epsilon = 10$), a pessimist will invest even though the current profits are equal to 0: Although the expected value of the investment projet is 0, and thus the value net of investment cost is at -10, a pessimist evaluates the highly ambiguous outside options as well very negative at $X = -10$. As optimism gradually increases, both options are evaluated higher. However, in this case, the relative effect of the outside option is predominant, so that an optimist faces a higher investment threshold π^* compared to a pessimist.

[Figure 6 goes here]

6 Flexible Investment Timing

So far, we have focussed on the optimal investment decision when the entrepreneur faces a pass-fail decision. We discarded the entrepreneur's possibility to put off the investment project for a while and to decide upon the venture at a later point of time. Yet, in many practical situations, investment projects have a higher degree of flexibility regarding the investment timing. Quite often the decision maker has the possibility to defer the project for a certain time period, in order to wait for some - hopefully - more favorable investment conditions. In this section, we thus extend our investment model by lifting the restriction of the immediate investment requirement.

6.1 The investment problem

Once we allow the investment to be carried out at later points of time, the determination of the optimal investment time becomes paramount. The investment problem changes accordingly.

Now, the investor wants to maximize the value of the project over the investment time τ. In absence of ambiguity, the problem is as follows:

$$F(\pi_t) = \max_{\tau \in [t,\infty)} E_t \left[\int_{\tau}^{\infty} e^{-\rho(s-t)} \pi_s ds - e^{-\rho(\tau-t)} I \right] \tag{15}$$

The problem in (15) is known as irreversible investment problem, see e.g. Dixit and Pindyck (1994). Given the infinite time horizon, the maximization problem has the form of a recursive optimal stopping problem, and can thus be solved by dynamic programming, leading to a analytical characterization of the solution. More precisely, it allows to derive a critical level of current profits π' (independent of time), that must be surpassed in order to invest. In contrast to π^*, which gives the critical level of profits when having only the possibility to invest immediately or to abandon the projet, π' reflects the possibility of delaying the investment project for a certain time. We have therefore $\pi' \geq \pi^*$. The difference captures the "value of waiting", i.e., having the possibility to carry out the project when investment conditions are better increases the investment threshold.

Now, in an ambiguous environment, the maximization problem of (15) has to reflect that the decision maker faces a set of probability distributions $\mathcal{P}$, as specified in section 4. Applying α-maxmin expected utility, we get the following maximization problem:

$$F(\pi_t|\alpha) = \max_{\tau \in [t,\infty)} \alpha\text{-}E_t \left[\int_{\tau}^{\infty} e^{-\rho(s-t)} \pi_s ds - e^{-\rho(\tau-t)} I \right] \tag{16}$$

However, it is generally not possible to derive a solution to this problem. Although we face an infinite time horizon and a constant level of ambiguity (specified by the constant ambiguity range K), and thus have at each point of time a priori an identical decision problem only depending on the state variable π_t, the optimization problem of (16) cannot be represented in a recursive way. The reason is that the preferences themselves have to be recursive for dynamic programming techniques to be applicable. Unfortunately, as discussed in section 7, α-MEU preferences do generally not exhibit such a recursive structure.

6.2 Special cases

There are however some special cases in which it is possible to derive a solution for the optimal investment strategy under ambiguity, as presented in (16). In the case of complete pessimism ($\alpha = 0$) or complete optimism ($\alpha = 1$) α-MEU preferences preserve a recursive structure, as shown in section 7. Hence, dynamic programming yields an analytical solution, similar to the standard real option theory. In their paper, Nishimura and Ozaki (2007) examine the worst case scenario, i.e. the pessimistic decision maker, which corresponds our case of $\alpha = 0$. The solution of the optimistic case is just analogous. We can hence state:

Proposition 3. (Flexible investment threshold): *Given a complete optimistic decision maker ($\alpha = 1$), or a complete pessimistic decision maker ($\alpha = 0$), the critical level of current profits π_t, that must be attained in order to invest is given as*

$$\pi' = \frac{b}{(b-1)\phi} I \tag{17}$$

where b is given by

$$b = \frac{1}{2} - \frac{\gamma}{\sigma^2} + \sqrt{\left(\frac{\gamma}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1$$

and γ is given by

$$\gamma = \alpha(\mu + \sigma\kappa) + (1-\alpha)(\mu - \sigma\kappa)$$

and ϕ is given by (9)*.*

Proof. If $\alpha = 0$, the proof is given by Proposition 2 in Nishimura and Ozaki (2007). In case of $\alpha = 1$ the proof is dual by replacing the infinimum term with the supremum term. □

Once π_t exceeds π', the option to invest is executed and investment is made. Note that equation (17) is rather similar to the usual expression for the investment threshold without ambiguity, where π' is given by: $\frac{b}{(b-1)} I(\rho - \mu)$. It depends on the parameters α and κ how the critical level deviates from the standard case. In the case of complete ambiguity aversion ($\alpha = 0$), the critical level π' is always higher in presence of uncertainty than without it. Because future prospects are evaluated very pessimistic, current conditions must reach higher levels in order to decide positively about the investment. Of course, in case of $\kappa = 0$, the critical levels coincide. It is also possible to derive the value of the option to invest:

Proposition 4. (Investment option value): *Let the objective level of ambiguity be specified by $K = [-\kappa, \kappa]$. Then, given α-MEU preferences and the rectangular structure of beliefs $\mathcal{P}$, the α-maxmin expected value of the investment option with an infinite profit stream π_t is given by (for $\alpha \in \{0, 1\}$):*

$$F(\pi_t|\alpha) = \begin{cases} \frac{(b-1)^{b-1}}{I^{(b-1)} b^b} (\pi_t \phi)^b = \frac{(b-1)^{b-1}}{I^{(b-1)} b^b} V(\pi_t|\alpha)^b & \text{for } \pi_t < \pi' \\ \pi_t \phi - I = V(\pi_t|\alpha) - I & \text{for } \pi_t \geq \pi' \end{cases} \tag{18}$$

6.3 Comparative statics

Again, we have a look at some comparative statics to get a better understanding of the model's predictions. Since we restrict the attitude towards ambiguity to the two extreme cases only, we present the effect of changes in the level of ambiguity (κ) only.

6.3.1 An increase in ambiguity

How is the decision maker's evaluation of the investment option affected by an perceived increase in ambiguity? Figure 7 displays the natural logarithm of the value of the option to invest $F(\pi_t|\alpha)$ as a function of the level on ambiguity (κ) for the case of complete optimism ($\alpha = 1$), and complete pessimism ($\alpha = 0$). In case of complete optimism, ambiguity increases the option value, in case of complete pessimism, ambiguity decreases its value. This is fairly similar to figure 1 which plots the value of the project $V(\pi_t|\alpha)$. This pattern is quite intuitive, since - besides the coefficient - the value of the option to invest equals the value of the installed investment to the power of b (see equation (18)). Since b is larger than 1, the responsiveness (positive or negative) of the value of the investment is enforced in the option formula.

[Figure 7 goes here]

Another interesting question is how the investment timing is affected by a perceived change in ambiguity. Since the timing of the investment is fully determined by the instant current profits π_t exceeding the threshold level π' for the first time, all that matters is to examine the relation between threshold level π' and ambiguity. Figure 8 displays the investment threshold π' as a function of the level of ambiguity for the case of complete optimism, and complete pessimism. In case of complete optimism, ambiguity decreases the investment threshold π', in case of complete pessimism, the presence of ambiguity increases its value. Intuitively, the reason for this effect is clear: Due to ambiguity, the optimist perceives the value of the investment project to get higher (lower for the pessimist), as seen before. Hence, it becomes more costly (less costly) to wait rather than to invest and receiving the stream of profits.

[Figure 8 goes here]

The case of an ambiguity loving entrepreneur has also another implication. Compared to the absence of ambiguity, the threshold decreases by more than 10%, thereby narrowing the gap between the investment threshold of the classical net present value (NPV) rule of investment (which is represented by the low straight line in the graph) and the irreversible investment approach. Although the difference is still large, other parameter values would reduce the difference further. Hence, ambiguity loving behavior is able to partially reconcile the usual NPV rule for investment with the irreversible investment approach.

7 Dynamic Inconsistency of α-MEU Preferences

The α-MEU preferences and the related neo-additive capacities (Chateauneuf et al., 2007) offer a convenient possibility to model decision making under ambiguity. Compared to the maximin preferences of the early Gilboa and Schmeidler (1989), they extend the model to include ambiguity loving features. Moreover, they allow for a separation between the level of ambiguity, as specified by the set of priors $\mathcal{P}$, and the ambiguity attitude reflected by the individual parameter α.

Unfortunately, preference models that can be expressed as a convex combination of several attitudes towards ambiguity have a crucial drawback: in general, they are not dynamically consistent. Preferences are said to be dynamically consistent if an *ex-ante* complete optimal decision plan based on prior beliefs is identical to the optimal decisions based on updated beliefs through a decision tree, and vice versa. If preferences are not dynamically consistent, intertemporal maximization problems as in (16) cannot be solved since optimal decision rules are no longer constant over time: the decision maker will revise the ex-ante optimal plan at later stages.

Although it is well-known that dynamic consistency is difficult to reconcile with ambiguity[13], this section shortly provides a formal analysis for the specific case of α-MEU and related preference models.

7.1 Recursive structure of preferences

For preferences to be dynamically consistent, they must have a recursive representation. However, α-MEU preferences do generally not exhibit such a recursive structure. For preferences to be recursive, the law of iterated expectations, $E_t[x] = E_t[E_s[x]]\ \forall s > t$, must be fulfilled. In the multiple priors model, an equivalent condition must hold, taking into account the multiplicity of probability measures at each point of time, and the expectation operator defined on these probability measures. In the case of $\alpha-$MEU preferences, this condition is given by $\alpha\text{-}E_t[x] = \alpha\text{-}E_t[\alpha\text{-}E_s[x]]\ \forall s > t$. Equivalently, using the explicit notation as presented in (4):

[13] See e.g. Epstein and Schneider (2003a), Hannay and Klibanoff (2006), Klibanoff et al. (2009), or Al-Najjar and Weinstein (2009).

$$\begin{aligned} & \alpha \sup_{p\in\mathcal{P}} E_t^p\left[x\right] + (1-\alpha) \inf_{p\in\mathcal{P}} E_t^p\left[x\right] \qquad (19) \\ = \; & \alpha \sup_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right] + (1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + (1-\alpha) \inf_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right] + (1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] \end{aligned}$$

where $\mathcal{P}'$ denotes the set of probability measures at time $s > t$, derived from $\mathcal{P}$ by the set of conditional probabilities imposed by rectangularity ($\mathcal{P}$ denoting as usual the probability set at time t). In general, the recursive structure imposed by this condition is not met. To see this, note that the second term can be transformed (see appendix C) into

$$\alpha^2 \sup_{p\in\mathcal{P}} E_t^p[x] + 2\,\alpha(1-\alpha) \sup_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p[x]\right] + (1-\alpha)^2 \inf_{p\in\mathcal{P}} E_t^p[x] \qquad (20)$$

which is generally different from the left hand side of (19). More intuitively, the decision maker evaluates x at each point of time as the weighted sum of the best and worst case scenario. So he evaluates at time $s > t$, the remaining decision tree under both the best case ($\sup_{p\in\mathcal{P}'} E_s[x]$) and the worst case scenario ($\inf_{p\in\mathcal{P}'} E_s[x]$), and combines them together to α-$E_s[x]$. However, when evaluating x at an earlier time t, the decision maker will not take into account the best case scenario at time t of the worst case scenario at time s, since the $\alpha-$MEU preferences do not allow for intertemporal weighting of best and worst cases (the term in the middle of expression (20)). Instead, $\alpha-$MEU preferences reflect only the intratemporal weighting of the best and the worst case. Hence, the decision maker evaluates x at time t for the best or the worst cases only, i.e he will consider at time t *only the best case of the best case scenario in* s *and the worst case of the worst case scenario in* s, i.e. the two other terms of (20). Consequently, α-$E_t[x]$ cannot use all the information as α-$E_t[\alpha$-$E_s[x]]$ can incorporate, so that both expressions differ from each other.

As a consequence, α-MEU preferences are generally not dynamically consistent. There are however special cases in which dynamic consistency can preserved. In case of complete pessimism ($\alpha = 0$) or complete optimism ($\alpha = 1$) the condition (19) is fulfilled, as can be easily verified in (20). The convex combination of the α-MEU preferences reduce then to a single term, consisting either of the best case scenario, or the worst case. As Chen and Epstein (2002) show, each of the two terms has a recursive structure itself - given the rectangularity assumption on the set $\mathcal{P}$. Another trivial case is when the set $\mathcal{P}$ is singleton, i.e., in the absence of ambiguity. In this case, the infinimum and supremum terms coincide.[14]

[14] If deviating from the strong rectangularity assumption, one can think of other conditions in which (19)

8 Conclusion

At the latest from the literature in behavioral economics and psychology we know that people tend to overestimate their competence. In ambiguous situations, this overconfidence can lead to ambiguity-loving behavior. This study shows that an entrepreneur's attitude towards ambiguity is crucial for his investment decision as well. By applying α-MEU preferences to the irreversible investment problem, we analyze investment behavior in situations that are objectively identical. This paper shows that the presence of ambiguity often leads to an increase in the subjective project value, and entrepreneurs are more eager to invest. In dynamic settings then, we show that ambiguity loving decision makers face a lower in investment threshold. Thereby we also reconcile the usual NPV rule for investment with the irreversible investment approach, whose investment thresholds are generally considered to be too high.

Although the positive impact of ambiguity love on investment decisions might not be surprising, we think that the results are nevertheless interesting. By showing that investment decisions are partly reversed even if investors are ambiguity averse except for a very small fraction of optimism, we show that the standard assumption of complete ambiguity aversion is not robust to small deviations.

The results of this paper have some interesting implications. First, the impact of an investor's ambiguity attitude on investment decisions highlights the importance of so-called soft factors, such as personal judgements, on investment in general. Our results suggest that the role of investor sentiment might have been underestimated in existing investment models. Although this study takes a micro perspective on investment decisions, it might also contribute to the analysis of the relation between investor confidence and investment cycles on a larger scale.[15] Second, the results suggest that the high-self confidence of entrepreneurs might have a rather positive impact on investment decisions - as opposed to what is sometimes believed. As Bhidé (2000) writes: "Even if objectively unwarranted, [excessive self-confidence] does not necessarily lead to overinvestment [...]. Rather it can offset excessive ambiguity aversion and thus mitigate underinvestment in uncertain businesses." Our paper then gives some theoretical underpinnings for this claim. Third, the results presented in this paper have also implications for other economic settings that are based on optimal stopping problems under ambiguity, such as labor search or wedding decisions. Our findings suggest that the negative relation between reservation

holds. As a simple example consider the case when $\mathcal{P}'$ reduces to a state dependent singleton. However, in the absence of the strong rectangularity structure, it is not possible to derive analytical solutions. In addition, such models rarely make economically sense - especially in the continuous time context.

[15] A prominent example is the R-word index as proposed by the Economist (2002), that predicts recessions by analyzing how often the word "recession" appears in newspapers.

wage and ambiguity as brought forward by Nishimura and Ozaki (2004b) only holds true for pessimistic job candidates. Optimistic individuals in contrast will wait longer before accepting a new position - or defer the decision to marry a spouse.

This paper should be considered only as a first step to analyze the impact of ambiguity attitudes on investment decisions. Many open issues remain, such as the reconciliation of dynamic consistency and extreme ambiguity attitudes, or an extension of the basic model to allow for learning under ambiguity.

Appendix A

When leaving the standard expected utility framework with Bayesian updating rules, dynamic consistent behavior is no longer automatically achieved: optimal ex-post behavior might differ from ex-ante optimal plans (Sarin and Wakker, 1998). There are various possibilities to restrict behavior in ambiguous environments to preserve dynamic consistency (Al-Najjar and Weinstein, 2009). Epstein and Schneider (2003b) propose a model that restricts information structures on the way through a decision tree. This approach restricts a decision maker's updated beliefs to a suitable collection of sets of one-step-ahead conditional probabilities that do not violate dynamically consistent behavior. More precisely, it ensures that the set of priors of one-step-ahead conditional probabilities is identical to the original set of priors. This condition is called *rectangularity* in the terminology of Epstein and Schneider (2003a), and *strongly rectangular* in the terminology of Nishimura and Ozaki (2007). Riedel (2009) calls it *time-consistency.* For are more detailed exposition, see also Asano (2005).

Rectangularity thus ensures that believes have a recursive structure, and thereby allows for intertemporal optimization problems to become dynamically consistent[16]. Rectangularity can be achieved by defining the set $\mathcal{P}$ of probability distributions with the help of suitable density generators (Chen and Epstein, 2002). In the following, we thus present the Chen and Epstein (2002) model of ambiguity in continuous time, i.e., the specification of the set $\mathcal{P}$.

Time t evolves over $[0, T]$ and uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{L}$ be the set of real-valued, measurable, and $\mathbb{F}$-adapted stochastic processes on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{L}^2$ be a subset of $\mathcal{L}$ which is defined by

$$\mathcal{L}^2 = \left\{ (\theta_t)_{0 \le t \le T} \in \mathcal{L} \,\middle|\, \int_0^T \theta_t^2 dt < +\infty \quad P - \text{a.s.} \right\}$$

A density generator is a stochastic process $\theta = (\theta_t) \in \mathcal{L}^2$ for which the process (z_t^θ) is a $\mathbb{F}$-martingale, where

$$(\forall t) \qquad z_t^\theta = \exp\left(-\frac{1}{2} \int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s \right)$$

A sufficient condition for (z_t^θ) to be a $\mathbb{F}$-martingale and thus for (θ_t) to be a density generator is Novikov's condition:

[16]However, as discussed in section 7, not only beliefs must be dynamically consistent, but the preferences themselves.

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < +\infty$$

With density generators we can construct other probability measures from a given probability measure:

$$(\forall A \in \mathcal{F}_T) \quad \mathcal{Q}^\theta(A) = \int_A z_T^\theta(\omega) dP(\omega)$$

where $\mathcal{Q}^\theta$ is the new probability measure which is absolutely continuous with respect to P. Thus, given a set Θ of density generators, the corresponding set of probability measures is

$$\mathcal{P} = \{\mathcal{Q}^\theta | \theta \in \Theta\}$$

Hence, ambiguity is characterized by $\mathcal{P}$ for some set Θ. Finally, we have to specify the set of density generators Θ that generate the set of probability measures $\mathcal{P}$. We rely on the definition of κ-ignorance and *IID ambiguity* by Chen and Epstein (2002) and specify Θ as follows:

$$\Theta = \{(\theta_t) \in \mathcal{L}^2 | \theta_t(\omega) \in [-\kappa, \kappa] \quad (m \otimes P) - a.s.\}$$

where m denotes the Lebesgue measure restricted on $\mathcal{B}([0, T])$.

This definition ensures that any element of Θ is restricted to the non-stochastic interval $K = [-\kappa, \kappa]$. This interval K can be interpreted as the objective, prevailing level of ambiguity. Consequently, the corresponding set of measures $\mathcal{P}$ is clustered within a constant ambiguity interval around the original (or objective) measure P.

Although we rely in this appendix for simplicity on a finite time horizon, the results also hold true for the infinite time horizon, see Nishimura and Ozaki (2007).

Appendix B

In this appendix, we prove Proposition 1 to derive the expected value of the investment project. The α-maxmin expected value of the project $V(\pi_t|\alpha)$ is given as follows:

$$V(\pi_t|\alpha) = \alpha \sup_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta} \left[\int_t^\infty e^{-\rho(s-t)} \pi_s ds \right] + (1-\alpha) \inf_{Q^\theta \in \mathcal{P}} E_t^{Q^\theta} \left[\int_t^\infty e^{-\rho(s-t)} \pi_s ds \right] \tag{7}$$

For notational convenience, we set $t = 0$, such that current profits are denoted π_0. Next, since we restrict the density generators θ to the non-stochastic interval $K = [-\kappa, \kappa]$, equation (7) can be rewritten as

$$V(\pi_0|\alpha) = \alpha \sup_{\theta \in K} E_0^\theta \left[\int_0^\infty e^{-\rho t} \pi_t dt \right] + (1-\alpha) \inf_{\theta \in K} E_0^\theta \left[\int_0^\infty e^{-\rho t} \pi_t dt \right]. \tag{21}$$

Next, we show that

$$\sup_{\theta \in K} E_0^\theta \left[\exp\left(\left(B_t^\theta - \int_0^t \theta_s ds \right) \sigma \right) \right] = E_0^{-\kappa} \left[\exp\left(\left(B_t^{-\kappa} + \kappa t \right) \sigma \right) \right] \tag{22}$$

where $E^{-\kappa}$ denotes the expectation with respect to the probability measure generated by the non-stochastic density generator $-\kappa \in K$. To see this, note that $\forall (\theta_t) \in K = [-\kappa, \kappa]$, it is true that:

$$\begin{aligned} E_0^\theta \left[\exp\left(\left(B_t^\theta - \int_0^t \theta_s ds \right) \sigma \right) \right] &\leq E_0^\theta \left[\exp\left(\left(B_t^\theta - \int_0^t -\kappa ds \right) \sigma \right) \right] \\ &= E_0^\theta \left[\exp\left(\left(B_t^\theta + \kappa t \right) \sigma \right) \right] \\ &= \exp\left(\left(\frac{1}{2} \sigma t + \kappa t \right) \sigma \right) = E_0^{-\kappa} \left[\exp\left(\left(B_t^{-\kappa} + \kappa t \right) \sigma \right) \right] \end{aligned}$$

Now we transform the supremum expression in the first term of equation (21) above as follows:

$$
\begin{aligned}
& \sup_{\theta\in K} E_0^\theta \left[\int_0^\infty e^{-\rho t}\pi_t dt\right] \\
= & \sup_{\theta\in K}\int_0^\infty E_0^\theta\left[e^{-\rho t}\pi_0 \exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)t-\sigma\int_0^t \theta_s ds+\sigma B_t^\theta\right)\right]dt \\
= & \int_0^\infty \pi_0 e^{\left(\mu-\rho-\frac{1}{2}\sigma^2\right)t}\sup_{\theta\in K}E_0^\theta\left[\exp\left(\left(B_t^\theta-\int_0^t\theta_s ds\right)\sigma\right)\right]dt \\
= & \pi_0\int_0^\infty e^{\left(\mu-\rho-\frac{1}{2}\sigma^2\right)t}E_0^{-\kappa}\left[\exp\left(\left(B_t^{-\kappa}+\kappa t\right)\sigma\right)\right]dt \\
= & \pi_0\int_0^\infty e^{\left(\mu-\rho-\frac{1}{2}\sigma^2\right)t}\exp\left(\sigma\left(\kappa t+\frac{1}{2}\sigma t\right)\right)dt \\
= & \pi_0\int_0^\infty e^{-(\rho-\kappa\sigma-\mu)t}dt \\
= & \frac{\pi_0}{\rho-(\mu+\kappa\sigma)}
\end{aligned} \tag{23}
$$

where we use the relation (22) to establish the third equality sign.

By replacing the supremum operator with infinimum operator, we can transform the second term of (21) into the following expression:

$$
\inf_{\theta\in K} E_0^\theta\left[\int_0^\infty e^{-\rho t}\pi_t dt\right] = \frac{\pi_0}{\rho-(\mu-\kappa\sigma)} \tag{24}
$$

Inserting the terms (23) and (24) into (21), we finally obtain following solution for the expected value of the installed investment project:

$$
\begin{aligned}
V(\pi_0|\alpha) &= \alpha\frac{\pi_0}{\rho-(\mu+\kappa\sigma)}+(1-\alpha)\frac{\pi_0}{\rho-(\mu-\kappa\sigma)} \\
&= \pi_0\left(\frac{\alpha}{\rho-(\mu+\kappa\sigma)}+\frac{(1-\alpha)}{\rho-(\mu-\kappa\sigma)}\right)
\end{aligned} \tag{25}
$$

which is identical to the expression (8) in Proposition 1 (for $t=0$). Here at the latest, we can see that the condition $\kappa<(\rho-\mu)/\sigma$ must be fulfilled that the problem makes sense.

Appendix C

In this appendix, we present the derivation of expression (20) from the right hand side of the condition (19). First note that for a random variable x, which is real-valued, measurable, and $\mathbb{F}$-adapted on $(\Omega, \mathcal{F}, P)$, and given that $\mathcal{P}$ is strongly rectangular, it holds $\forall s > t$:

$$\sup_{p\in\mathcal{P}} E_t^p \left[\sup_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] = \sup_{p\in\mathcal{P}} E_t^p[x] \tag{26}$$

For a proof, see Lemma B3 in Nishimura and Ozaki (2007). Similarly, by replacing the supremum operator with infinimum operator, one can show that:

$$\inf_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] = \inf_{p\in\mathcal{P}} E_t^p[x] \tag{27}$$

Next, we use the minimax theorem for continuous stochastic processes under multiple priors and strong rectangularity:

$$\sup_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p[x]\right] = \inf_{p\in\mathcal{P}} E_t^p \left[\sup_{p\in\mathcal{P}'} E_s^p[x]\right] \tag{28}$$

For a proof, see e.g. Proposition 5.14 in Karatzas and Kou (1998). Finally, we can present the full derivation:

$$\begin{aligned}
& \alpha \sup_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right] + (1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + (1-\alpha) \inf_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right] + (1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] \\
= \;& \alpha \sup_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + \alpha \sup_{p\in\mathcal{P}} E_t^p \left[(1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + \\
& (1-\alpha) \inf_{p\in\mathcal{P}} E_t^p \left[\alpha \sup_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + (1-\alpha) \inf_{p\in\mathcal{P}} E_t^p \left[(1-\alpha) \inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] \\
= \;& \alpha^2 \sup_{p\in\mathcal{P}} E_t^p \left[\sup_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + \alpha(1-\alpha) \sup_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + \\
& (1-\alpha)\alpha \inf_{p\in\mathcal{P}} E_t^p \left[\sup_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] + (1-\alpha)^2 \inf_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p\left[x\right]\right] \\
= \;& \alpha^2 \sup_{p\in\mathcal{P}} E_t^p[x] + \alpha(1-\alpha) \left(\sup_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p[x]\right] + \inf_{p\in\mathcal{P}} E_t^p \left[\sup_{p\in\mathcal{P}'} E_s^p[x]\right]\right) + (1-\alpha)^2 \inf_{p\in\mathcal{P}} E_t^p[x] \\
= \;& \alpha^2 \sup_{p\in\mathcal{P}} E_t^p[x] + 2\,\alpha(1-\alpha) \sup_{p\in\mathcal{P}} E_t^p \left[\inf_{p\in\mathcal{P}'} E_s^p[x]\right] + (1-\alpha)^2 \inf_{p\in\mathcal{P}} E_t^p[x]
\end{aligned}$$

where we used the relations (26) and (27) to establish the third equality sign, and the minimax theorem (28) to finally obtain expression (20) in the last line.

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Figures

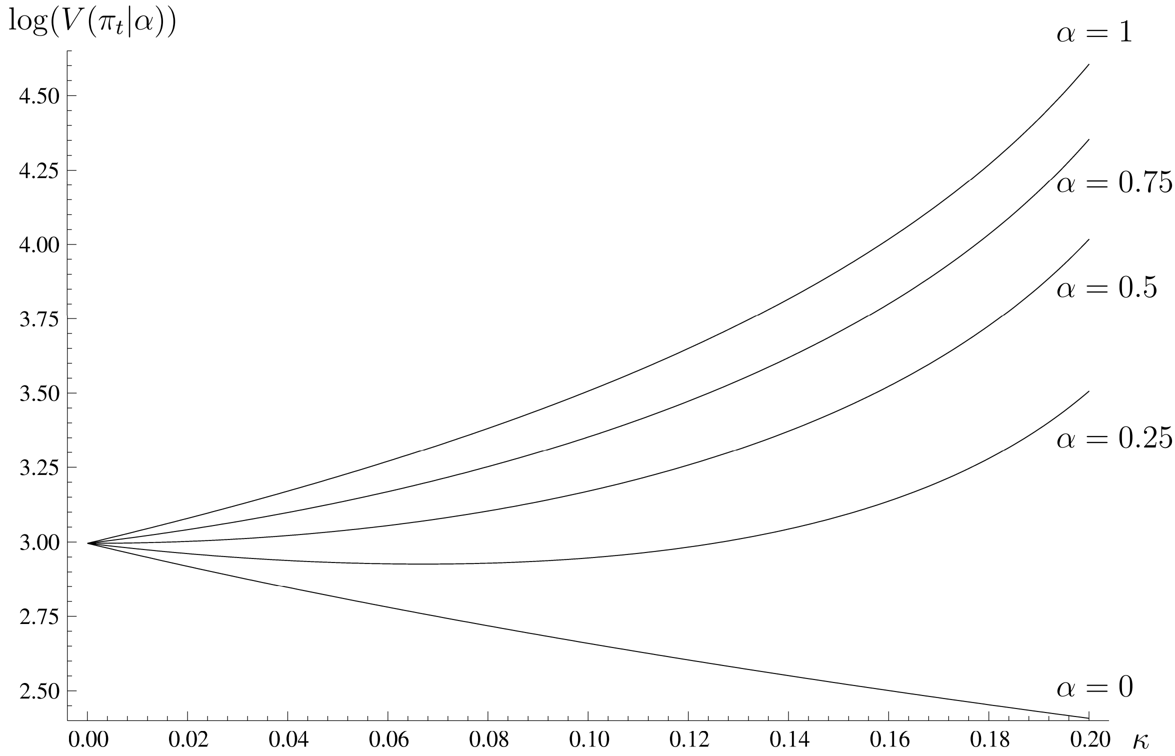

Figure 1: The natural logarithm of the value of the installed investment project $V(\pi_t|\alpha)$ as a function of the level on ambiguity (κ) for different values of optimism (α). In case of complete optimism ($\alpha = 1$), ambiguity increases the project value, in case of complete pessimism ($\alpha = 0$), ambiguity decreases its value. For all parameter values in-between, it depends on the level of ambiguity, whether increasing ambiguity rises or lowers the project value. The value of the installed project value is given by:

$$V(\pi_t|\alpha) = \pi_t \left(\frac{\alpha}{\rho - (\mu + \kappa\sigma)} + \frac{1-\alpha}{\rho - (\mu - \kappa\sigma)} \right) \tag{8}$$

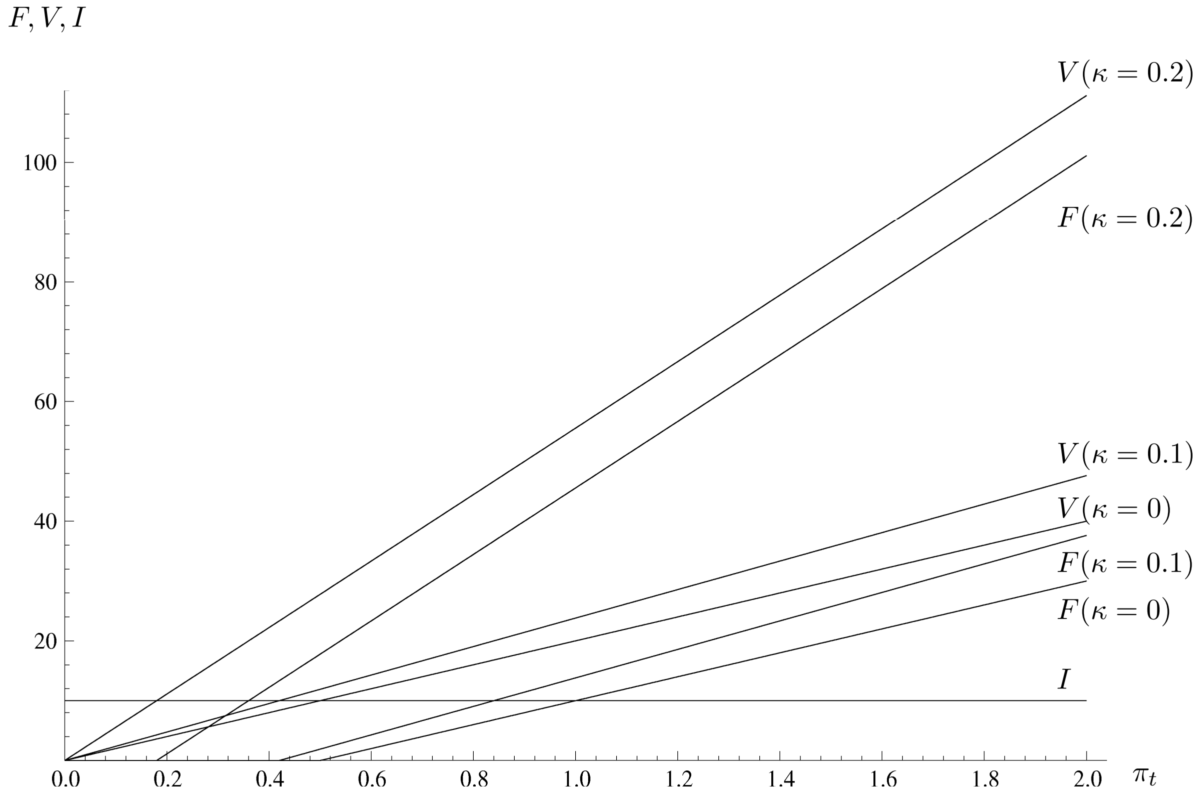

Figure 2: The value of the investment project $V(\pi_t|\alpha)$, and the option to invest $F(\pi_t|\alpha)$ as a function of the current profit level π_t for different degrees of ambiguity. The level of optimism is fixed at $\alpha = 0.5$. The value of the option to invest is given by:

$$F(\pi_t|\alpha) = \max\{V(\pi_t|\alpha) - I, 0\} = \max\{\pi_t\phi - I, 0\} \tag{10}$$

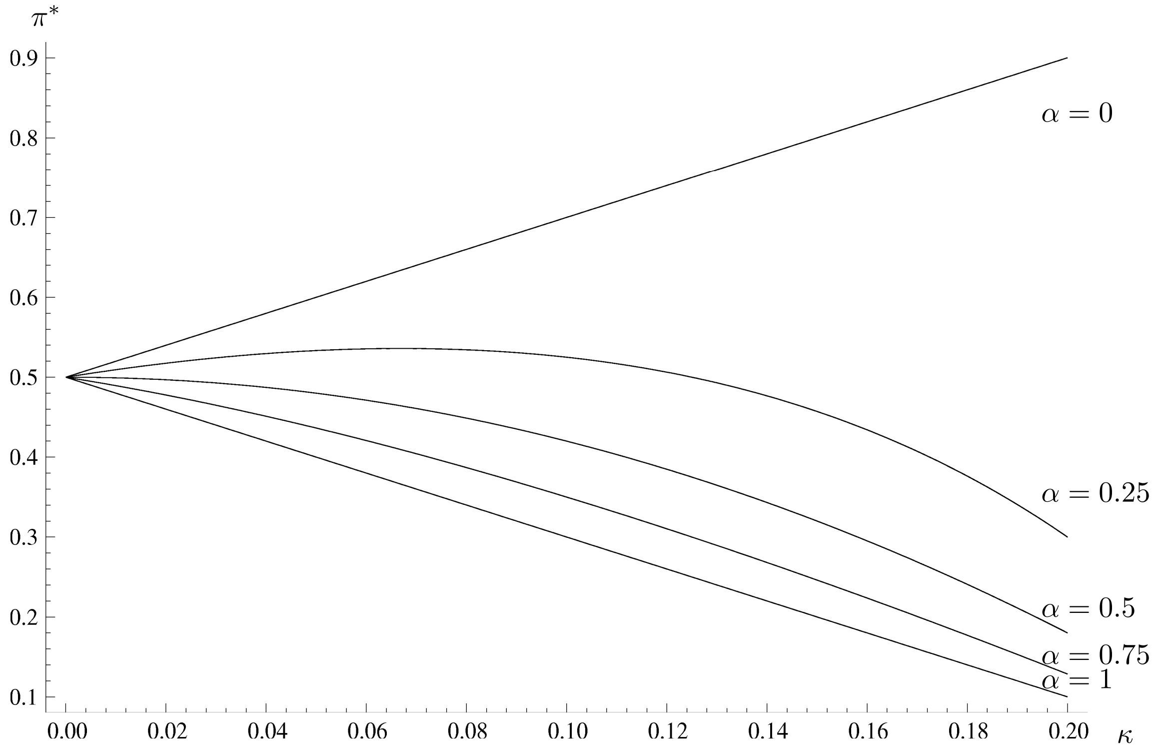

Figure 3: The investment threshold (π^*) as a function of the level of ambiguity (κ) for different values of optimism (α). In case of complete optimism ($\alpha = 1$), ambiguity decreases the investment threshold π^*, in case of complete pessimism ($\alpha = 0$), the presence of ambiguity increases its value. For parameter values in-between, ambiguity decreases the investment threshold as well, but not as much as in the optimistic case. In the absence of ambiguity, the threshold lies at 1. The function for the threshold level π^* is given by:

$$\pi^* = \frac{I}{\phi} = \frac{I}{\frac{\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{1-\alpha}{\rho-(\mu-\kappa\sigma)}} \tag{11}$$

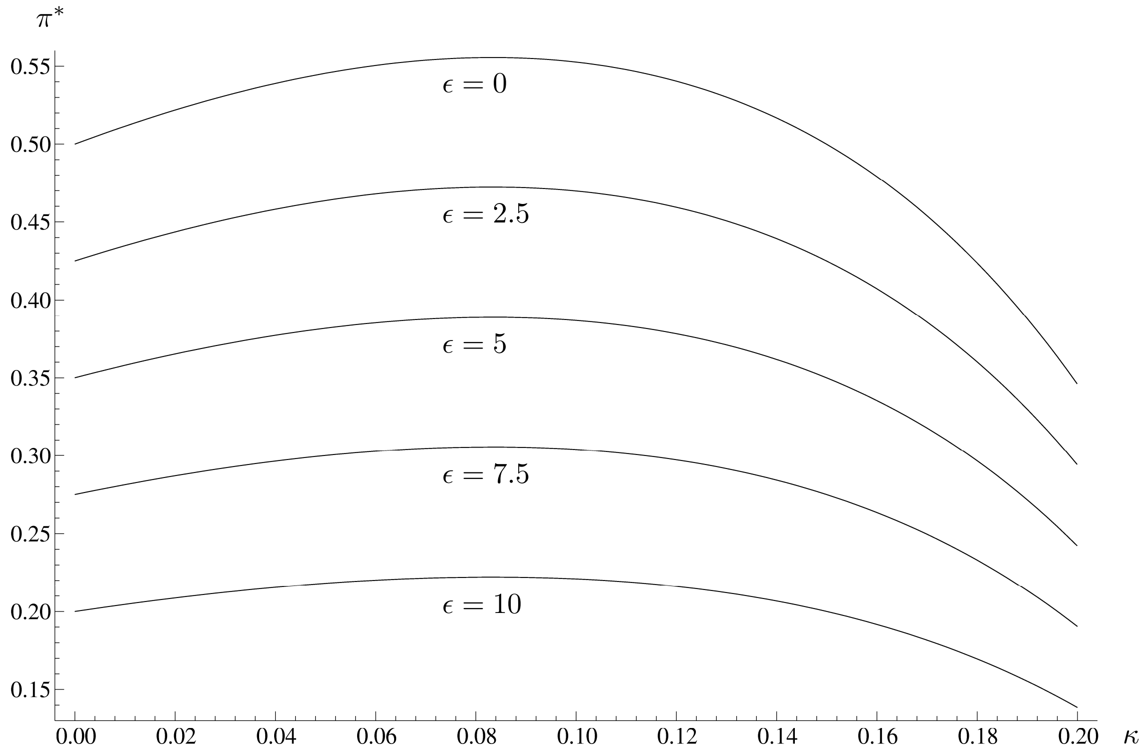

Figure 4: The investment threshold (π^*) as a function of the level of ambiguity (κ) for different degrees of ambiguity of the outside option as measured by ϵ. The level of optimism is fixed at α=0.2, i.e., a rather pessimistic entrepreneur. The function for the threshold level π^* with outside option is given by:

$$\pi^* = \frac{I + X}{\phi} = \frac{I + \epsilon(2\alpha - 1)}{\frac{\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{1-\alpha}{\rho-(\mu-\kappa\sigma)}} \tag{12}$$

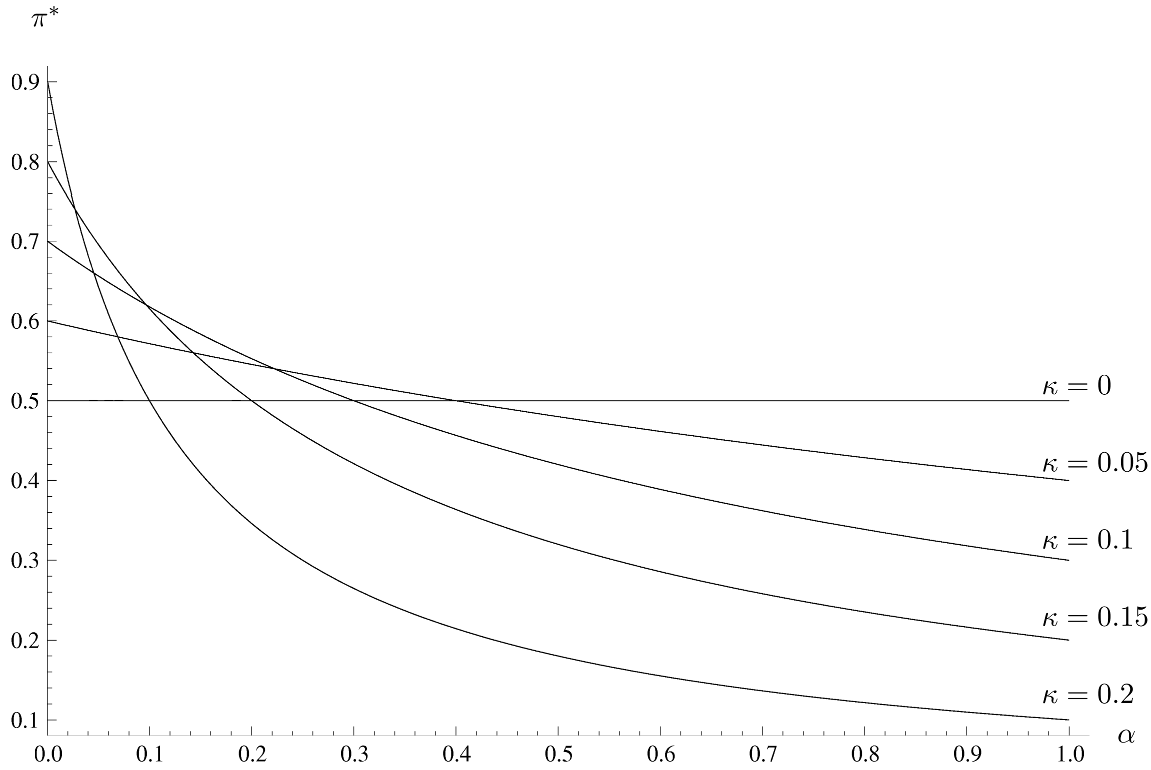

Figure 5: The investment threshold (π^*) as a function of the value of optimism (α) for different levels of ambiguity (κ). For all positive degrees of ambiguity, decreasing optimism lowers the threshold level. For low values of α, i.e. an pessimist, the threshold is higher than without ambiguity; for high values of α, i.e. an optimist, the threshold is below. The function for the threshold level without outside option π^* is given by:

$$\pi^* = \frac{I}{\phi} = \frac{I}{\frac{\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{1-\alpha}{\rho-(\mu-\kappa\sigma)}} \tag{11}$$

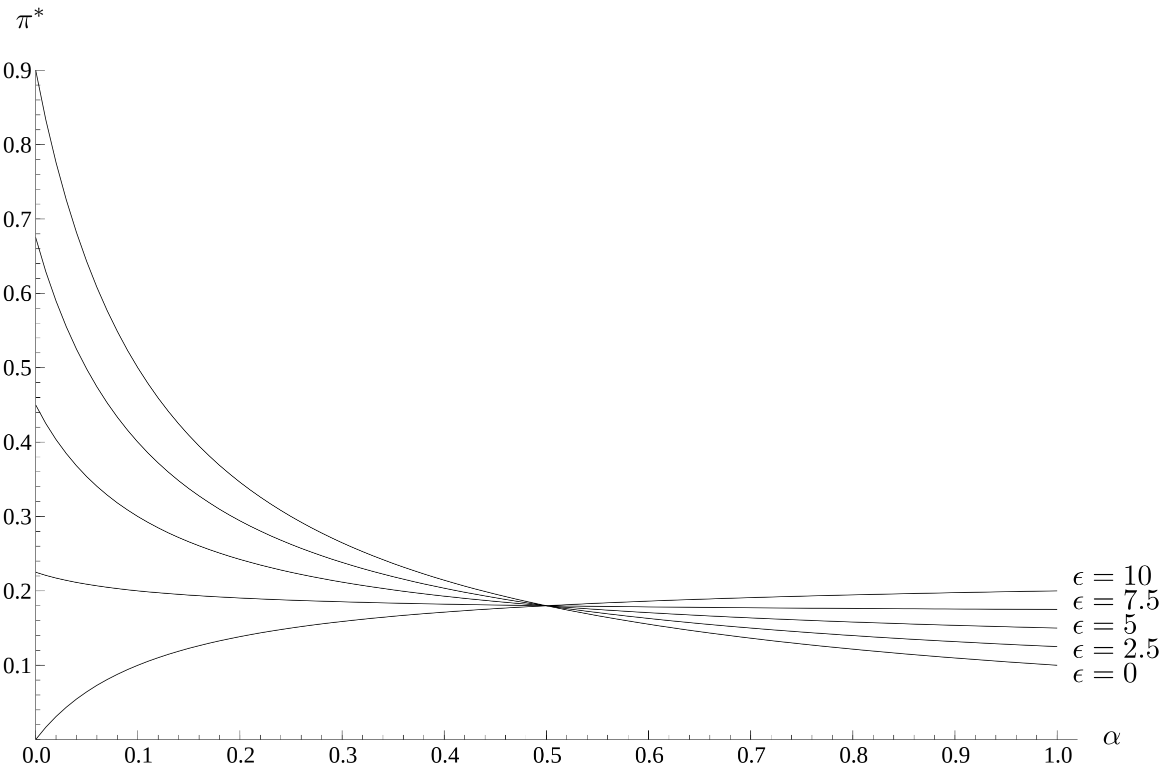

Figure 6: The investment threshold π^* as a function of optimism α for various levels of objective ambiguity of the outside option (as measured by ϵ). The function for the threshold level π^* is given by:

$$\pi^* = \frac{I + X}{\phi} = \frac{I + \delta\epsilon(2\alpha - 1)}{\frac{1-\delta}{\rho-\mu} + \frac{\delta\alpha}{\rho-(\mu+\kappa\sigma)} + \frac{\delta(1-\alpha)}{\rho-(\mu-\kappa\sigma)}} \tag{12}$$

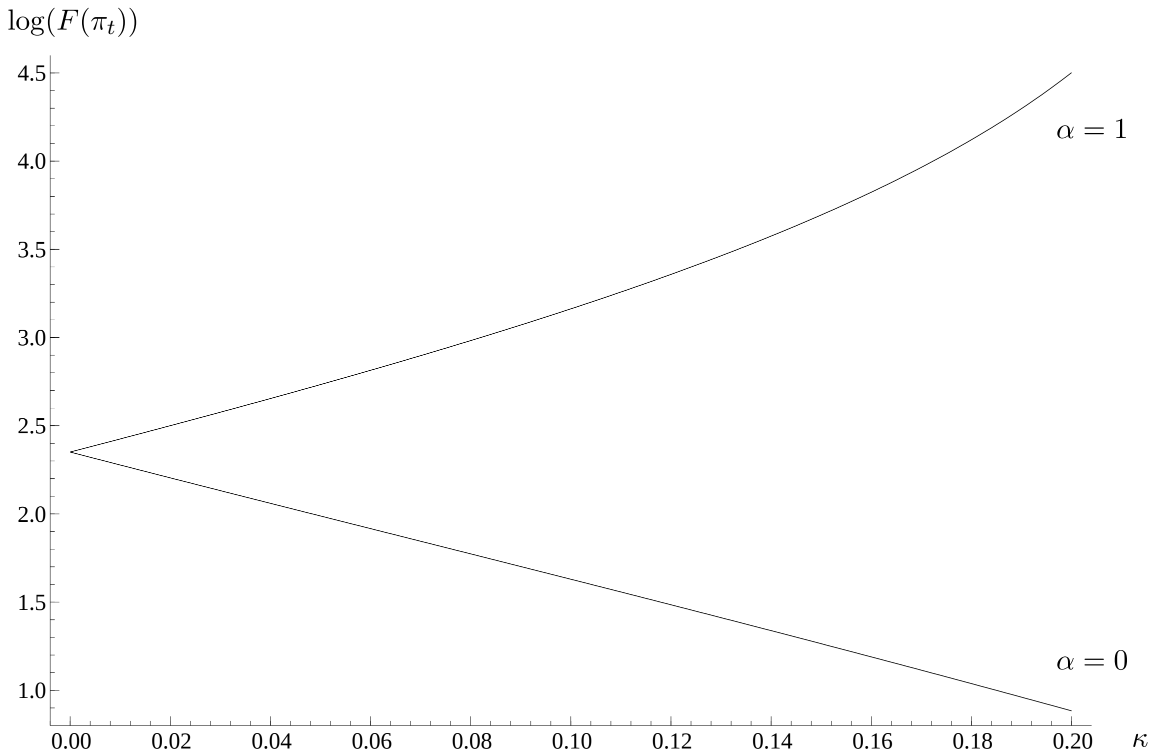

Figure 7: The natural logarithm of the value of the option to invest $F(\pi_t)$ as a function of the level of ambiguity (κ) for the case of complete optimism ($\alpha = 1$), and complete pessimism ($\alpha = 0$). In case of complete optimism ($\alpha = 1$), ambiguity increases the option value, in case of complete pessimism ($\alpha = 0$), ambiguity decreases its value. For $\alpha \in \{0, 1\}$ the value of the option to invest is given by:

$$F(\pi_t|\alpha) = \begin{cases} \frac{(b-1)^{b-1}}{I^{(b-1)}b^b}(\pi_t\phi)^b = \frac{(b-1)^{b-1}}{I^{(b-1)}b^b}V(\pi_t|\alpha)^b & \text{for } \pi_t < \pi' \\ \pi_t\phi - I = V(\pi_t|\alpha) - I & \text{for } \pi_t \geq \pi' \end{cases} \tag{18}$$

Since we assume $\pi_t = 1$, we always have $\pi_t < \pi'$ in this illustration. This can be seen on figure 8 or be verified using equation (17)

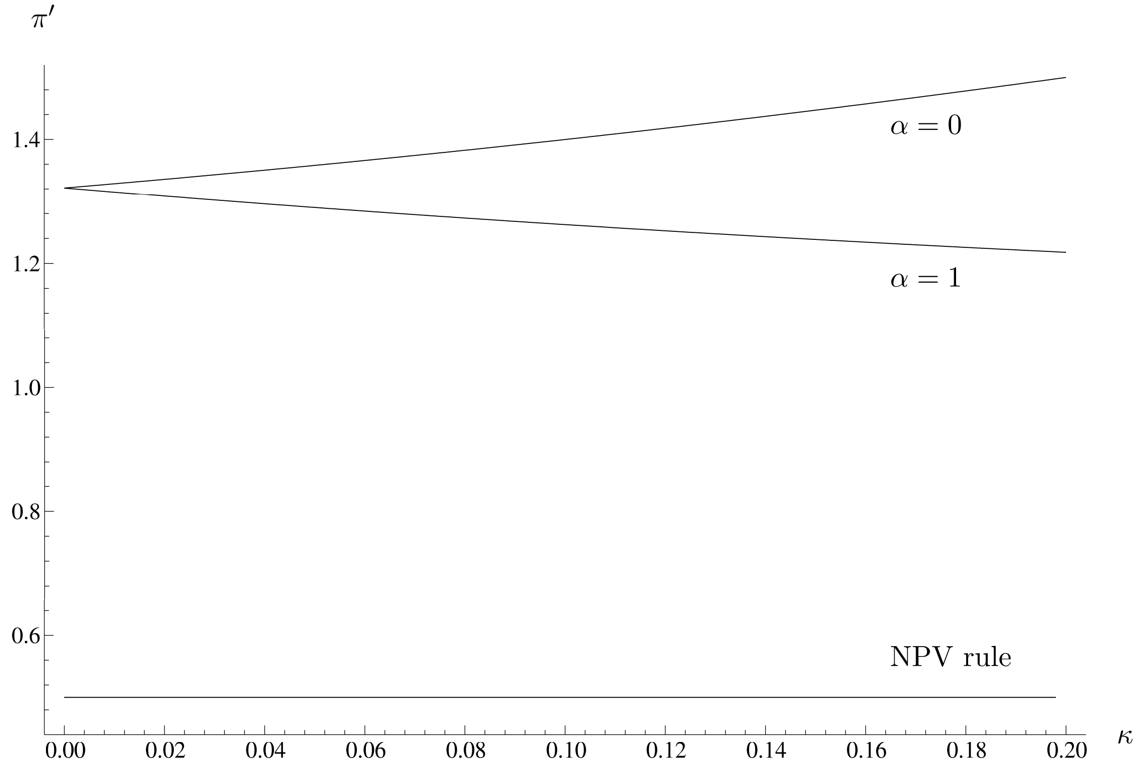

Figure 8: The investment threshold (π') as a function of the level of ambiguity (κ) for the case of complete optimism ($\alpha = 1$), and complete pessimism ($\alpha = 0$). In case of complete optimism ($\alpha = 1$), ambiguity decreases the investment threshold π', in case of complete pessimism ($\alpha = 0$), the presence of ambiguity increases its value. The investment threshold for the classical NPV rule lies in this example at $\pi' = I(\rho - \mu) = 0.5$. The function for the threshold level of π' is given by:

$$\pi' = \frac{b}{(b-1)\phi} I \tag{17}$$